Filomat 32:17 (2018), 6145–6153
https://doi.org/10.2298/FIL1817145A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Dilations of $*$-Frames and Their Operator Duals

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Abstract. Theory of frame has been extended from Hilbert spaces to various spaces, specially, to Hilbert C^*-modules. The paper introduces operator duals for frames with C^*-valued bounds in Hilbert C^*-modules. It is shown that operator duals for frames (*-frames) share several useful properties with ordinary duals. Some operator duals are constructed by a given operator dual or a given $*$-frame. Also, a characterization of all of operator duals for a given $*$-frame is obtained. Moreover, dilations of frames and their operator duals are studied. And, an application for frames and operator duals is suggested.

1. Introduction

Frames have been first introduced in 1952 by Duffin-Schaeffer [4]. They abstracted the fundamental notion of Gabor [6] to study signal processing. The theory of frames has been rapidly generalized and, until 2005, various generalizations consisting of vectors in Hilbert spaces or Hilbert C^*-modules have been developed. It is well known that the theory of Hilbert C^*-modules has applications in the study of locally compact quantum groups, complete maps between C^*-algebras, non-commutative geometry, and KK-theory. There are many differences between Hilbert C^*-modules and Hilbert spaces. It is expected that some problems about frames for Hilbert C^*-modules to be more complicated than those for Hilbert spaces. This makes the study of the frames for Hilbert C^*-modules important and interesting. So, frames in Hilbert spaces have been extended to frames in Hilbert C^*-modules [5]. In this case, studying of frames is not easily with respect to the Hilbert case because in Hilbert C^*-module case is needed to compare and to work with elements in a C^*-algebra. Alijani-Dehghan [1], have studied frames in Hilbert C^*-modules with C^*-valued bounds. They have given interesting results about frames and C^*-valued bounds. One of the most important of frame theory's applications is decomposition of elements of framework space. In this decomposition, coefficients are given by duals of frames. Dehghan-Hassankhani [3], have extended duals for frames and have introduced operator duals or generalized duals for frames in Hilbert spaces. We study operator duals and their properties for frames with C^*-valued bounds in Hilbert C^*-modules.

In [7], the authors explained the history of dilation of frames as follows. "The first result about the dilation of frames in Hilbert spaces was given in [8]. It was proved that every frame in a Hilbert space is a direct summand of a Riesz basis, in other words, each frame is a compression of a Riesz basis of a larger space, [8]. This helps us to understand the concept of frame from the geometric point of view. In [5], it was shown that this is still true for the modular frames. In particular, it was proved in [5], that each Parseval

2010 *Mathematics Subject Classification*. Primary 47C15 ; Secondary 46C20, 42C15.
Keywords. Dilation, $*$-frame, Hilbert C^*-module, operator dual frame.
Received: 15 July 2015; Revised: 21 October 2018; Accepted: 15 November 2018
Communicated by Dragan S. Djordjević
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frame of Hilbert C^*-modules can be dilated to an orthonormal basis. More generally, a dual frame pair in a Hilbert space can be dilated to a Riesz basis and its dual Riesz basis [2]. " Afterwards, the authors [7] have shown that these results are valid for Hilbert C^*-modules frames with different techniques with respect to Hilbert spaces case. We consider dilation for $*$-frames and their operator duals. However, the proof is independent on frame bounds, then the result is true for both frames and $*$-frames.

The paper is organized as follows. The reminder of section contains a brief account of definitions and basic properties of C^*-algebras, Hilbert C^*-modules, and frames.

The second section explains main results of the paper. Operator duals for $*$-frames are introduced and are compared to ordinary duals of frames in Hilbert C^*-modules.(Every ordinary dual is an operator dual.) special $\mathcal{A}$-linear combinations of ordinary duals are operator duals. Some operator duals are constructed by a given operator dual; $\mathcal{A}$-valude multiple of an operator dual, transfer of an operator dual by an adjointable and invertible operator, and summation of two operator duals. Moreover, a collection of operator duals in a Hilbert $\mathcal{A}$-module $\mathcal{A}$ is obtained by an operator dual of a Hilbert $\mathcal{A}$-module $\mathcal{H}$. Operator duals of tensor product of $*$-frames are studied. The interesting result is given. A $*$-frame (primary $*$-frame) is transformed by an adjointable and invertible operator and then an operator dual $*$-frame (second $*$-frame) for the primary $*$-frame is obtained. Also, the set of all operator duals of second $*$-frame is in one to one corresponds with the set of all operator duals of the primary $*$-frame. In continue, the all of operator duals of a given $*$-frame are characterized by adjointable and invertible operators on $\mathcal{H}$ and adjointable operators from $\mathcal{H}$ into $l_2(\mathcal{A})$. At the end of paper, we studied dilation of $*$-frames and their operator duals in a larger than Hilbert C^*-module. Also, a suggestion is given about an application of operator duals and frames in cryptography. It seems that operator duals are better than ordinary duals when they are used for public keys and private keys.

In continue, some definitions of Hilbert C^*-modules and frames and their properties are recalled. For more detail, we refer the interested reader to [9], [10], and [11].

Let $\mathcal{A}$ be a unital C^*-algebra and $a \in \mathcal{A}$. The nonzero element a is called strictly nonzero if zero dosn't belong to the spectrom of a, $(\sigma(a))$. Let $\mathcal{B}$ be an another unital C^*-algebra. The tensor product of algebras $\mathcal{A}$ and $\mathcal{B}$ is the completion of $\mathcal{A} \otimes_{alg} \mathcal{B}$ with the spatial norm and the following operation and involution,

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb' \ , \ (a \otimes b)^* = a^* \otimes b^*, \quad \forall a \otimes b, a' \otimes b' \in \mathcal{A} \otimes \mathcal{B}.$$

Hence $\mathcal{A} \otimes \mathcal{B}$ is a C^*-algebra such that $\|a \otimes b\| = \|a\|.\|b\|$ for $a \otimes b \in \mathcal{A} \otimes \mathcal{B}$. If $0 \leq a_1 \leq a_2$ in $\mathcal{A}$ and $0 \leq b_1 \leq b_2$ in $\mathcal{B}$, then $0 \leq a_1 \otimes b_1 \leq a_2 \otimes b_2$, (see [9], Lemma 4.3).

Suppose $\mathcal{A}$ is a C^*-algebra. A linear space $\mathcal{H}$ which is also an algebraic (left) $\mathcal{A}$-module together with an $\mathcal{A}$-inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{A}$ and possesses the following properties is called a pre-Hilbert C^*-module,

(*i*) $\langle f, f \rangle \geq 0$, for any $f \in \mathcal{H}$.
(*ii*) $\langle f, f \rangle = 0$ if and only if $f = 0$.
(*iii*) $\langle f, g \rangle = \langle g, f \rangle^*$, for any $f, g \in \mathcal{H}$.
(*iv*) $\langle \lambda f, h \rangle = \lambda \langle f, h \rangle$, for any $\lambda \in \mathbb{C}$ and $f, h \in \mathcal{H}$.
(*v*) $\langle af + bg, h \rangle = a\langle f, h \rangle + b\langle g, h \rangle$, for any $a, b \in \mathcal{A}$ and $f, g, h \in \mathcal{H}$.

If $\mathcal{H}$ is a Banach space with respect to the induced norm by the $\mathcal{A}$-valued inner product, then $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is called a Hilbert C^*-module over $\mathcal{A}$ or, simply, a Hilbert $\mathcal{A}$-module.

The C^*-algebra $\mathcal{A}$ can be recognized as a Hilbert $\mathcal{A}$-module with the inner product $\langle a, b \rangle = ab^*$. The standard Hilbert $\mathcal{A}$-module $l_2(\mathcal{A})$ is defined by

$$l_2(\mathcal{A}) := \{\{a_j\}_{j \in \mathbb{N}} \subseteq \mathcal{A} : \sum_{j \in \mathbb{N}} a_j a_j^* \ converges \ in \ \mathcal{A}\},$$

with $\mathcal{A}$-inner product $\langle \{a_j\}_{j \in \mathbb{N}}, \{b_j\}_{j \in \mathbb{N}} \rangle = \sum_{j \in \mathbb{N}} a_j b_j^*$.

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle_1)$ and $(\mathcal{K}, \langle \cdot, \cdot \rangle_2)$ be two Hilbert $\mathcal{A}$-modules. The class of all of adjointable maps from H into K is denoted by $B_*(H, K)$.

The useful result is presented in below that illustrates about upper and lower bounds of self adjoint and invertible operators corresponding to a given adjointable operator.

Lemma 1.1. *(see [1].) Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules and $T \in B_*(\mathcal{H}, \mathcal{K})$. Then*

(i) If T is injective and T has closed range, then the adjointable map T^*T is invertible and $\|(T^*T)^{-1}\|^{-1} \leq T^*T \leq \|T\|^2$.

(ii) If T is surjective, then the adjointable map TT^* is invertible and $\|(TT^*)^{-1}\|^{-1} \leq TT^* \leq \|T\|^2$.

The notion of frames for Hilbert spaces have been extended by Frank-Larson [5] to the notion of frames in Hilbert $\mathcal{A}$-modules. In that studying, the frame bounds were real. Afterwards, Alijani-Dehghan [1] have been studied frames in a Hilbert $\mathcal{A}$-module with $\mathcal{A}$-valued bounds as a countable family $\{f_j\}_{j\in J}$ in a Hilbert $\mathcal{A}$-module $\mathcal{H}$ such that there exist two strictly nonzero elements A and B in $\mathcal{A}$ satisfying in the following inequality.

$$A\langle f, f\rangle A^* \leq \sum_{j\in J} \langle f, f_j\rangle\langle f_j, f\rangle \leq B\langle f, f\rangle B^*, \qquad \forall f \in \mathcal{H}. \tag{1.1}$$

The sequence $\{f_j\}_{j\in J}$ is called to be a $*$-frame and the elements A and B are said the lower and upper $*$-frame bounds, respectively. Moreover, they defined operators corresponding to a $*$-frame in a Hilbert $\mathcal{A}$-module.

Let $\{f_j\}_{j\in J}$ be a $*$-frame for $\mathcal{H}$ with lower and upper $*$-frame bounds A and B, respectively. The frame transform or pre-frame operator $T : \mathcal{H} \longrightarrow l_2(\mathcal{A})$ is defined by $T(f) = \{\langle f, f_j\rangle\}_{j\in J}$ and it is an injective and closed range adjointable $\mathcal{A}$-module map and $\|T\| \leq \|B\|$. The adjoint operator T^* is surjective and it is given by $T^*(e_j) = f_j$ for $j \in J$ where $\{e_j\}_{j\in J}$ is the standard basis for $l_2(\mathcal{A})$. Also, the frame operator $S : \mathcal{H} \longrightarrow \mathcal{H}$ is defined by $Sf = T^*Tf = \sum_{j\in J}\langle f, f_j\rangle f_j$ that is positive, invertible and adjointable and the inequality $\|A^{-1}\|^{-2} \leq \|S\| \leq \|B\|^2$ holds, and the reconstruction formula $f = \sum_{j\in J}\langle f, S^{-1}f_j\rangle f_j$ holds for all $f \in \mathcal{H}$, [1].

In [1], the relations between frames and $*$-frames are finding; for example, every frame is a $*$-frame and for every $*$-frame exists real bounds. Then $*$-frames can be studied as frames with different bounds. In applications of frames, existing of optimal bounds plays an important role. By some examples, it has been seen that $\mathcal{A}$-valued bounds may be more suitable than real valued bounds for a $*$-frame. In other words, there are tight $*$-frames that are not tight frames (with real valued bounds). Now, we study duals for $*$-frames because the results for $\mathcal{A}$-valued bounds are valid for real-valued bounds.

In the frame theory, a collection of frames corresponding to a given frame is defined and has a special relation with respect to the first frame. They are called dual frames.

Let $\{f_j\}_{j\in J}$ be a $*$-frame for $\mathcal{H}$ with frame operator S. If there exists a $*$-frame $\{g_j\}_{j\in J}$ for $\mathcal{H}$ such that $f = \sum_{j\in J}\langle f, g_j\rangle f_j$ for $f \in \mathcal{H}$, then $*$-frame $\{g_j\}_{j\in J}$ is called a dual frame of $\{f_j\}_{j\in J}$. By the reconstruction formula, $f = \sum_{j\in J}\langle f, S^{-1}f_j\rangle f_j$ for $f \in \mathcal{H}$ then the spacial dual $*$-frame $\{S^{-1}f_j\}_{j\in J}$ is said to be the canonical dual frame of $\{f_j\}_{j\in J}$.

Also, Riesz bases for Hilbert C^*-modules have been defined. A frame $\{f_j\}_{j\in J}$ is a Riesz basis for $\mathcal{H}$ if every $\mathcal{A}$-linear combination $\sum_{j\in S} a_j f_j$ is equal to zero, then each summand $a_j f_j$ is also equal to zero when $a_j \in \mathcal{A}$ for all $j \in S$, and every finite subset S of J. In [7], dual of Riesz bases in Hilbert C*-modules have been studied and the following result has been obtained.

Theorem 1.2. *Suppose that $\mathcal{H}$ is a finitely or countably Hilbert $\mathcal{A}$-module. If $\{f_j\}_{j\in J}$ is a frame of $\mathcal{H}$ with pre-frame operator $T_{\mathcal{F}}$, then the following statements are equivalent.*

i. $\{f_j\}_{j\in J}$ has a unique dual frame.

ii. the pre-frame operator $T_{\mathcal{F}}$ is surjective.

iii. If $\sum_{j\in J} c_j f_j = 0$ for some sequence $\{c_j\}_{j\in J} \in l_2(\mathcal{A})$, then $c_j = 0, \forall j \in J$.

Since the proof of Theorem 1.2 is independent on type bounds, it can be used for $*$-frames.

The reminder of paper consists the definition of operator duals for $*$-frames and their properties. Among the following results, let J be a countably index set and let $\mathcal{H}$ be a finitely or countably generated Hilbert C^*-module on a unital C^*-algebra $\mathcal{A}$.

2. Operator duals

Every element of a Hilbert space (or Hilbert C^*-module) has a decomposition with respect to frames of the space. Dual frames are important in this decomposition because coefficients are given from a dual frame. In [3], the authors extended dual frames to generalized dual frames in Hilbert spaces. In this section, the generalized duals for a given $*$-frame will be considered. Also, we study their properties and characterize all operator dual $*$-frames associated to a given $*$-frame in a Hilbert C^*-module.

Definition 2.1. *Let $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ be two $*$-frames for $\mathcal{H}$. If there exists an invertible adjointable $\mathcal{A}$-module map Γ on $\mathcal{H}$ such that*

$$f = \sum_{j\in J}\langle \Gamma f, g_j\rangle f_j, \quad \forall f \in \mathcal{H}, \tag{2.1}$$

then $\{g_j\}_{j\in J}$ is called to be an operator dual of $\{f_j\}_{j\in J}$.

Remark 2.2. *Every $*$-frame $\{f_j\}_{j\in J}$ with frame operator S is an operator dual for itself. In order to see this, set $\Gamma := S^{-1}$ and the reconstruction formula concludes it.*

Remark 2.3. *Every dual $*$-frame $\{g_j\}_{j\in J}$ of $*$-frame $\{f_j\}_{j\in J}$ is an operator dual when $\Gamma = I$, I is identity operator on $\mathcal{H}$.*

Remark 2.4. *Let $\mathcal{G} = \{g_j\}_{j\in J}$ be an operator dual of a $*$-frame $\mathcal{F} = \{f_j\}_{j\in J}$ in $\mathcal{H}$. Then for some invertible adjointable map $\Gamma \in B_*(\mathcal{H})$,*

$$f = \sum_{j\in J}\langle \Gamma f, g_j\rangle f_j, \quad \forall f \in \mathcal{H}.$$

The equality shows that $I = (T^*_{\mathcal{F}}T_{\mathcal{G}})\Gamma$ where I is identity map on $\mathcal{H}$, and $T_{\mathcal{F}}$ and $T_{\mathcal{G}}$ are pre-frame operators of $\mathcal{F}$ and $\mathcal{G}$, respectively. Therefore, the operator Γ is unique and $\Gamma^{-1} = T^*_{\mathcal{F}}T_{\mathcal{G}}$.

By Remark 2.4, we say that $\{g_j\}_{j\in J}$ is an operator dual for $\{f_j\}_{j\in J}$ with the corresponding operator Γ.

In more, we mention that the operator duality relation of $*$-frames is symmetric. It is considered in the next remark.

Remark 2.5. *If $\mathcal{G} = \{g_j\}_{j\in J}$ is an operator dual of a given $*$-frame $\mathcal{F} = \{f_j\}_{j\in J}$ with the corresponding operator Γ, then $\{f_j\}_{j\in J}$ is an operator dual for $\{g_j\}_{j\in J}$ with the corresponding operator Γ^*. In order to see this, assume that $T_{\mathcal{F}}$ and $T_{\mathcal{G}}$ are pre-frame operators of $\mathcal{F}$ and $\mathcal{G}$, respectively. By the definition of operator duals, we have*

$$I = \sum_{j\in J}\langle \Gamma f, g_j\rangle f_j = (T^*_{\mathcal{F}}T_{\mathcal{G}})\Gamma.$$

Since Γ is invertible, $\Gamma^{-1} = T^*_{\mathcal{F}}T_{\mathcal{G}}$ and

$$I = \Gamma(T^*_{\mathcal{F}}T_{\mathcal{G}}) = (T^*_{\mathcal{G}}T_{\mathcal{F}})\Gamma^* = \sum_{j\in J}\langle \Gamma^* f, f_j\rangle g_j.$$

By using the last remark and some properties of the pre-frame operator, the following Lemma is obtained.

Lemma 2.6. *Let $\mathcal{F} = \{f_j\}_{j\in J}$ and $\mathcal{G} = \{g_j\}_{j\in J}$ be $*$-Bessel sequences for $\mathcal{H}$ with the pre-frame operators $T_{\mathcal{F}}$ and $T_{\mathcal{G}}$, respectively. Assume that Γ is an invertible and adjointable $\mathcal{A}$-module map on $\mathcal{H}$. Then for $f \in \mathcal{H}$, the following statements are equivalent.*

i. $f = \sum_{j\in J}\langle \Gamma f, g_j\rangle f_j$.

ii. $f = \sum_{j\in J}\langle \Gamma^* f, f_j\rangle g_j$.

In case that one of the above equalities is satisfied, $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are operator dual $*$-frames. Moreover, if B is an upper bound for $\{f_j\}_{j\in J}$ and S is frame operator of $\{f_j\}_{j\in J}$, then $B\|S^{-1}\|^{-\frac{1}{2}}\|T_{\mathcal{F}}\|^{-1}\|\Gamma\|^{-1}$ is a lower bound for $\{g_j\}_{j\in J}$.

Proof. The equivalency of two conditions is given from Remark 2.6. Now, let B be a $*$-Bessel bound for $\{f_j\}_{j\in J}$ and the pare i holds. By the definition of $*$-Bessel sequence $\{f_j\}_{j\in J}$ and $T^*_{\mathcal{F}}T_{\mathcal{G}}\Gamma = id_{\mathcal{H}}$, we can write for $f \in \mathcal{H}$,

$$\langle T_{\mathcal{F}}f, T_{\mathcal{F}}f\rangle \leq B\langle f, f\rangle B^* = B\langle T^*_{\mathcal{F}}T_{\mathcal{G}}\Gamma f, T^*_{\mathcal{F}}T_{\mathcal{G}}\Gamma f\rangle B^* \leq B\|T_{\mathcal{F}}\|^2\langle T_{\mathcal{G}}\Gamma f, T_{\mathcal{G}}\Gamma f\rangle B^*, \tag{2.2}$$

Using Lemma 1.1,

$$\|(T^*_{\mathcal{F}}T_{\mathcal{F}})^{-1}\|^{-1}\langle f, f\rangle \leq \langle T_{\mathcal{F}}f, T_{\mathcal{F}}f\rangle, \qquad \forall f \in \mathcal{H}. \tag{2.3}$$

It follows from Lemma 1.1, (2.2), and (2.3) that for $f \in \mathcal{H}$,

$$\|S^{-1}\|^{-1}\|\Gamma\Gamma^*\|^{-1}\langle f, f\rangle \leq \|S^{-1}\|^{-1}\langle \Gamma^{-1}f, \Gamma^{-1}f\rangle \leq B\|T_{\mathcal{F}}\|^2\langle T_{\mathcal{G}}f T_{\mathcal{G}}f\rangle B^*,$$

$$(B^{-1}\|S^{-1}\|^{-\frac{1}{2}}\|T_{\mathcal{F}}\|^{-1}\|\Gamma\|^{-1})\langle f, f\rangle(B^{-1}\|S^{-1}\|^{-\frac{1}{2}}\|T_{\mathcal{F}}\|^{-1}\|\Gamma\|^{-1})^* \leq \langle T_{\mathcal{G}}f, T_{\mathcal{G}}f\rangle.$$

Therefore, $B\|S^{-1}\|^{-\frac{1}{2}}\|\Gamma\|^{-1}\|T_{\mathcal{F}}\|^{-1}$ is a lower $*$-frame bound for $\{g_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ is a $*$-frame. Similarly, $\{f_j\}_{j\in J}$ is also a $*$-frame. □

Example 2.7. *Let $\{\{g_{ij}\}_{j\in J} : i = 1, ..., t\}$ be a finite set of dual $*$-frames for $\{f_j\}_{j\in J}$ and let $\{\alpha_i\}_{i=1}^t$ be a set of strictly nonzero elements in center of $\mathcal{A}$. Then $\{\sum_{i=1}^t \alpha_i g_{ij}\}_{j\in J}$ is an operator dual for $\{f_j\}_{j\in J}$ with the corresponding operator $\Gamma = (\sum_{i=1}^t \alpha_i^*)^{-1}I$, when I is identity operator on $\mathcal{H}$.*

Proof. For $f \in \mathcal{H}$,

$$\sum_{j\in J}\langle \Gamma^* f, f_j\rangle(\sum_{i=1}^t \alpha_i g_{ij}) = (\sum_{i=1}^t \alpha_i)^{-1}(\sum_{i=1}^t \alpha_i)\sum_{j\in J}\langle f, f_j\rangle g_{ij} = f.$$

□

For rest, in the reminder of paper, the pair $(\{g_j\}_{j\in J}, \Gamma)$ will be used for an operator dual $\{g_j\}_{j\in J}$ with the corresponding operator Γ.

If $\{g_j\}_{j\in J}$ is an operator dual of a $*$-frame $\{f_j\}_{j\in J}$, we can obtain some new operator duals for $\{f_j\}_{j\in J}$; for example, its $\mathcal{A}$-valued multiply, its transformation by an invertible operator. Also, the sum of two operator duals is also an operator dual. The following proposition illustrates them.

Proposition 2.8. *Let $(\{g_j\}_{j\in J}, \Gamma)$ be an operator dual of a $*$-frame $\{f_j\}_{j\in J}$. Then*

i. For strictly nonzero element α in center of $\mathcal{A}$, the pair $(\{\alpha g_j\}_{j\in J}, \alpha^{-1}\Gamma)$ is an operator dual for $\{f_j\}_{j\in J}$.

ii. If Υ is an invertible and adjointable operator on $\mathcal{H}$, then $(\{\Upsilon g_j\}_{j\in J}, (\Upsilon)^{-1}\Gamma)$ is an operator dual for $\{f_j\}_{j\in J}$.

iii. The sequence $\{g_j\}_{j\in J}$ is a dual of $\{\Gamma^* f_j\}_{j\in J}$.

iv. Assume that $(\{h_j\}_{j\in J}, \Lambda)$ is another operator dual of $\{f_j\}_{j\in J}$. So $(\{g_j + h_j\}_{j\in J}, (\Gamma^{-1} + \Lambda^{-1})^{-1})$ is an operator dual for $\{f_j\}_{j\in J}$.

The proof of statements is easily by the definition of operator duality.

A collection of $*$-frames for Hilbert $\mathcal{A}$-module $\mathcal{A}$ is constructed by a given $*$-frame of a Hilbert $\mathcal{A}$-module $\mathcal{H}$, [1]. Now, we obtain some their duals by operator duals of the primary $*$-frame.

Proposition 2.9. *Let $(\{g_j\}_{j\in J}, \Gamma)$ be an operator dual of $\{f_j\}_{j\in J}$ for $\mathcal{H}$. If f is an element of $\mathcal{H}$ such that $\langle f, f\rangle$ is a strictly nonzero element in the center of $\mathcal{A}$, then $\{\langle g_j, (\langle f, f\rangle)^{-1}\Gamma f\rangle\}_{j\in J}$ is a dual of $\{\langle f_j, f\rangle\}_{j\in J}$.*

Proof. Suppose that $a \in \mathcal{A}$,

$$\begin{aligned}\sum_{j\in J}\langle a, \langle g_j, \langle f, f\rangle^{-1}\Gamma f\rangle\rangle\langle f_j, f\rangle &= \sum_{j\in J} a\langle\langle f, f\rangle^{-1}\Gamma f, g_j\rangle\langle f_j, f\rangle \\ &= a\langle f, f\rangle^{-1}\langle\sum_{j\in J}\langle \Gamma f, g_j\rangle f_j, f\rangle \\ &= a\langle f, f\rangle^{-1}\langle f, f\rangle = a.\end{aligned}$$

□

In [1], $*$-frames for the tensor product of Hilbert C^*-modules are considered. The following proposition finds operator duals of them.

Proposition 2.10. *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert C^*-modules over unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively, and let $\{f_j\}_{j\in J}$ and $\{k_j\}_{j\in J}$ be two $*$-frames for $\mathcal{H}$ and $\mathcal{K}$, respectively. If $(\{g_i\}_{i\in J},\Gamma)$ and $(\{h_j\}_{j\in J},\Upsilon)$ are operator duals for $\{f_j\}_{j\in J}$ and $\{k_j\}_{j\in J}$, then $(\{g_i\otimes h_j\}_{i,j\in J},\Gamma\otimes\Upsilon)$ is an operator dual for $*$-frame $\{f_j\otimes k_j\}_{j\in J}$.*

Proof. For $f\otimes k\in\mathcal{H}\otimes\mathcal{K}$,

$$\begin{aligned}\sum_{i,j\in J}\langle(\Gamma\otimes\Upsilon)(f\otimes k),g_i\otimes h_j\rangle f_i\otimes k_j &= \sum_{i,j\in J}\langle\Gamma f\otimes\Upsilon k,g_i\otimes h_j\rangle f_i\otimes k_j\\ &= \sum_{i,j\in J}(\langle\Gamma f,g_j\rangle\otimes\langle\Upsilon k,h_j\rangle)f_i\otimes k_j\\ &= \sum_{i\in J}\langle\Gamma f,g_j\rangle f_i\otimes\sum_{j\in J}\langle\Upsilon k,h_j\rangle k_j = f\otimes k.\end{aligned}$$

□

In [1], Alijani-Dehghan have obtained a $*$-frame corresponding to a surjective operator and a given $*$-frame. Now, we could have a class of $*$-frames corresponding to a given $*$-frame such that the set of operator duals of $*$-frames of this class is isomorphic to the set of operator duals of a primary $*$-frame. These results are considered in the following.

Proposition 2.11. *Let $\{f_j\}_{j\in J}$ be a $*$-frame for $\mathcal{H}$ with frame operator S. If θ is an adjointable and invertible operator on $\mathcal{H}$, then $(\{\theta f_j\}_{j\in J},(\theta^{-1})^*S^{-1})$ is an operator dual for $\{f_j\}_{j\in J}$.*

Proof. Let $f\in\mathcal{H}$.

$$\sum_{j\in J}\langle(S^{-1}\theta^{-1})f,f_j\rangle\theta f_j=\theta(\sum_{j\in J}\langle(S^{-1}\theta^{-1})f,f_j\rangle f_j)=\theta(\theta^{-1}f)=f.$$

□

In this time, this question is interesting that "Is the vice versa of the above proposition valid?". It means that "Is every operator dual of $\{f_j\}_{j\in J}$ constructed by an adjointable and invertible operator on $\mathcal{H}$?" It is not true because each ordinary dual of $\{f_j\}_{j\in J}$ is an operator dual with corresponding operator I, I is identity operator. But the given operator dual in Proposition 2.11 has an attractive property. The operator duals of the given $*$-frame in Proposition 2.11 are in one to one correspondence to operator duals of $\{f_j\}_{j\in J}$

Proposition 2.12. *Let $\{f_j\}_{j\in J}$ be a $*$-frame and let θ be an adjointable and invertible operator on $\mathcal{H}$. Then sets of operator duals of $\{f_j\}_{j\in J}$ and $\{\theta f_j\}_{j\in J}$ are in one to one correspondence.*

Proof. First, suppose that $(\{g_j\}_{j\in J},\Gamma)$ is an operator dual for $\{f_j\}_{j\in J}$. For $f\in\mathcal{H}$, we obtain

$$f=\sum_{j\in J}\langle\Gamma^*f,f_j\rangle g_j=\sum_{j\in J}\langle\theta^*(\theta^{-1})^*\Gamma^*f,f_j\rangle g_j=\sum_{j\in J}\langle(\theta^{-1})^*\Gamma^*f,\theta f_j\rangle g_j.$$

So $(\{g_j\}_{j\in J},\Gamma\theta^{-1})$ is an operator dual for $\{\theta f_j\}_{j\in J}$. Now, if $(\{g_j\}_{j\in J},\Gamma)$ is an operator dual of $\{\theta f_j\}_{j\in J}$, then $(\{g_j\}_{j\in J},\Gamma^*\theta)$ is an operator dual of $\{f_j\}_{j\in J}$ because

$$f=\sum_{j\in J}\langle\Gamma f,\theta f_j\rangle g_j=\sum_{j\in J}\langle\theta^*\Gamma f,f_j\rangle g_j,\quad\forall f\in\mathcal{H}.$$

□

Now, we are ready to characterize the set of all of operator duals for a given $*$-frame.

Proposition 2.13. *Let $\mathcal{F} = \{f_j\}_{j\in J}$ be a $*$-frame for $\mathcal{H}$ with pre-frame operator $T_{\mathcal{F}}$ and frame operator S. Then the set of all of operator duals of $\{f_j\}_{j\in J}$ is precisely the set of the following sequences.*

$$\{g_j\}_{j\in J} = \{\Gamma f_j + \varphi e_j - \sum_{i\in J}\langle S^{-1}f_j, f_i\rangle \varphi e_j\}_{j\in J},$$

where $\{e_j\}_{j\in J}$ is the standard orthonormal basis for $l_2(\mathcal{A})$, $\varphi \in B_*(\mathcal{H}, l_2(\mathcal{A}))$, and Γ is an invertible adjointable operator on $\mathcal{H}$.

Proof. Assume that $\{g_j\}_{j\in J}$ is a sequence as above. Then its pre-frame operator is $T_{\mathcal{G}} = T_{\mathcal{F}}\Gamma + \varphi - T_{\mathcal{F}}S^{-1}T^*_{\mathcal{F}}\varphi$ and so

$$\begin{aligned}(S\Gamma)^{-1}(T^*_{\mathcal{F}}T_{\mathcal{G}}) &= (S\Gamma)^{-1}(T^*_{\mathcal{F}}T_{\mathcal{F}}\Gamma + T^*_{\mathcal{F}}\varphi - T^*_{\mathcal{F}}T_{\mathcal{F}}S^{-1}T^*_{\mathcal{F}}\varphi)\\ &= (S\Gamma)^{-1}(T^*_{\mathcal{F}}T_{\mathcal{F}}S^{-1}S\Gamma + T^*_{\mathcal{F}}\varphi - T^*_{\mathcal{F}}T_{\mathcal{F}}S^{-1}T^*_{\mathcal{F}}\varphi)\\ &= (S\Gamma)^{-1}(S\Gamma) = I.\end{aligned}$$

By the similar relation with the given equality in Remark 2.5, it concludes that $\{g_j\}_{j\in J}$ is an operator dual for $\{f_j\}_{j\in J}$ with the corresponding operator $(S\Gamma)^{-1}$. □

In [7], the dilation of frames and dual frames in Hilbert C^*-modules have been considered. Same results are valid for frames ($*$-frames) and their operator duals. Some parts of the proof of Theorem 2.5 [7], can not be directly applied for studying this subject that about operator duals in Hilbert $C*$-modules but the some parts are similar that they are omitted. More precisely, we use Lemma 1.1 in the proof of invertibility of $T^*_U T_U$, and other some parts of the proof of Theorem 2.5 [7] are omitted.

Theorem 2.14. *Let $(\{g_j\}_{j\in J}, \Gamma)$ be an operator dual of $*$-frame $\{f_j\}_{j\in J}$ for $\mathcal{H}$. Then there exits a Hilbert $\mathcal{A}$-module $\mathcal{K} \supseteq \mathcal{H}$ and a Riesz basis $\{u_j\}_{j\in J}$ of $\mathcal{K}$ which has a unique dual $\{v_j\}_{j\in J}$ and satisfies $Pu_j = f_j$ and $Pv_j = g_j$, for all $j \in J$, where P is the projection from $\mathcal{K}$ onto $\mathcal{H}$.*

Proof. Assume that $T_{\mathcal{F}}$, $T_{\mathcal{G}}$ and $S_{\mathcal{F}}$, $S_{\mathcal{G}}$ are pre-frame operators and frame operators of $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$, respectively . Also, the orthogonal projections onto range of $T_{\mathcal{F}}$, $R(T_{\mathcal{F}})$, and range of $T_{\mathcal{G}}$, $R(T_{\mathcal{G}})$, are $P_{\mathcal{F}}$ and $P_{\mathcal{G}}$, respectively. Now, for $f \in \mathcal{H}$,

$$\begin{aligned}\langle T_{\mathcal{G}}f, T_{\mathcal{G}}S^{-1}_{\mathcal{G}}g_j\rangle = \langle T^*_{\mathcal{G}}T_{\mathcal{G}}f, S^{-1}_{\mathcal{G}}g_j\rangle &= \langle S^{-1}_{\mathcal{G}}S_{\mathcal{G}}f, g_j\rangle\\ &= \langle f, g_j\rangle = \langle T_{\mathcal{G}}f, e_j\rangle = \langle T_{\mathcal{G}}f, P_{\mathcal{G}}e_j\rangle,\end{aligned}$$

so

$$P_{\mathcal{G}}e_j = T_{\mathcal{G}}S^{-1}_{\mathcal{G}}g_j, \quad \forall j \in J, \tag{2.4}$$

where $\{e_j\}_{j\in J}$ is the standard orthonormal basis of $l_2(\mathcal{A})$. By the equation (2.4), for $f \in \mathcal{H}$, we give

$$\begin{aligned}P_{\mathcal{G}}T_{\mathcal{F}}(\Gamma^* f) &= P_{\mathcal{G}}(\sum_{j\in J}\langle \Gamma^* f, f_j\rangle e_j)\\ &= \sum_{j\in J}\langle \Gamma^* f, f_j\rangle P_{\mathcal{G}}e_j\\ &= \sum_{j\in J}\langle \Gamma^* f, f_j\rangle T_{\mathcal{G}}S^{-1}_{\mathcal{G}}g_j\\ &= T_{\mathcal{G}}S^{-1}_{\mathcal{G}}(\sum_{j\in J}\langle \Gamma^* f, f_j\rangle g_j) = T_{\mathcal{G}}S^{-1}_{\mathcal{G}}f.\end{aligned}$$

Set

$$\mathcal{K} = \mathcal{H} \oplus P_{\mathcal{G}}^{\perp} l_2(\mathcal{A}), \quad u_j = f_j \oplus P_{\mathcal{G}}^{\perp} e_j, \quad \forall j \in J.$$

If T_U is a pre-frame operator of sequence $\{u_j\}_{j\in J}$, then $T_U(f \oplus w) = T_{\mathcal{F}} f + w$ and

$$\|T_U(f \oplus w)\| = \|T_{\mathcal{F}} f + w\| \leq B(\|f\| + \|w\|) = B\|f \oplus w\|, \quad \forall f \oplus w \in \mathcal{K},$$

for some $B > 0$, and so $\{u_j\}_{j\in J}$ is a Bessel sequence. Now, we show that T_U has the closed range. Suppose $\{\eta_n\}_{n\in\mathbb{N}} \subseteq R(T_U)$ such that $\eta_n \overset{n\to\infty}{\longrightarrow} \eta$. Since Γ is invertible and adjointable,

$$\exists \Gamma^* f_n \oplus w_n \in \mathcal{H} \oplus P_{\mathcal{G}}^{\perp}(l_2(\mathcal{A}); \;\; T_U(\Gamma^* f_n \oplus w_n) = \eta_n.$$

On the other hand,

$$T_U(\Gamma^* f_n \oplus w_n) = T_{\mathcal{F}}(\Gamma^* f_n) + w_n = \eta_n \overset{n\to\infty}{\longrightarrow} \eta,$$

and Remark 2.4 obtains

$$f_n = T_{\mathcal{G}}^* T_{\mathcal{F}} \Gamma^* f_n = T_{\mathcal{G}}^*(T_{\mathcal{F}} \Gamma^* f_n + w_n) \overset{n\to\infty}{\longrightarrow} T_{\mathcal{G}}^* \eta.$$

And $R(T_U)$ is closed because $R(T_{\mathcal{F}})$ is closed. Also, T_U^* has the closed range. In this step, the injectivity of T_U^* will be obtained. If $T_U^*(\sum_{j\in J} a_j e_j) = 0$, then

$$0 = \sum_{j\in J} a_j (f_j \oplus P_{\mathcal{G}}^{\perp} e_j) = \sum_{j\in J} a_j f_j \oplus P_{\mathcal{G}}^{\perp}(\sum_{j\in J} a_j e_j).$$

It concludes that

i. $\sum_{j\in J} a_j f_j = 0$.

ii. $P_{\mathcal{G}}^{\perp}(\sum_{j\in J} a_j e_j) = 0$.

From *ii*,

$$\sum_{j\in J} a_j e_j \in R(T_{\mathcal{G}}) \Longrightarrow \exists h \in \mathcal{H}; \;\; T_{\mathcal{G}} h = \sum_{j\in J} a_j e_j,$$

and on the other hand,

$$T_{\mathcal{G}} h = \sum_{j\in J} \langle h, g_j \rangle e_j \Longrightarrow a_j = \langle h, g_j \rangle, \;\; \forall j \in J.$$

From *i*,

$$0 = \sum_{j\in J} a_j f_j = \sum_{j\in J} \langle h, g_j \rangle f_j = \sum_{j\in J} \langle \Gamma\Gamma^{-1} h, g_j \rangle f_j = \Gamma^{-1} h.$$

Since Γ is injective, $h = 0$ and $a_j = 0$ for $j \in J$. So $\sum_{j\in J} a_j e_j = 0$, and T_U^* is injective. The operator $T_U^* T_U$ is an invertible selfadjoint operator such that it has an upper bound and a lower bound by Lemma 1.1, and also $\{u_j\}_{j\in J}$ is a frame for $\mathcal{K}$ with frame operator $S_U = T_U^* T_U$. The remainder of proof is similar to Theorem 2.5, [7]. □

Suggestion 2.15. *At the end of this manuscript, we discuss about an application of frames and operator duals. We start with a question fora system of a given frame for $\{f_j\}_{j\in J}$ and a pair of its operator dual $(\{g_j\}_{j\in J}, \Gamma)$. "If the invertible operator Γ is identify, can the sequence $\{g_j\}_{j\in J}$ been introduced?". The answer is negative because we saw that all of ordinary duals of a given $*$-frame are operator duals with the corresponding operator identity. Therefore, if the invertible operator Γ is identify, we can not obtain $\{g_j\}_{j\in J}$. It seems that this property is important in cryptography. In cryptography, two spaces are used; private key space and public key space. A cryptosystem is to be secure, if it must be unable to decrypt the massage. Some cryptosystems use from sequences. Now, these cryotosystems can use from frames and their duals; means that let $(\{g_j\}_{j\in J}, \Gamma)$ be an operator dual of $\{f_j\}_{j\in J}$. Then private key is $\{g_j\}_{j\in J}$ and public key is Γ. So, in this case, the domain of chosen private keys and public keys is larger than the case that they are chosen of frames and their ordinary duals.*

References

[1] A. Alijani and M. A. Dehghan, $*$-Frames in Hilbert C^*-modules, U.P.B. Sci. Bull., Series A, 73 (4) (2011) 89-106.
[2] P. Casazza, D. Han, and D. Larson, Frames in Banach spaces, Contemp. Math., 247 (1999) 149–181.
[3] M. A. Dehghan and M. A. Hasankhani Fard, G-Dual frames in Hilbert spaces, U.P.B. Sci. Bull., Series A, 75 (1) (2013) 129–140.
[4] R. Duffin and A. Schaeffer, A class of nonharmonic Fourier series, Trans. Am. Math. Soc., 72 (1952) 341–366.
[5] M. Frank and D. R. Larson, Frames in Hilbert C^*-modules and C^*-algebra, J. Operator theory, 48 (2002) 273-314.
[6] D. Gabor, Theory of communications, J. Electr. Eng. (London), 49 (3) (1946) 429–457.
[7] D. Han, W. Jing, D. Larson, and R. Mohapatra, Riesz bases and their dual modular frames in Hilbert C^*-modules, J. Math. Anal. Appl., 343 (2008) 246-256.
[8] D. Han and D. Larson, Frames, bases and group representations, Mem. Am. Math. Soc., 147 (697) (2000) 1–94.
[9] E. C. Lance, Hilbert C^*-modules, A Toolkit for Operator Algebraists: University of Leeds, Cambridge University Press, 1995.
[10] G. J. Morphy, C^*-Algebras and Operator Theory, San Diego, California, Academic Press, 1990.
[11] N. E. Wegge Olsen, K-Theory and C^*-Algebras, A Friendly Approch, Oxford University press, Oxford, England, 1993.