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Invertibility of Multipliers in Hilbert C^*-Modules

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Abstract. In this paper, we present some sufficient conditions under which Bessel multipliers in Hilbert C^*-modules with semi-normalized symbols are invertible and we calculate the inverses. Especially we consider the invertibility of Bessel multipliers when the elements of their symbols are positive and when their Bessel sequences are equivalent, duals, modular Riesz bases or stable under small perturbations. We show that in these cases the inverse of a Bessel multiplier can be represented as a Bessel multiplier.

1. Introduction and preliminaries

Bessel multipliers in Hilbert spaces were introduced by Balazs in [3]. Bessel multipliers are operators defined by a fixed multiplication pattern which is inserted between the analysis and synthesis operators. Bessel multipliers have useful applications, for example they are used for solving approximation problems and they are used in acoustics as a way to implement time-varying filters and recently have found applications in psychoacoustics and denoising. For more information about the stated applications, see [4, 7, 17, 21, 26]. Multipliers have been investigated for Bessel fusion sequences in Hilbert spaces [16] (called Bessel fusion multipliers) and for generalized Bessel sequences in Hilbert spaces [23] (called g-Bessel multipliers). Also multipliers were introduced for p-Bessel sequences in Banach spaces [24] and for continuous frames [6]. Recently the present author and A. Khosravi generalized Bessel multipliers, g-Bessel multipliers and Bessel fusion multipliers to Hilbert C^*-modules and many important results obtained for Bessel multipliers in Hilbert and Banach spaces were generalized to Hilbert C^*-modules (see [14]). In this paper, we consider the invertibility of multipliers in Hilbert C^*-modules. As we know, the invertibility of the operators related to frames has great importance in frame theory mostly because of the reconstruction of signals. Thus studying the conditions under which Bessel multipliers are invertible can be interesting. As a special case, we see in [19] that $(a, m)-$approximate duals generate an invertible multiplier. We mention that $(a, m)-$approximate duals are standard frames that imply the distance (with respect to the norm) between the identity operator on the Hilbert C^*-module and the operator defined by multiplying the Bessel multiplier with symbol m by an element a in the center of the C^*-algebra is strictly less than one. Therefore in this case, the inverse of the multiplier can be calculated using Neumann algorithm. In this paper, we study the invertibility of multipliers in more general cases.

Suppose that $\mathfrak{A}$ is a unital C^*–algebra and E is a left $\mathfrak{A}$–module such that the linear structures of $\mathfrak{A}$ and E are compatible. E is a pre-Hilbert $\mathfrak{A}$–module if E is equipped with an $\mathfrak{A}$–valued inner product $\langle .,. \rangle : E \times E \longrightarrow \mathfrak{A}$, such that

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(i) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$, for each $\alpha, \beta \in \mathbb{C}$ and $x, y, z \in E$;
(ii) $\langle ax, y\rangle = a\langle x, y\rangle$, for each $a \in \mathfrak{A}$ and $x, y \in E$;
(iii) $\langle x, y\rangle = \langle y, x\rangle^*$, for each $x, y \in E$;
(iv) $\langle x, x\rangle \geq 0$, for each $x \in E$ and if $\langle x, x\rangle = 0$, then $x = 0$.

For each $x \in E$, we define $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$ and $|x| = \langle x, x\rangle^{\frac{1}{2}}$. If E is complete with $\|.\|$, it is called a *Hilbert* $\mathfrak{A}$*–module* or a *Hilbert* C^**–module* over $\mathfrak{A}$. We call $\mathcal{Z}(\mathfrak{A}) = \{a \in \mathfrak{A} : ab = ba, \forall b \in \mathfrak{A}\}$, the *center* of $\mathfrak{A}$. Note that if $a \in \mathcal{Z}(\mathfrak{A})$, then $a^* \in \mathcal{Z}(\mathfrak{A})$, and if a is an invertible element of $\mathcal{Z}(\mathfrak{A})$, then $a^{-1} \in \mathcal{Z}(\mathfrak{A})$, also if a is a positive element of $\mathcal{Z}(\mathfrak{A})$, since $a^{\frac{1}{2}}$ is in the closure of the set of polynomials in a, we have $a^{\frac{1}{2}} \in \mathcal{Z}(\mathfrak{A})$ (see [20]).
A Hilbert $\mathfrak{A}$−module E is *finitely generated* if there exists some set $\{x_1, \ldots, x_n\}$ in E such that every element $x \in E$ can be expressed as an $\mathfrak{A}$−linear combination $x = \sum_{i=1}^n a_i x_i, a_i \in \mathfrak{A}$. A Hilbert $\mathfrak{A}$−module E is *countably generated* if there exists a countable set $\{x_i\}_{i\in I} \subseteq E$ such that E equals the norm-closure of the $\mathfrak{A}$−linear hull of $\{x_i\}_{i\in I}$.
Let E and F be Hilbert $\mathfrak{A}$–modules. An operator $T : E \longrightarrow F$ is called *adjointable* if there exists an operator $T^* : F \longrightarrow E$ such that $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$, for each $x \in E$ and $y \in F$. Every adjointable operator T is bounded and $\mathfrak{A}$–linear (that is, $T(ax) = aT(x)$ for each $x \in E$ and $a \in \mathfrak{A}$). We denote the set of all adjointable operators from E into F by $\mathfrak{L}(E, F)$. Note that $\mathfrak{L}(E, E)$ is a C^*–algebra and it is denoted by $\mathfrak{L}(E)$. For more details about Hilbert C^*−modules, see [15].
Frames in Hilbert C^*−modules were introduced in [10]:

Definition 1.1. *Let E be a Hilbert* $\mathfrak{A}$*−module. A family* $\mathcal{F} = \{f_i\}_{i\in I} \subseteq E$ *is a* frame *for E, if there exist real constants* $0 < A_{\mathcal{F}} \leq B_{\mathcal{F}} < \infty$, *such that for each* $x \in E$,

$$A_{\mathcal{F}}\langle x, x\rangle \leq \sum_{i\in I}\langle x, f_i\rangle\langle f_i, x\rangle \leq B_{\mathcal{F}}\langle x, x\rangle,$$

i.e., there exist real constants $0 < A_{\mathcal{F}} \leq B_{\mathcal{F}} < \infty$, *such that the series* $\sum_{i\in I}\langle x, f_i\rangle\langle f_i, x\rangle$ *converges in the ultraweak operator topology to some element in the universal enveloping Von Neumann algebra of* $\mathfrak{A}$ *such that the inequality holds, for each* $x \in E$. *The numbers* $A_{\mathcal{F}}$ *and* $B_{\mathcal{F}}$ *are called the* lower *and* upper bound *of the frame, respectively. In this case we call it an* $(A_{\mathcal{F}}, B_{\mathcal{F}})$ frame. *If only the second inequality is required, we call it a* Bessel sequence. *If the sum converges in norm, the frame is called* standard.

Frames in Hilbert C^*−modules are not trivial generalizations of Hilbert space frames because of the complex structure of C^*−algebras. As we know, many important results obtained in Hilbert spaces do not hold in Hilbert C^*−modules. For example, any closed linear subspace in a Hilbert space has an orthogonal complement. But not every closed submodule of a Hilbert C^*−module is complemented. Moreover, the Riesz representation theorem for continuous functionals on Hilbert spaces does not hold in Hilbert C^*−modules, and so there exist nonadjointable bounded linear operators on Hilbert C^*−modules (see [15]). Therefore it is expected that problems about frames in Hilbert C^*−modules are more complicated than those in Hilbert spaces.
Now we recall the definition of g-frames in Hilbert C^*−modules from [12]:
A sequence $\Lambda = \{\Lambda_i \in \mathfrak{L}(E, E_i) : i \in I\}$ is called a *g-frame* for E with respect to $\{E_i : i \in I\}$ if there exist real constants $A_\Lambda, B_\Lambda > 0$ such that

$$A_\Lambda\langle x, x\rangle \leq \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle \leq B_\Lambda\langle x, x\rangle,$$

for each $x \in E$. In this case we call it an (A_Λ, B_Λ) *g-frame*. If only the second-hand inequality is required, then Λ is called a *g-Bessel sequence*. Note that *standard* g-frames are defined similar to the standard frames. If $\{E_i : i \in I\}$ is a sequence of Hilbert $\mathfrak{A}$−modules, then $\oplus_{i\in I}E_i$ which is the set

$$\oplus_{i\in I}E_i = \Big\{\{x_i\}_{i\in I} : x_i \in E_i \text{ and } \sum_{i\in I}\langle x_i, x_i\rangle \text{ is norm convergent in } \mathfrak{A}\Big\},$$

is a Hilbert $\mathfrak{A}$−module with pointwise operations and $\mathfrak{A}$-valued inner product

$$\langle\{x_i\}_{i\in I}, \{y_i\}_{i\in I}\rangle = \sum_{i\in I}\langle x_i, y_i\rangle.$$

For a standard g-Bessel sequence Λ, the operator $T_\Lambda : E \longrightarrow \oplus_{i\in I}E_i$ which is defined by $T_\Lambda(x) = \{\Lambda_i x\}_{i\in I}$ is called the *analysis operator* of Λ. T_Λ is adjointable with $T_\Lambda^*(\{x_i\}_{i\in I}) = \sum_{i\in I}\Lambda_i^*(x_i)$, for each $\{x_i\}_{i\in I} \in \oplus_{i\in I}E_i$. Now we define the operator $S_\Lambda : E \longrightarrow E$ by $S_\Lambda x = T_\Lambda^* T_\Lambda(x) = \sum_{i\in I}\Lambda_i^*\Lambda_i(x)$. If Λ is a standard (A_Λ, B_Λ) g-frame, then $A_\Lambda Id_E \leq S_\Lambda \leq B_\Lambda Id_E$. The operator S_Λ is called the *g-frame operator* of Λ.

Recall that if $\Lambda = \{\Lambda_i\}_{i\in I}$ and $\Gamma = \{\Gamma_i\}_{i\in I}$ are standard $g-$Bessel sequences such that $\sum_{i\in I}\Gamma_i^*\Lambda_i x = x$ or equivalently $\sum_{i\in I}\Lambda_i^*\Gamma_i x = x$, for each $x \in E$, then Γ (resp. Λ) is called a *g-dual* of Λ (resp. Γ). Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be an (A_Λ, B_Λ) standard g-frame. We call $\widetilde{\Lambda} = \{\Lambda_i S_\Lambda^{-1}\}_{i\in I}$ the *canonical g-dual* of Λ which is an $(\frac{1}{B_\Lambda}, \frac{1}{A_\Lambda})$ standard g-frame.

Note that $\mathcal{F} = \{f_i\}_{i\in I}$ is a standard Bessel sequence (resp. frame) if and only if $\Lambda_{\mathcal{F}} = \{\Lambda_{f_i}\}_{i\in I}$ is a standard g-Bessel sequence (resp. g-frame), where $\Lambda_{f_i}(x) = \langle x, f_i\rangle$, for each $x \in E$ (see [12, Example 3.1]). This shows that each Bessel sequence (resp. frame) generates a g-Bessel sequence (resp. g-frame). For a standard Bessel sequence $\mathcal{F} = \{f_i\}_{i\in I}$, we denote $T_{\Lambda_{\mathcal{F}}}$ and $S_{\Lambda_{\mathcal{F}}}$ by $T_{\mathcal{F}}$ and $S_{\mathcal{F}}$, respectively.

We denote the *canonical dual* of a standard frame $\mathcal{F} = \{f_i\}_{i\in I}$ by $\widetilde{\mathcal{F}} = \{\widetilde{f_i}\}_{i\in I}$, where $\widetilde{f_i} = S_{\mathcal{F}}^{-1} f_i$. Also, *duals* for two standard Bessel sequences $\mathcal{F} = \{f_i\}_{i\in I}$ and $\mathcal{G} = \{g_i\}_{i\in I}$ can be defined using the generated g-Bessel sequences, so $\mathcal{G}$ (resp. $\mathcal{F}$) is a dual of $\mathcal{F}$ (resp. $\mathcal{G}$) if $x = \sum_{i\in I}\langle x, f_i\rangle g_i$ or equivalently $x = \sum_{i\in I}\langle x, g_i\rangle f_i$, for each $x \in E$ (see [11, Proposition 3.8]). For more results about frames and their generalizations in Hilbert C^*-modules, see [1, 10–12, 22, 27].

Recall that $\ell^\infty(I, \mathfrak{A})$ is $\left\{\{a_i\}_{i\in I} \subseteq \mathfrak{A} : \|\{a_i\}_{i\in I}\|_\infty = \sup\{\|a_i\| : i \in I\} < \infty\right\}$. If $m = \{m_i\}_{i\in I}$ is a sequence in $\ell^\infty(I, \mathfrak{A})$ with $m_i \in \mathcal{Z}(\mathfrak{A})$, for each $i \in I$, then m is called a *symbol*. If c is an element in $\mathcal{Z}(\mathfrak{A})$ and $m_i = c$, for each $i \in I$, then m is denoted by $m = \{c\}$.

Proposition 1.2. [14] *Let $m = \{m_i\}_{i\in I}$ be a symbol. Then the operator $\mathcal{M}_m$ defined on $\oplus_{i\in I}E_i$ by $\mathcal{M}_m(\{x_i\}_{i\in I}) = \{m_i x_i\}_{i\in I}$ is well-defined, adjointable with $\mathcal{M}_m^* = \mathcal{M}_{m^*}$ and $\|\mathcal{M}_m\| \leq \|m\|_\infty$, where $m^* = \{m_i^*\}_{i\in I}$.*

Definition 1.3. [14] *Let $\Lambda = \{\Lambda_i\}_{i\in I}$, $\Gamma = \{\Gamma_i\}_{i\in I}$ be standard g-Bessel sequences for E with respect to $\{E_i\}_{i\in I}$ and let $m = \{m_i\}_{i\in I}$ be a symbol. Then the operator $M_{m,\Gamma,\Lambda} : E \longrightarrow E$ which is defined by $M_{m,\Gamma,\Lambda} = T_\Gamma^*\mathcal{M}_m T_\Lambda$ is called the* g-Bessel multiplier *for the g-Bessel sequences Λ and Γ with symbol m. We have $M_{m,\Gamma,\Lambda}(x) = \sum_{i\in I} m_i\Gamma_i^*\Lambda_i(x)$. Also $\|M_{m,\Gamma,\Lambda}\| \leq \sqrt{B_\Lambda B_\Gamma}\|m\|_\infty$ and $M_{m,\Gamma,\Lambda}^* = M_{m^*,\Lambda,\Gamma}$.*

Definition 1.4. [14] *Let E_1, E_2 be Hilbert $\mathfrak{A}-$modules, $m = \{m_i\}_{i\in I}$ a symbol and let $\mathcal{F} = \{f_i\}_{i\in I} \subseteq E_1$, $\mathcal{G} = \{g_i\}_{i\in I} \subseteq E_2$ be standard Bessel sequences. We call the operator $M_{m,\mathcal{G},\mathcal{F}} : E_1 \longrightarrow E_2$ which is defined by $M_{m,\mathcal{G},\mathcal{F}} = T_{\Lambda_{\mathcal{G}}}^*\mathcal{M}_m T_{\Lambda_{\mathcal{F}}} = T_{\mathcal{G}}^*\mathcal{M}_m T_{\mathcal{F}}$, the* Bessel multiplier *for the Bessel sequences $\mathcal{F}$ and $\mathcal{G}$ with symbol m. It is easy to see that $M_{m,\mathcal{G},\mathcal{F}}(x) = \sum_{i\in I} m_i\langle x, f_i\rangle g_i$.*

As we said before, in frame theory, the inverses of the operators related to Bessel sequences and frames play an important role mostly because of the reconstruction of signals. Note that if the Bessel multiplier $M_{m,\mathcal{G},\mathcal{F}}$ is invertible, then every $x \in E$ is reconstructed as

$$\sum_{i\in I} m_i\langle M_{m,\mathcal{G},\mathcal{F}}^{-1}x, f_i\rangle g_i = x = \sum_{i\in I} m_i\langle x, f_i\rangle M_{m,\mathcal{G},\mathcal{F}}^{-1} g_i.$$

We can conclude from the above relation that $\{m_i M_{m,\mathcal{G},\mathcal{F}}^{-1} g_i\}_{i\in I}$ is a dual for $\{f_i\}_{i\in I}$, so every invertible multiplier generates a dual. Our purpose is to obtain some sufficient conditions under which Bessel multipliers are invertible and find a representation of the inverse of an invertible multiplier as a Bessel multiplier.

In this paper $n = \{n_i\}_{i\in I}$ is a symbol with this property that there exists some positive number A such that $A1_{\mathfrak{A}} \leq |n_i|$, for each $i \in I$. n with these properties is called a *semi-normalized symbol*. Also in this note $m = \{m_i\}_{i\in I}$ is a semi-normalized symbol such that m_i's are positive elements in $\mathfrak{A}$ (there exists some positive number A such that $A1_{\mathfrak{A}} \leq m_i$, for each $i \in I$). $\mathcal{F} = \{f_i\}_{i\in I}$ and $\mathcal{G} = \{g_i\}_{i\in I}$ are always assumed to be sequences in a Hilbert C^*-module E, so $M_{m,\mathcal{G},\mathcal{F}} \in \mathfrak{L}(E)$. All C^*-algebras are unital and all Hilbert C^*-modules are finitely or countably generated. All frames, g-frames and Bessel sequences are standard and all index sets are finite or countable subsets of $\mathbb{N}$.

2. Invertibility of Bessel multipliers with equivalent sequences

In this section, we consider the invertibility of Bessel multipliers when their Bessel sequences are equivalent. First we recall the following definition from [2]:

Definition 2.1. *Let T be an invertible operator in $\mathfrak{L}(E)$.*

(i) *We say that $\Gamma = \{\Gamma_i \in \mathfrak{L}(E, E_i)\}_{i\in I}$ and $\Lambda = \{\Lambda_i \in \mathfrak{L}(E, E_i)\}_{i\in I}$ are $T-$ equivalent if $\Gamma_i = \Lambda_i T$, for each $i \in I$.*

(ii) *Let $\mathcal{G} = \{g_i\}_{i\in I}, \mathcal{F} = \{f_i\}_{i\in I} \subseteq E$. We say that $\mathcal{G}$ and $\mathcal{F}$ are $T-$ equivalent if $g_i = Tf_i$, for each $i \in I$.*

Theorem 2.2. *Let Λ be a standard g-frame and let Γ and Λ be $T-$ equivalent. Then*

(i) *Γ is a standard g-frame and $M_{m,\Lambda,\Gamma}$, $M_{m,\Gamma,\Lambda}$ are invertible with $M^{-1}_{m,\Lambda,\Gamma} = T^{-1}S^{-1}_{m^{\frac{1}{2}}\cdot\Lambda}$ and $M^{-1}_{m,\Gamma,\Lambda} = S^{-1}_{m^{\frac{1}{2}}\cdot\Lambda}(T^{-1})^*$, where $m^{\frac{1}{2}}\cdot\Lambda = \{m_i^{\frac{1}{2}}\Lambda_i\}_{i\in I}$.*

(ii) *If $c \in \mathcal{Z}(\mathfrak{A})$ is positive and invertible, then $M^{-1}_{\{c\},\Lambda,\Gamma} = M_{\{c^{-1}\},\widetilde{\Gamma},\widetilde{\Lambda}}$.*

(iii) $M^{-1}_{\{1_{\mathfrak{A}}\},\Lambda,\Lambda} = M_{\{1_{\mathfrak{A}}\},\widetilde{\Lambda},\widetilde{\Lambda}}$.

Proof. (i) Let Λ be an (A_Λ, B_Λ) standard g-frame. It follows from the relation

$$\frac{A_\Lambda\|x\|^2}{\|T^{-1}\|^2} \leq A_\Lambda\|Tx\|^2 \leq \left\|\sum_{i\in I}\langle\Lambda_i Tx, \Lambda_i Tx\rangle\right\| \leq B_\Lambda\|T\|^2\|x\|^2,$$

for each $x \in E$, the equality $\Gamma_i = \Lambda_i T$ and Theorem 3.1 in [27] that Γ is a standard g-frame with bounds $\frac{A_\Lambda}{\|T^{-1}\|^2}$ and $B_\Lambda\|T\|^2$. Since m is semi-normalized with positive elements and Λ is a standard g-frame, Proposition 3.7 in [14] implies that $M_{m,\Lambda,\Lambda}$ is invertible. Now we have

$$M_{m,\Lambda,\Gamma}x = \sum_{i\in I} m_i\Lambda_i^*\Gamma_i x = \sum_{i\in I} m_i\Lambda_i^*\Lambda_i Tx = M_{m,\Lambda,\Lambda}Tx.$$

Because $M_{m,\Lambda,\Lambda}$ and T are invertible, $M_{m,\Lambda,\Gamma}$ is invertible with $M^{-1}_{m,\Lambda,\Gamma} = T^{-1}M^{-1}_{m,\Lambda,\Lambda}$ and the equality $M^*_{m,\Gamma,\Lambda} = M_{m^*,\Lambda,\Gamma} = M_{m,\Lambda,\Gamma}$ yields that $M_{m,\Gamma,\Lambda}$ is also invertible with $M^{-1}_{m,\Gamma,\Lambda} = (M^{-1}_{m,\Lambda,\Gamma})^*$. Since $M_{m,\Lambda,\Lambda} = S_{m^{\frac{1}{2}}\cdot\Lambda}$ (see the proof of [14, Proposition 3.7] and also note that in this case, Λ is called weighted in the sense of [5]), we get $M^{-1}_{m,\Lambda,\Gamma} = T^{-1}S^{-1}_{m^{\frac{1}{2}}\cdot\Lambda}$ and $M^{-1}_{m,\Gamma,\Lambda} = S^{-1}_{m^{\frac{1}{2}}\cdot\Lambda}(T^{-1})^*$.

(ii) It follows from part (i) that Γ is a standard g-frame. Let $m_i = c$, for each $i \in I$. Then $\|c^{-1}\|^{-1}1_{\mathfrak{A}} \leq m_i$, for each $i \in I$ and $S_{m^{\frac{1}{2}}\cdot\Lambda}x = c\sum_{i\in I}\Lambda_i^*\Lambda_i x = cS_\Lambda x$, so $S^{-1}_{m^{\frac{1}{2}}\cdot\Lambda} = c^{-1}S^{-1}_\Lambda$. Thus by part (i), $M^{-1}_{\{c\},\Lambda,\Gamma} = c^{-1}T^{-1}S^{-1}_\Lambda$. It is easy to obtain from $\Gamma_i = \Lambda_i T$ that $S_\Gamma = T^*S_\Lambda T$ and $\widetilde{\Gamma}_i = \Gamma_i S^{-1}_\Gamma = \widetilde{\Lambda}_i T^{*-1}$. Then

$$M_{\{c^{-1}\},\widetilde{\Gamma},\widetilde{\Lambda}}x = \sum_{i\in I} c^{-1}T^{-1}\widetilde{\Lambda}_i^*\Lambda_i S^{-1}_\Lambda x = c^{-1}T^{-1}S^{-1}_\Lambda x.$$

Therefore $M_{\{c^{-1}\},\widetilde{\Gamma},\widetilde{\Lambda}} = c^{-1}T^{-1}S^{-1}_\Lambda = M^{-1}_{\{c\},\Lambda,\Gamma}$.

(iii) The result follows from part (ii) by considering $c = 1_{\mathfrak{A}}$. □

Remark 2.3. *Let $\mathcal{F} = \{f_i\}_{i\in I}$ and $\mathcal{G} = \{g_i\}_{i\in I} \subseteq E$ be standard Bessel sequences. If $\phi_i, \psi_i : E \longrightarrow \mathfrak{A}$ are defined by $\phi_i(x) = \langle x, f_i\rangle$, $\psi_i(x) = \langle x, g_i\rangle$, then $\Phi = \{\phi_i\}_{i\in I}$ and $\Psi = \{\psi_i\}_{i\in I}$ are standard g-Bessel sequences and in this case $M_{m,\Psi,\Phi} = M_{m,\mathcal{G},\mathcal{F}}$ ([14, Remark 3.6]). Also if $\mathcal{G}$ and $\mathcal{F}$ are $T-$equivalent, then*

$$\psi_i(x) = \langle x, g_i\rangle = \langle x, Tf_i\rangle = \langle T^*x, f_i\rangle = \phi_i T^*(x).$$

Hence Ψ and Φ are T^*-equivalent.

The above remark shows that if two standard Bessel sequences are $T-$equivalent, then the standard g-Bessel sequences induced by them are T^*-equivalent. Now using Theorem 2.2 and Remark 2.3, we get the following result which is a generalization of Corollary 4.5 and Example 4.1 in [8] to Hilbert C^*-modules.

Corollary 2.4. *Let $\mathcal{F}$ be a standard frame and let $\mathcal{G}$ and $\mathcal{F}$ be $T-$ equivalent. Then*

(i) *$\mathcal{G}$ is a standard frame and $M_{m,\mathcal{F},\mathcal{G}}$, $M_{m,\mathcal{G},\mathcal{F}}$ are invertible with* $M^{-1}_{m,\mathcal{F},\mathcal{G}} = (T^{-1})^* S^{-1}_{m^{\frac{1}{2}}\cdot\mathcal{F}}$ *and* $M^{-1}_{m,\mathcal{G},\mathcal{F}} = S^{-1}_{m^{\frac{1}{2}}\cdot\mathcal{F}} T^{-1}$, *where* $m^{\frac{1}{2}} \cdot \mathcal{F} = \{m_i^{\frac{1}{2}} f_i\}_{i\in I}$.

(ii) *If $c \in \mathcal{Z}(\mathfrak{A})$ is positive and invertible, then* $M^{-1}_{\{c\},\mathcal{F},\mathcal{G}} = M_{\{c^{-1}\},\widetilde{\mathcal{G}},\widetilde{\mathcal{F}}}$.

(iii) $M^{-1}_{\{1_{\mathfrak{A}}\},\mathcal{F},\mathcal{F}} = M_{\{1_{\mathfrak{A}}\},\widetilde{\mathcal{F}},\widetilde{\mathcal{F}}}$.

Proposition 2.5. *Let Λ and Γ be standard g-frames. If Γ and $\{n_i^*\Lambda_i\}_{i\in I}$ or Λ and $\{n_i\Gamma_i\}_{i\in I}$ are equivalent, then $M_{n,\Lambda,\Gamma}$ is invertible with $M^{-1}_{n,\Lambda,\Gamma} = M_{n^{-1},\widetilde{\Gamma},\widetilde{\Lambda}}$, where $n^{-1} = \{n_i^{-1}\}_{i\in I}$.*

Proof. Since n is a semi-normalized symbol, similar to the first part of the proof of Theorem 4.3 (iii) in [14], we obtain that $n^{-1} = \{n_i^{-1}\}_{i\in I}$ is a symbol. Let Γ and $\{n_i^*\Lambda_i\}_{i\in I}$ be $T-$equivalent. We have $\Gamma_i = (n_i^*\Lambda_i)T$ and it is easy to see that $\{n_i^*\Lambda_i\}_{i\in I}$ is a standard g-Bessel sequence and $S_\Gamma = T^* S_{n^*\cdot\Lambda} T$. Because S_Γ, T and T^* are invertible, $S_{n^*\cdot\Lambda}$ is also invertible. Now we have $M_{n,\Lambda,\Gamma}x = \sum_{i\in I} n_i \Lambda_i^* n_i^* \Lambda_i T x = S_{n^*\cdot\Lambda} T x$, and

$$\begin{aligned} M_{n^{-1},\widetilde{\Gamma},\widetilde{\Lambda}}x &= \sum_{i\in I} n_i^{-1} T^{-1} S^{-1}_{n^*\cdot\Lambda} (T^*)^{-1} \Gamma_i^* \Lambda_i S_\Lambda^{-1} x \\ &= \sum_{i\in I} n_i^{-1} T^{-1} S^{-1}_{n^*\cdot\Lambda} (T^*)^{-1} T^* (n_i \cdot \Lambda_i^*) \Lambda_i S_\Lambda^{-1} x = T^{-1} S^{-1}_{n^*\cdot\Lambda} x. \end{aligned}$$

Hence $M_{n,\Lambda,\Gamma}$ is invertible with $M^{-1}_{n,\Lambda,\Gamma} = M_{n^{-1},\widetilde{\Gamma},\widetilde{\Lambda}}$. The result for the case that Λ and $\{n_i\Gamma_i\}_{i\in I}$ are equivalent is obtained with a similar proof. □

The following result is a generalization of Proposition 4.7 in [8] to Hilbert C^*-modules.

Corollary 2.6. *Let $\mathcal{F}$ and $\mathcal{G}$ be standard frames. If $\mathcal{G}$ and $\{n_i f_i\}_{i\in I}$ or $\mathcal{F}$ and $\{n_i^* g_i\}_{i\in I}$ are equivalent, then $M_{n,\mathcal{F},\mathcal{G}}$ is invertible with $M^{-1}_{n,\mathcal{F},\mathcal{G}} = M_{n^{-1},\widetilde{\mathcal{G}},\widetilde{\mathcal{F}}}$.*

3. Inversion of multipliers using perturbed sequences, duals and modular Riesz bases

In this section, we consider the invertibility of multipliers when their Bessel sequences are duals, modular Riesz bases or stable under small perturbations.
Recall from [18] that two standard g-Bessel sequences Λ and Γ are *approximately dual g*-frames if $\|Id_E - T_\Lambda^* T_\Gamma\| < 1$ or equivalently $\|Id_E - T_\Gamma^* T_\Lambda\| < 1$. In this case, we say that Γ (resp. Λ) is an *approximate g-dual* of Λ (resp. Γ).

Proposition 3.1. [18] *Let Λ be a standard g-frame and Λ^d be a g-dual of Λ. If Γ is a sequence such that $\Gamma - \Lambda = \{\Gamma_i - \Lambda_i\}_{i\in I}$ is a standard g-Bessel sequence with $B_{\Gamma-\Lambda}B_{\Lambda^d} < 1$, then Γ is a standard g-frame with upper bound $\sqrt{B_\Lambda} + \sqrt{B_{\Gamma-\Lambda}}$ and Γ and Λ^d are approximately dual g-frames.*

Lemma 3.2. *Let Λ be a standard g-frame. If Γ is a sequence such that $\Gamma - \Lambda$ is a g-Bessel sequence with $\frac{B_{\Gamma-\Lambda}}{A_\Lambda} < 1$, then Γ is a standard g-frame and Γ and $\widetilde{\Lambda}$ are approximately dual g-frames.*

Proof. It is enough to consider $\Lambda^d = \widetilde{\Lambda}$ and $B_{\Lambda^d} = \frac{1}{A_\Lambda}$ in Proposition 3.1. □

Lemma 3.3. [9] *Let X and Y be Banach spaces, $U : X \longrightarrow Y$ be a bounded, invertible operator. If $V : X \longrightarrow Y$ is an operator such that*

$$\|Ux - Vx\| \leq \lambda_1\|Ux\| + \lambda_2\|Vx\|,$$

for some $0 \leq \lambda_1, \lambda_2 < 1$ and for each $x \in X$, then V is invertible with $\|V^{-1}y\| \leq \frac{1+\lambda_2}{1-\lambda_1}\|U^{-1}\|\|y\|$, for each $y \in Y$.

In the following theorem A is a positive number such that $A1_{\mathfrak{A}} \leq m_i$, for each $i \in I$.

Theorem 3.4. *Let Λ be a standard g-frame. Suppose that $\Gamma - \Lambda$ is a standard g-Bessel sequence with $B_{\Gamma-\Lambda} < \frac{A_\Lambda^2}{B_\Lambda}$ and $\frac{\|m\|_\infty}{A} < \frac{A_\Lambda}{\sqrt{B_{\Gamma-\Lambda}B_\Lambda}}$. Then Γ is a standard g-frame and the multipliers $M_{m,\Lambda,\Gamma}$ and $M_{m,\Gamma,\Lambda}$ are invertible with*

$$\frac{\|x\|}{\|m\|_\infty(B_\Lambda + \sqrt{B_\Lambda B_{\Gamma-\Lambda}})} \leq \|M^{-1}x\| \leq \frac{\|x\|}{AA_\Lambda - \|m\|_\infty \sqrt{B_\Lambda B_{\Gamma-\Lambda}}},$$

and $M^{-1} = \sum_{n=0}^{\infty}[S^{-1}_{m^{\frac{1}{2}}\cdot\Lambda}(S_{m^{\frac{1}{2}}\cdot\Lambda} - M)]^n S^{-1}_{m^{\frac{1}{2}}\cdot\Lambda}$, where M is $M_{m,\Lambda,\Gamma}$ or $M_{m,\Gamma,\Lambda}$.

Proof. It follows from the first part of the proof of Proposition 3.7 in [14] that

$$AA_\Lambda\langle x, x\rangle \leq \sum_{i\in I}\langle m_i^{\frac{1}{2}}\Lambda_i x, m_i^{\frac{1}{2}}\Lambda_i x\rangle \leq \|m\|_\infty B_\Lambda\langle x, x\rangle.$$

Thus

$$AA_\Lambda Id_E \leq S_{m^{\frac{1}{2}}\cdot\Lambda} \leq \|m\|_\infty B_\Lambda Id_E$$

and Theorem 2.2.5 part (4) in [20] implies that

$$\frac{1}{\|m\|_\infty B_\Lambda} Id_E \leq S^{-1}_{m^{\frac{1}{2}}\cdot\Lambda} \leq \frac{1}{AA_\Lambda} Id_E,$$

so (using Theorem 2.2.5 part (3) in [20])

$$\frac{1}{\|m\|_\infty B_\Lambda} \leq \|S^{-1}_{m^{\frac{1}{2}}\cdot\Lambda}\| \leq \frac{1}{AA_\Lambda}$$

and $A_\Lambda A \leq \|S^{-1}_{m^{\frac{1}{2}}\cdot\Lambda}\|^{-1}$. Since $A_\Lambda \leq B_\Lambda$, we have $B_{\Gamma-\Lambda} < \frac{A_\Lambda^2}{B_\Lambda} \leq A_\Lambda$. Because $\frac{B_{\Gamma-\Lambda}}{A_\Lambda} < 1$, by Lemma 3.2, Γ is a standard g-frame. We have $\|M_{m,\Lambda,\Gamma-\Lambda}\| \leq \|m\|_\infty \sqrt{B_\Lambda B_{\Gamma-\Lambda}}$, so for each $x \in E$,

$$\|M_{m,\Lambda,\Gamma}x - S_{m^{\frac{1}{2}}\cdot\Lambda}x\| = \|M_{m,\Lambda,\Gamma-\Lambda}x\| \leq \|m\|_\infty \sqrt{B_\Lambda B_{\Gamma-\Lambda}}\|x\|.$$

We have

$$\|m\|_\infty \sqrt{B_\Lambda B_{\Gamma-\Lambda}} < A_\Lambda A \leq \|S^{-1}_{m^{\frac{1}{2}}\cdot\Lambda}\|^{-1},$$

(the first inequality is valid because of the relation $\frac{\|m\|_\infty}{A} < \frac{A_\Lambda}{\sqrt{B_{\Gamma-\Lambda}B_\Lambda}}$ stated in the assumption) so

$$\|M_{m,\Lambda,\Gamma} - S_{m^{\frac{1}{2}}\cdot\Lambda}\| < \|S^{-1}_{m^{\frac{1}{2}}\cdot\Lambda}\|^{-1}.$$

Thus $\|S^{-1}_{m^{\frac{1}{2}}\cdot\Lambda}M_{m,\Lambda,\Gamma} - Id_E\| < 1$, so by Newmann algorithm $S^{-1}_{m^{\frac{1}{2}}\cdot\Lambda}M_{m,\Lambda,\Gamma}$ is invertible with

$$(S^{-1}_{m^{\frac{1}{2}}\cdot\Lambda}M_{m,\Lambda,\Gamma})^{-1} = \sum_{n=0}^{\infty}(Id_E - S^{-1}_{m^{\frac{1}{2}}\cdot\Lambda}M_{m,\Lambda,\Gamma})^n,$$

consequently $M_{m,\Lambda,\Gamma}$ is invertible with

$$M^{-1}_{m,\Lambda,\Gamma} = \Big[\sum_{n=0}^{\infty}(Id_E - S^{-1}_{m^{\frac{1}{2}}\cdot\Lambda}M_{m,\Lambda,\Gamma})^n\Big]S^{-1}_{m^{\frac{1}{2}}\cdot\Lambda} = \sum_{n=0}^{\infty}[S^{-1}_{m^{\frac{1}{2}}\cdot\Lambda}(S_{m^{\frac{1}{2}}\cdot\Lambda} - M_{m,\Lambda,\Gamma})]^n S^{-1}_{m^{\frac{1}{2}}\cdot\Lambda},$$

and the equality $M_{m,\Gamma,\Lambda} = M^*_{m,\Lambda,\Gamma}$ yields that $M_{m,\Gamma,\Lambda}$ is also invertible with

$$M^{-1}_{m,\Gamma,\Lambda} = \sum_{n=0}^{\infty}[S^{-1}_{m^{\frac{1}{2}}\cdot\Lambda}(S_{m^{\frac{1}{2}}\cdot\Lambda} - M_{m,\Gamma,\Lambda})]^n S^{-1}_{m^{\frac{1}{2}}\cdot\Lambda}.$$

By Proposition 3.1, $\sqrt{B_\Lambda} + \sqrt{B_{\Gamma-\Lambda}}$ is an upper bound for Γ, so

$$\|M_{m,\Lambda,\Gamma}\| \leq \|m\|_\infty \sqrt{B_\Lambda}(\sqrt{B_\Lambda} + \sqrt{B_{\Gamma-\Lambda}}).$$

Using $\|x\| = \|M_{m,\Lambda,\Gamma}M^{-1}_{m,\Lambda,\Gamma}x\| \leq \|M_{m,\Lambda,\Gamma}\|\|M^{-1}_{m,\Lambda,\Gamma}x\|$, we get

$$\|M^{-1}_{m,\Lambda,\Gamma}x\| \geq \frac{1}{\|M_{m,\Lambda,\Gamma}\|}\|x\| \geq \frac{\|x\|}{\|m\|_\infty \sqrt{B_\Lambda}(\sqrt{B_\Lambda} + \sqrt{B_{\Gamma-\Lambda}})}.$$

Now the remainder can be obtained from the relation

$$\begin{aligned} \|S_{m^{\frac{1}{2}}\cdot\Lambda}x - M_{m,\Lambda,\Gamma}x\| &\leq \|m\|_\infty \sqrt{B_\Lambda B_{\Gamma-\Lambda}}\|x\| \\ &\leq \|m\|_\infty \sqrt{B_\Lambda B_{\Gamma-\Lambda}}\|S^{-1}_{m^{\frac{1}{2}}\cdot\Lambda}\|\|S_{m^{\frac{1}{2}}\cdot\Lambda}x\|, \end{aligned}$$

Lemma 3.3 with $U = S_{m^{\frac{1}{2}}\cdot\Lambda}$, $V = M_{m,\Lambda,\Gamma}$, $\lambda_1 = \|m\|_\infty \sqrt{B_\Lambda B_{\Gamma-\Lambda}}\|S^{-1}_{m^{\frac{1}{2}}\cdot\Lambda}\|$, $\lambda_2 = 0$ and using the inequality $\|S^{-1}_{m^{\frac{1}{2}}\cdot\Lambda}\|^{-1} \geq AA_\Lambda$. □

Corollary 3.5. *Let $\mathcal{F}$ be a standard frame. Suppose that $\mathcal{G} - \mathcal{F} = \{g_i - f_i\}_{i\in I}$ is a standard Bessel sequence with $B_{\mathcal{G}-\mathcal{F}} < \frac{A^2_{\mathcal{F}}}{B_{\mathcal{F}}}$ and $\frac{\|m\|_\infty}{A} < \frac{A_{\mathcal{F}}}{\sqrt{B_{\mathcal{G}-\mathcal{F}}B_{\mathcal{F}}}}$. Then $\mathcal{G}$ is a standard frame and the multipliers $M_{m,\mathcal{F},\mathcal{G}}$ and $M_{m,\mathcal{G},\mathcal{F}}$ are invertible with*

$$\frac{\|x\|}{\|m\|_\infty(B_{\mathcal{F}} + \sqrt{B_{\mathcal{F}}B_{\mathcal{G}-\mathcal{F}}})} \leq \|M^{-1}x\| \leq \frac{\|x\|}{AA_{\mathcal{F}} - \|m\|_\infty \sqrt{B_{\mathcal{F}}B_{\mathcal{G}-\mathcal{F}}}},$$

and $M^{-1} = \sum_{n=0}^{\infty}[S^{-1}_{m^{\frac{1}{2}}\cdot\mathcal{F}}(S_{m^{\frac{1}{2}}\cdot\mathcal{F}} - M)]^n S^{-1}_{m^{\frac{1}{2}}\cdot\mathcal{F}}$, where M is $M_{m,\mathcal{F},\mathcal{G}}$ or $M_{m,\mathcal{G},\mathcal{F}}$.

Corollary 3.6. *Let Λ be a standard g-frame. If Γ is a sequence such that there exists $\lambda \in [0, \frac{A^2_\Lambda}{B_\Lambda})$ with $\left\|\sum_{i\in I}\langle(\Gamma_i - \Lambda_i)x, (\Gamma_i - \Lambda_i)x\rangle\right\| \leq \lambda\|x\|^2$, for each $x \in E$ and $\frac{\|m\|_\infty\sqrt{\lambda}}{A} < \frac{A_\Lambda}{\sqrt{B_\Lambda}}$, then Γ is a standard g-frame and $M_{m,\Lambda,\Gamma}$, $M_{m,\Gamma,\Lambda}$ are invertible with*

$$\frac{\|x\|}{\|m\|_\infty(B_\Lambda + \sqrt{\lambda B_\Lambda})} \leq \|M^{-1}x\| \leq \frac{\|x\|}{AA_\Lambda - \|m\|_\infty \sqrt{\lambda B_\Lambda}},$$

where M is $M_{m,\Lambda,\Gamma}$ or $M_{m,\Gamma,\Lambda}$.

Proof. It is enough to consider $B_{\Gamma-\Lambda} = \lambda$ in Theorem 3.4. □

The following result is a generalization of Proposition 4.1 in [25] to Hilbert C^*−modules.

Corollary 3.7. *Let $\mathcal{F}$ be a standard frame. If $\mathcal{G}$ is a sequence such that there exists $\lambda \in [0, \frac{A^2_{\mathcal{F}}}{B_{\mathcal{F}}})$ with $\left\|\sum_{i\in I}|\langle x, (g_i - f_i)\rangle|^2\right\| \leq \lambda\|x\|^2$, for each $x \in E$ and $\frac{\|m\|_\infty\sqrt{\lambda}}{A} < \frac{A_{\mathcal{F}}}{\sqrt{B_{\mathcal{F}}}}$, then $\mathcal{G}$ is a standard frame and $M_{m,\mathcal{F},\mathcal{G}}$ and $M_{m,\mathcal{G},\mathcal{F}}$ are invertible with*

$$\frac{\|x\|}{\|m\|_\infty(B_{\mathcal{F}} + \sqrt{\lambda B_{\mathcal{F}}})} \leq \|M^{-1}x\| \leq \frac{\|x\|}{AA_{\mathcal{F}} - \|m\|_\infty \sqrt{\lambda B_{\mathcal{F}}}},$$

for each $x \in E$, where M denotes anyone of $M_{m,\mathcal{F},\mathcal{G}}$ and $M_{m,\mathcal{G},\mathcal{F}}$.

In the next proposition, we get some results about the stability of a standard g-frame under small perturbations and consider the invertibility of the multiplier generated by the standard g-frame and the perturbed sequence.

Proposition 3.8. *Let $\Phi = \{\phi_i \in \mathfrak{L}(E, E_i) : i \in I\}$ be a standard g-frame. If $\Psi = \{\psi_i \in \mathfrak{L}(E, E_i) : i \in I\}$ is a sequence such that there exists $\lambda \in [0, \frac{A_\Phi^2}{B_\Phi})$ with $\left\| \sum_{i\in I} \langle (n_i\psi_i - \phi_i)x, (n_i\psi_i - \phi_i)x\rangle \right\| \leq \lambda \|x\|^2$, for each $x \in E$, then $n \cdot \Psi$ and Ψ are standard g-frames and $M_{n,\Phi,\Psi}$ and $M_{n^*,\Psi,\Phi}$ are invertible with*

$$\frac{\|x\|}{B_\Phi + \sqrt{\lambda B_\Phi}} \leq \|M^{-1}x\| \leq \frac{\|x\|}{A_\Phi - \sqrt{\lambda B_\Phi}},$$

for each $x \in E$, where M denotes anyone of $M_{n,\Phi,\Psi}$ and $M_{n^*,\Psi,\Phi}$.

Proof. Suppose that $\Lambda = \Phi$, $\Gamma = n \cdot \Psi$, $m_i = 1_{\mathfrak{A}}$ for each $i \in I$ and $A = 1$. Since $\left\| \sum_{i\in I} \langle (\Gamma_i - \Lambda_i)x, (\Gamma_i - \Lambda_i)x\rangle \right\| \leq \lambda \|x\|^2$ and $\frac{\|m\|_\infty \sqrt{\lambda}}{A} < \frac{A_\Lambda}{\sqrt{B_\Lambda}}$, by Corollary 3.6, $\Gamma = n \cdot \Psi$ is a standard g-frame and $M_{m,\Lambda,\Gamma} = M_{n,\Phi,\Psi}$ and $M_{m,\Gamma,\Lambda} = M_{n^*,\Psi,\Phi}$ are invertible with

$$\frac{\|x\|}{B_\Phi + \sqrt{\lambda B_\Phi}} \leq \|M^{-1}x\| \leq \frac{\|x\|}{A_\Phi - \sqrt{\lambda B_\Phi}},$$

for each $x \in E$, where M denotes anyone of $M_{n,\Phi,\Psi}$ and $M_{n^*,\Psi,\Phi}$. Because n is semi-normalized, there exists some positive number D such that $D1_{\mathfrak{A}} \leq |n_i| \leq \|n\|_\infty 1_{\mathfrak{A}}$, for each $i \in I$. Therefore the relation

$$D^2 \left\| \sum_{i\in I} \langle \psi_i x, \psi_i x\rangle \right\| \leq \left\| \sum_{i\in I} \langle n_i\psi_i x, n_i\psi_i x\rangle \right\| \leq \|n\|_\infty^2 \left\| \sum_{i\in I} \langle \psi_i x, \psi_i x\rangle \right\|,$$

and Theorem 3.1 in [27] yield that Ψ is a standard g-frame. □

Corollary 3.9. *Let $\mathcal{F}$ be a standard frame. If $\mathcal{G}$ is a sequence such that there exists $\lambda \in [0, \frac{A_\mathcal{F}^2}{B_\mathcal{F}})$ with $\left\| \sum_{i\in I} |\langle x, (n_i^* g_i - f_i)\rangle|^2 \right\| \leq \lambda \|x\|^2$, for each $x \in E$, then $n^* \cdot \mathcal{G}$ and $\mathcal{G}$ are standard frames and $M_{n,\mathcal{F},\mathcal{G}}$ and $M_{n^*,\mathcal{G},\mathcal{F}}$ are invertible with*

$$\frac{\|x\|}{B_\mathcal{F} + \sqrt{\lambda B_\mathcal{F}}} \leq \|M^{-1}x\| \leq \frac{\|x\|}{A_\mathcal{F} - \sqrt{\lambda B_\mathcal{F}}},$$

for each $x \in E$, where M denotes anyone of $M_{n,\mathcal{F},\mathcal{G}}$ and $M_{n^*,\mathcal{G},\mathcal{F}}$.

Proposition 3.10. *Suppose that $\Phi = \{\phi_i \in \mathfrak{L}(E, E_i) : i \in I\}$ is a standard g-frame and $\Psi = \{\psi_i \in \mathfrak{L}(E, E_i) : i \in I\}$ is a sequence such that there exists $\lambda \in [0, \frac{A_\Phi^2}{B_\Phi})$ with $\left\| \sum_{i\in I} \langle (\psi_i - \phi_i)x, (\psi_i - \phi_i)x\rangle \right\| \leq \lambda \|x\|^2$, for each $x \in E$. Let $\alpha < \frac{A_\Phi - \sqrt{\lambda B_\Phi}}{B_\Phi + \sqrt{\lambda B_\Phi}}$ with $|n_i - 1_{\mathfrak{A}}| \leq \alpha 1_{\mathfrak{A}}$, for each $i \in I$. Then Ψ is a standard g-frame and $M_{n,\Phi,\Phi}$, $M_{n,\Phi,\Psi}$ and $M_{n,\Psi,\Phi}$ are invertible with*

$$\frac{\|x\|}{(\alpha + 1)B_\Phi} \leq \|M_{n,\Phi,\Phi}^{-1}x\| \leq \frac{\|x\|}{A_\Phi - \alpha B_\Phi},$$

$$\frac{\|x\|}{(\alpha + 1)(B_\Phi + \sqrt{\lambda B_\Phi})} \leq \|M^{-1}x\| \leq \frac{\|x\|}{A_\Phi - \alpha B_\Phi - (\alpha + 1)\sqrt{\lambda B_\Phi}},$$

$$M_{n,\Phi,\Phi}^{-1} = \sum_{n=0}^{\infty} [S_\Phi^{-1}(S_\Phi - M_{n,\Phi,\Phi})]^n S_\Phi^{-1}$$

and

$$M^{-1} = \sum_{n=0}^{\infty} [M_{n,\Phi,\Phi}^{-1}(M_{n,\Phi,\Phi} - M)]^n M_{n.\Phi,\Phi}^{-1},$$

where M is $M_{n,\Phi,\Psi}$ or $M_{n,\Psi,\Phi}$.

Proof. First suppose that $\lambda = 0$. Then $\Psi = \Phi$. Now let $m = \{m_i\}_{i\in I}$ with $m_i = 1_{\mathfrak{A}}$, for each $i \in I$, $A = 1$, $\Gamma = n \cdot \Phi$ and $\Lambda = \Phi$. Then

$$\Big\| \sum_{i\in I} \langle (\Gamma_i - \Lambda_i)x, (\Gamma_i - \Lambda_i)x \rangle \Big\| \leq B_\Phi \|n_i - 1_{\mathfrak{A}}\|^2 \|x\|^2 \leq B_\Phi \alpha^2 \|x\|^2.$$

Therefore we can set $B_{\Gamma-\Lambda} = B_\Phi \alpha^2$. Then

$$B_{\Gamma-\Lambda} = B_\Phi \alpha^2 < B_\Phi \Big(\frac{A_\Phi - \sqrt{\lambda B_\Phi}}{B_\Phi + \sqrt{\lambda B_\Phi}} \Big)^2 \leq B_\Phi (\frac{A_\Phi}{B_\Phi})^2 = \frac{A_\Lambda^2}{B_\Lambda}.$$

Now the result follows from Theorem 3.4 using the equalities $M_{m,\Lambda,\Gamma} = M_{\{1_{\mathfrak{A}}\},\Phi,n\cdot\Phi} = M_{n,\Phi,\Phi}$ and $S_{m^{\frac{1}{2}}\cdot\Lambda} = S_\Phi$. Now let $\lambda > 0$. By considering $B_{\Psi-\Phi} = \lambda$, we get $B_{\Psi-\Phi} < \frac{A_\Phi^2}{B_\Phi}$. Now similar to the proof of Theorem 3.4, we get Ψ is a standard g-frame with $B_\Psi = \sqrt{B_\Phi} + \sqrt{\lambda}$. For each $x \in E$, we have

$$\|M_{n,\Phi,\Phi}x - M_{n,\Phi,\Psi}x\| = \|M_{n,\Phi,\Phi-\Psi}x\| \leq \|n\|_\infty \sqrt{B_\Phi B_{\Psi-\Phi}} \|x\|.$$

Since $|n_i - 1_{\mathfrak{A}}| \leq \alpha 1_{\mathfrak{A}}$, we get $\|n_i - 1_{\mathfrak{A}}\| \leq \alpha$, for each $i \in I$ and $\|n\|_\infty \leq \alpha + 1$. Then

$$\|M_{n,\Phi,\Phi}x - M_{n,\Phi,\Psi}x\| \leq (\alpha+1)\sqrt{B_\Phi \lambda}\|x\| \leq (\alpha+1)\sqrt{B_\Phi\lambda}\|M_{n,\Phi,\Phi}^{-1}\|\|M_{n,\Phi,\Phi}x\|. \tag{1}$$

Because $\alpha < \frac{A_\Phi - \sqrt{\lambda B_\Phi}}{B_\Phi + \sqrt{\lambda B_\Phi}}$ and $\|M_{n,\Phi,\Phi}^{-1}\| \leq \frac{1}{A_\Phi - \alpha B_\Phi}$, we obtain that

$$(\alpha+1)\sqrt{B_\Phi \lambda} < A_\Phi - \alpha B_\Phi \leq \frac{1}{\|M_{n,\Phi,\Phi}^{-1}\|},$$

so $(\alpha+1)\sqrt{B_\Phi\lambda}\|M_{n,\Phi,\Phi}^{-1}\| < 1$ and $\|M_{n,\Phi,\Phi} - M_{n,\Phi,\Psi}\| < \frac{1}{\|M_{n,\Phi,\Phi}^{-1}\|}$. Now the result is obtained using Newmann algorithm with $\|Id_E - M_{n,\Phi,\Phi}^{-1}M_{n,\Phi,\Psi}\| < 1$, using (1) by considering $\lambda_1 = (\alpha+1)\sqrt{B_\Phi\lambda}\|M_{n,\Phi,\Phi}^{-1}\|$, $U = M_{n,\Phi,\Phi}$, $\lambda_2 = 0$ and $V = M_{n,\Phi,\Psi}$ in Lemma 3.3 and using the inequalities $\|M_{n,\Phi,\Psi}\| \leq (\alpha+1)\sqrt{B_\Phi}(\sqrt{B_\Phi} + \sqrt{\lambda})$ and $\frac{1}{(\alpha+1)\sqrt{B_\Phi}(\sqrt{B_\Phi}+\sqrt{\lambda})} \leq \frac{1}{\|M_{n,\Phi,\Psi}\|} \leq \|M_{n,\Phi,\Psi}^{-1}\|$. The result for $M_{n,\Psi,\Phi}$ is obtained similarly. □

The next corollary is a generalization of Proposition 4.2 in [25] to Hilbert C^*−modules.

Corollary 3.11. *Suppose that $\mathcal{F}$ is a standard frame and $\mathcal{G}$ is a sequence such that there exists $\lambda \in [0, \frac{A_\mathcal{F}^2}{B_\mathcal{F}})$ with $\Big\| \sum_{i\in I} |\langle x, g_i - f_i \rangle|^2 \Big\| \leq \lambda \|x\|^2$, for each $x \in E$. Let $\alpha < \frac{A_\mathcal{F} - \sqrt{\lambda B_\mathcal{F}}}{B_\mathcal{F} + \sqrt{\lambda B_\mathcal{F}}}$ with $|n_i - 1_{\mathfrak{A}}| \leq \alpha 1_{\mathfrak{A}}$, for each $i \in I$. Then $\mathcal{G}$ is a standard frame and $M_{n,\mathcal{F},\mathcal{F}}$, $M_{n,\mathcal{F},\mathcal{G}}$ and $M_{n,\mathcal{G},\mathcal{F}}$ are invertible with*

$$\frac{\|x\|}{(\alpha+1)B_\mathcal{F}} \leq \|M_{n,\mathcal{F},\mathcal{F}}^{-1}x\| \leq \frac{\|x\|}{A_\mathcal{F} - \alpha B_\mathcal{F}},$$

$$\frac{\|x\|}{(\alpha+1)(B_\mathcal{F} + \sqrt{\lambda B_\mathcal{F}})} \leq \|M^{-1}x\| \leq \frac{\|x\|}{A_\mathcal{F} - \alpha B_\mathcal{F} - (\alpha+1)\sqrt{\lambda B_\mathcal{F}}},$$

$$M_{n,\mathcal{F},\mathcal{F}}^{-1} = \sum_{n=0}^{\infty} [S_\mathcal{F}^{-1}(S_\mathcal{F} - M_{n,\mathcal{F},\mathcal{F}})]^n S_\mathcal{F}^{-1}$$

and

$$M^{-1} = \sum_{n=0}^{\infty} [M_{n,\mathcal{F},\mathcal{F}}^{-1}(M_{n,\mathcal{F},\mathcal{F}} - M)]^n M_{n.\mathcal{F},\mathcal{F}}^{-1},$$

where M is $M_{n,\mathcal{F},\mathcal{G}}$ or $M_{n,\mathcal{G},\mathcal{F}}$.

Proposition 3.12. *Let Λ be a standard g-frame. Assume that Γ and Λ are $T-$equivalent and there exists a nonnegative number α such that $|n_i - 1_{\mathfrak{A}}| \leq \alpha 1_{\mathfrak{A}} < \frac{A_\Lambda}{B_\Lambda} 1_{\mathfrak{A}}$, for each $i \in I$. Then Γ is a standard g-frame and $M_{n,\Lambda,\Gamma}$, $M_{n,\Gamma,\Lambda}$ are invertible with $M^{-1}_{n,\Lambda,\Gamma} = T^{-1} M^{-1}_{n,\Lambda,\Lambda}$ and $M^{-1}_{n,\Gamma,\Lambda} = M^{-1}_{n,\Lambda,\Lambda}(T^{-1})^*$.*

Proof. By Theorem 2.2, Γ is a standard g-frame. Consider $\Phi = \Psi = \Lambda$ in Proposition 3.10, so we can assume that $\lambda = 0$. Hence by Proposition 3.10, $M_{n,\Lambda,\Lambda}$ is invertible. Now the remainder follows from the equalities $M_{n,\Lambda,\Gamma} = M_{n,\Lambda,\Lambda} T$ and $M_{n,\Gamma,\Lambda} = T^* M_{n,\Lambda,\Lambda}$. □

The following result is a generalization of Proposition 4.3 in [25] to Hilbert C^*-modules.

Corollary 3.13. *Let $\mathcal{F}$ be a standard frame. Assume that $\mathcal{G}$ and $\mathcal{F}$ are $T-$equivalent and there exists a nonnegative number α such that $|n_i - 1_{\mathfrak{A}}| \leq \alpha 1_{\mathfrak{A}} < \frac{A_\mathcal{F}}{B_\mathcal{F}} 1_{\mathfrak{A}}$, for each $i \in I$. Then $\mathcal{G}$ is a standard frame and $M_{n,\mathcal{F},\mathcal{G}}$, $M_{n,\mathcal{G},\mathcal{F}}$ are invertible with $M^{-1}_{n,\mathcal{F},\mathcal{G}} = (T^{-1})^* M^{-1}_{n,\mathcal{F},\mathcal{F}}$ and $M^{-1}_{n,\mathcal{G},\mathcal{F}} = M^{-1}_{n,\mathcal{F},\mathcal{F}} T^{-1}$.*

Proposition 3.14. *Suppose that Φ is a standard g-frame and Ψ is a sequence such that $\left\| \sum_{i\in I} \langle (n_i\psi_i - \phi_i^d)x, (n_i\psi_i - \phi_i^d)x \rangle \right\| \leq \lambda \|x\|^2$, for each $x \in E$, where $\lambda \in [0, \frac{1}{B_\Phi})$ and $\Phi^d = \{\Phi_i^d\}_{i\in I}$ is a g-dual of Φ. If M is $M_{n,\Phi,\Psi}$ or $M_{n^*,\Psi,\Phi}$, then $n \cdot \Psi$ is a standard g-frame and M is invertible with*

$$\frac{\|x\|}{1 + \sqrt{\lambda B_\Phi}} \leq \|M^{-1}x\| \leq \frac{\|x\|}{1 - \sqrt{\lambda B_\Phi}},$$

for each $x \in E$.

Proof. Let $\Gamma = n \cdot \Psi$, $\Lambda = \Phi^d$, $\Lambda^d = \Phi$ and $B_{\Gamma-\Lambda} = \lambda$. Since $B_{\Gamma-\Lambda} B_{\Lambda^d} < 1$, then by Proposition 3.1, Γ is a standard g-frame. For each $x \in E$, we have

$$\begin{aligned} \|M_{n,\Phi,\Psi}x - x\| &= \|M_{n,\Phi,\Psi}x - M_{\{1_{\mathfrak{A}}\},\Phi,\Phi^d}x\| \\ &= \|M_{\{1_{\mathfrak{A}}\},\Phi,n.\Psi}x - M_{\{1_{\mathfrak{A}}\},\Phi,\Phi^d}\| = \|M_{\{1_{\mathfrak{A}}\},\Phi,(n.\Psi-\Phi^d)}\| \leq \sqrt{B_\Phi \lambda}\|x\|. \end{aligned}$$

Hence $\|M_{n,\Phi,\Psi} - Id_E\| < 1$, so by Newmann algorithm $M_{n,\Phi,\Psi}$ is invertible. Because $\|M_{n,\Phi,\Psi}x\| - \|x\| \leq \|M_{n,\Phi,\Psi}x - x\| \leq \sqrt{B_\Phi\lambda}\|x\|$, we get $\|M_{n,\Phi,\Psi}\| \leq (1 + \sqrt{B_\Phi\lambda})$, so $\frac{1}{1+\sqrt{B_\Phi\lambda}} \leq \frac{1}{\|M_{n,\Phi,\Psi}\|} \leq \|M^{-1}_{n,\Phi,\Psi}\|$. The remainder follows from Lemma 3.3 by considering $U = Id_E$, $\lambda_1 = \sqrt{B_\Phi\lambda}$, $\lambda_2 = 0$ and $V = M_{n,\Phi,\Psi}$. The result for $M_{n^*,\Psi,\Phi}$ is obtained similarly using

$$\|M_{n^*,\Psi,\Phi} - Id_E\| = \|(M_{n,\Phi,\Psi} - Id_E)^*\| = \|M_{n,\Phi,\Psi} - Id_E\|.$$

□

The next corollary is a generalization of Proposition 4.5 in [25] to Hilbert C^*-modules.

Corollary 3.15. *Suppose that $\mathcal{F}$ is a standard frame and $\mathcal{G}$ is a sequence such that there exists some $\lambda \in [0, \frac{1}{B_\mathcal{F}})$ with $\left\| \sum_{i\in I} |\langle x, n_i^* g_i - f_i^d \rangle|^2 \right\| \leq \lambda \|x\|^2$, for each $x \in E$, where $\mathcal{F}^d = \{f_i^d\}_{i\in I}$ is a dual of $\mathcal{F}$. If M is $M_{n,\mathcal{F},\mathcal{G}}$ or $M_{n^*,\mathcal{G},\mathcal{F}}$, then $n^* \cdot \mathcal{G}$ is a standard frame and M is invertible with*

$$\frac{\|x\|}{1 + \sqrt{\lambda B_\mathcal{F}}} \leq \|M^{-1}x\| \leq \frac{\|x\|}{1 - \sqrt{\lambda B_\mathcal{F}}},$$

for each $x \in E$.

Proposition 3.16. *Let Φ be a standard g-frame and let Φ^d be a g-dual for Φ. If there exists a nonnegative number $\alpha < \frac{1}{\sqrt{B_\Phi B_{\Phi^d}}}$ such that $|n_i - 1_{\mathfrak{A}}| \le \alpha 1_{\mathfrak{A}}$, then M_{n,Φ,Φ^d} and $M_{n,\Phi^d,\Phi}$ are invertible with*

$$\frac{\|x\|}{1+\alpha\sqrt{B_\Phi B_{\Phi^d}}} \le \|M^{-1}x\| \le \frac{\|x\|}{1-\alpha\sqrt{B_\Phi B_{\Phi^d}}}, \tag{2}$$

for each $x \in E$, where M is M_{n,Φ,Φ^d} or $M_{n,\Phi^d,\Phi}$ and

$$M^{-1}_{n,\Phi,\Phi^d} = \sum_{n=0}^{\infty} (M_{\{(1_{\mathfrak{A}}-n_i)\}_{i\in I},\Phi,\Phi^d})^n \tag{3}$$

and

$$M^{-1}_{n,\Phi^d,\Phi} = \sum_{n=0}^{\infty} (M_{\{(1_{\mathfrak{A}}-n_i)\}_{i\in I},\Phi^d,\Phi})^n.$$

Proof. If $\alpha = 0$, then $n_i = 1_{\mathfrak{A}}$, for each $i \in I$, so $M_{n,\Phi,\Phi^d} = Id_E = M_{n,\Phi^d,\Phi}$. Let $\alpha > 0$. Then for each $x \in E$, we have

$$\begin{aligned} \|M_{n,\Phi,\Phi^d}x - x\| &= \|M_{n,\Phi,\Phi^d}x - M_{\{1_{\mathfrak{A}}\},\Phi,\Phi^d}x\| \\ &\le \|\{n_i - 1_{\mathfrak{A}}\}_{i\in I}\|_\infty \sqrt{B_\Phi B_{\Phi^d}}\|x\| \le \alpha\sqrt{B_\Phi B_{\Phi^d}}\|x\|. \end{aligned}$$

Now (2) is obtained from Lemma 3.3 by considering $\lambda_1 = \alpha\sqrt{B_\Phi B_{\Phi^d}}$, $\lambda_2 = 0$, $U = Id_E$, $V = M_{n,\Phi,\Phi^d}$ and the relations

$$\|M_{n,\Phi,\Phi^d}x\| \le \|x\| + \|M_{n,\Phi,\Phi^d}x - x\| \le (\alpha\sqrt{B_\Phi B_{\Phi^d}} + 1)\|x\|$$

and $\frac{1}{\|M_{n,\Phi,\Phi^d}\|} \le \|M^{-1}_{n,\Phi,\Phi^d}\|$. Since

$$\|M_{\{(1_{\mathfrak{A}}-n_i)\}_{i\in I},\Phi,\Phi^d}\| = \|Id_E - M_{n,\Phi,\Phi^d}\| < 1,$$

Newmann algorithm yields (3). The result for $M_{n,\Phi^d,\Phi}$ can be obtained with a similar proof. □

The following result is a generalization of Proposition 4.4 in [25] to Hilbert C^*−modules.

Corollary 3.17. *Let $\mathcal{F}$ be a standard frame and let $\mathcal{F}^d$ be a dual for $\mathcal{F}$. If there exists a nonnegative number $\alpha < \frac{1}{\sqrt{B_\mathcal{F} B_{\mathcal{F}^d}}}$ such that $|n_i - 1_{\mathfrak{A}}| \le \alpha 1_{\mathfrak{A}}$, then $M_{n,\mathcal{F},\mathcal{F}^d}$ and $M_{n,\mathcal{F}^d,\mathcal{F}}$ are invertible with*

$$\frac{\|x\|}{1+\alpha\sqrt{B_\mathcal{F} B_{\mathcal{F}^d}}} \le \|M^{-1}x\| \le \frac{\|x\|}{1-\alpha\sqrt{B_\mathcal{F} B_{\mathcal{F}^d}}},$$

for each $x \in E$, where M is $M_{n,\mathcal{F},\mathcal{F}^d}$ or $M_{n,\mathcal{F}^d,\mathcal{F}}$ and

$$M^{-1}_{n,\mathcal{F},\mathcal{F}^d} = \sum_{n=0}^{\infty} (M_{\{(1_{\mathfrak{A}}-n_i)\}_{i\in I},\mathcal{F},\mathcal{F}^d})^n$$

and

$$M^{-1}_{n,\mathcal{F}^d,\mathcal{F}} = \sum_{n=0}^{\infty} (M_{\{(1_{\mathfrak{A}}-n_i)\}_{i\in I},\mathcal{F}^d,\mathcal{F}})^n.$$

Proposition 3.18. *Let Λ and Γ be standard g-frames and let $M_{n,\Lambda,\Gamma}$ be invertible. Then $M^{-1}_{n,\Lambda,\Gamma}$ is a g-Bessel multiplier for Λ^d and Γ^d with symbol n^{-1}, where Γ^d and Λ^d are g-duals of Γ and Λ, respectively.*

Proof. Since n_i's are elements of $\mathcal{Z}(\mathfrak{A})$, the operator $n_i\Gamma_i$ is adjointable with $(n_i\Gamma_i)^* = n_i^*\Gamma_i^*$. Now for each $x \in E$, we have $\sum_{i\in I} \Lambda_i^*[(n_i\Gamma_i) \circ M_{n,\Lambda,\Gamma}^{-1}]x = \sum_{i\in I} n_i\Lambda_i^*\Gamma_i M_{n,\Lambda,\Gamma}^{-1}x = x$, so $\Lambda^d = \{(n_i\Gamma_i) \circ M_{n,\Lambda,\Gamma}^{-1}\}_{i\in I}$ is a g-dual for Λ. Now for each $\{x_i\}_{i\in I} \in \oplus_{i\in I}E_i$, we get

$$T_{\Lambda^d}^* \mathcal{M}_{n^{*-1}}(\{x_i\}_{i\in I}) = \sum_{i\in I}[(n_i\Gamma_i) \circ M_{n,\Lambda,\Gamma}^{-1}]^*(n_i^{*-1}x_i) = M_{n^*,\Gamma,\Lambda}^{-1}T_\Gamma^*(\{x_i\}_{i\in I}),$$

so $T_{\Lambda^d}^*\mathcal{M}_{n^{*-1}} = M_{n^*,\Gamma,\Lambda}^{-1}T_\Gamma^*$. Now let Γ^d be a g-dual of Γ. Then $T_\Gamma^*T_{\Gamma^d} = Id_E$, so

$$T_{\Lambda^d}^* \circ \mathcal{M}_{n^{*-1}} \circ T_{\Gamma^d} = M_{n^*,\Gamma,\Lambda}^{-1}T_\Gamma^*T_{\Gamma^d} = M_{n^*,\Gamma,\Lambda}^{-1}.$$

Thus

$$M_{n,\Lambda,\Gamma}^{-1} = (M_{n^*,\Gamma,\Lambda}^{-1})^* = T_{\Gamma^d}^*\mathcal{M}_{n^{-1}}T_{\Lambda^d} = M_{n^{-1},\Gamma^d,\Lambda^d}.$$

This completes the proof. □

The next result is a generalization of Theorem 1.1 in [8] to Hilbert C^*– modules.

Corollary 3.19. *Let $\mathcal{F}$ and $\mathcal{G}$ be standard frames and let $M_{n,\mathcal{F},\mathcal{G}}$ be invertible. Then $M_{n,\mathcal{F},\mathcal{G}}^{-1}$ is a Bessel multiplier for $\mathcal{F}^d$ and $\mathcal{G}^d$ with symbol $n^{-1} = \{n_i^{-1}\}_{i\in I}$, where $\mathcal{G}^d$ and $\mathcal{F}^d$ are duals of $\mathcal{G}$ and $\mathcal{F}$, respectively.*

Now we recall the following definition from [13]:

Definition 3.20. *A standard g-frame Λ is a* modular g-Riesz basis *if it has the following property: if $\sum_{i\in\Omega} \Lambda_i^*g_i = 0$, where $g_i \in E_i$ and $\Omega \subseteq I$, then $g_i = 0$, for each $i \in \Omega$.*

The following proposition is a generalization of Theorem 4.3 (iii) in [14] to standard g-frames.

Proposition 3.21. *Let Λ and Γ be modular g-Riesz bases. Then $M_{n,\Gamma,\Lambda}$ is invertible with $M_{n,\Gamma,\Lambda}^{-1} = M_{n^{-1},\widetilde{\Lambda},\widetilde{\Gamma}}$.*

Proof. Let $x \in E$. For each $i \in I$, we have $\sum_{k\in I} \Gamma_k^*\widetilde{\Gamma}_k\Gamma_i^*x = \Gamma_i^*x$, so $\sum_{k\in I} \Gamma_k^*f_k = 0$, where $f_k = \widetilde{\Gamma}_k\Gamma_i^*x$, for $k \neq i$ and $f_i = \widetilde{\Gamma}_i\Gamma_i^*x - x$. Because Γ is a modular g-Riesz basis, $\widetilde{\Gamma}_k\Gamma_i^*x = 0$, for $k \neq i$ and $\widetilde{\Gamma}_i\Gamma_i^*x = x$. Now we have

$$\begin{aligned} M_{n^{-1},\widetilde{\Lambda},\widetilde{\Gamma}} \circ M_{n,\Gamma,\Lambda}x &= M_{n^{-1},\widetilde{\Lambda},\widetilde{\Gamma}}(\sum_{k\in I} n_k\Gamma_k^*\Lambda_kx) = \sum_{i\in I}\sum_{k\in I} n_i^{-1}n_k\widetilde{\Lambda_i}^*\widetilde{\Gamma}_i\Gamma_k^*\Lambda_kx \\ &= \sum_{i\in I} n_i^{-1}n_i\widetilde{\Lambda_i}^*\Lambda_ix = \sum_{i\in I}\widetilde{\Lambda_i}^*\Lambda_ix = x. \end{aligned}$$

With a similar proof we can get $M_{n,\Gamma,\Lambda} \circ M_{n^{-1},\widetilde{\Lambda},\widetilde{\Gamma}}x = x$, so $M_{n,\Gamma,\Lambda}^{-1} = M_{n^{-1},\widetilde{\Lambda},\widetilde{\Gamma}}$. □

Corollary 3.22. *Let Λ and Γ be modular g-Riesz bases and let T, S be invertible elements in $\mathfrak{L}(E)$. Then $M_{n,\Gamma S,\Lambda T}$ is invertible with $M_{n,\Gamma S,\Lambda T}^{-1} = M_{n^{-1},\widetilde{\Lambda T},\widetilde{\Gamma S}} = T^{-1}M_{n^{-1},\widetilde{\Lambda},\widetilde{\Gamma}}S^{*-1}$, where $\Gamma S = \{\Gamma_iS\}_{i\in I}$ and $\Lambda T = \{\Lambda_iT\}_{i\in I}$.*

Proof. Using Theorem 2.2, we obtain that ΛT is a standard g-frame with operator $S_{\Lambda T} = T^*S_\Lambda T$, so $\widetilde{\Lambda T} = \{(\Lambda_iT)S_{\Lambda T}^{-1}\}_{i\in I} = \{\widetilde{\Lambda}_iT^{*-1}\}_{i\in I}$. Similarly we have $\widetilde{\Gamma S} = \{\widetilde{\Gamma}_iS^{*-1}\}_{i\in I}$. It is easy to see that $M_{n^{-1},\widetilde{\Lambda T},\widetilde{\Gamma S}} = T^{-1}M_{n^{-1},\widetilde{\Lambda},\widetilde{\Gamma}}S^{*-1}$. Because by the above proposition the operator $M_{n^{-1},\widetilde{\Lambda},\widetilde{\Gamma}}$ is the inverse of $M_{n,\Gamma,\Lambda}$, we get $M_{n^{-1},\widetilde{\Lambda T},\widetilde{\Gamma S}}$ is the inverse of $S^*M_{n,\Gamma,\Lambda}T = M_{n,\Gamma S,\Lambda T}$ and the proof is completed. □

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