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Three Step Algorithm for Weighted Resolvent Average of a Finite Family of Monotone Operators

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Abstract. In this paper, we introduce a composite iterative method for finding a common zero point of weighted resolvent average of a finite family of monotone operators. Furthermore, the strong convergence of the proposed iterative method is established. Finally, our results are illustrated by some numerical examples.

1. Introduction

Let H be a real Hilbert space with the norm $\|.\|$ and the inner product $\langle .,. \rangle$. Let A be a set-valued mapping with the *domain* $\operatorname{dom} A = \{x \in H : A(x) \neq \emptyset\}$ and the *range* $\operatorname{ran} A = \{u \in H : \exists x \in \operatorname{dom} A, u \in A(x)\}$. The *graph* of A is the set $\operatorname{gra} A = \{(x,u) \in H \times H : x \in \operatorname{dom} A, u \in A(x)\}$. An operator $A : H \multimap H$ is said to be *monotone* if

$$\langle x - y, u - v \rangle \geq 0, \ \forall\, (x,u), (y,v) \in \operatorname{gra} A.$$

A monotone operator A is called *maximal monotone* if there exists no monotone operator B such that $\operatorname{gra} A$ is a proper subset of $\operatorname{gra} B$. The *resolvent* of A is the mapping $J_A = (A + \mathrm{Id})^{-1}$.

Recall [2] that a map $T : H \to H$ is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall\, x, y \in H.$$

A point $x \in H$ is said to be a *fixed point* of the operator $T : H \to H$, if $Tx = x$. The set of all fixed points of T is denoted by $\operatorname{Fix}(T)$, i.e.,

$$\operatorname{Fix}(T) = \{x \in H : Tx = x\}.$$

Let us consider the zero point problem for monotone operator A on a real Hilbert space H, i.e., finding a point $x \in \operatorname{dom} A$ such that $0 \in A(x)$. It was first introduced by Martinet [12] in 1970. Rockafellar [16] defined the proximal point algorithm of Martinet by generalizing a sequence $\{x_n\}$ such that

$$x_{n+1} = J_{s_n A} x_n + e_n, \ n \in \mathbb{N},$$

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for arbitrary point $x_0 \in H$, where $\{e_n\}$ is a sequence of errors and $\{s_n\} \subseteq (0,\infty)$. The sequence $\{x_n\}$ is known to converge weakly to a zero of A, if $\liminf_{n\to\infty} s_n > 0$ and $\sum_{n=0}^{\infty} \|e_n\| < \infty$, see [16], but fails in general to converge strongly [6]. Xu [21] investigated a modified version of the initial proximal point algorithm studied by Rockafellar with $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = \beta_n x_0 + (1-\beta_n) J_{s_n A} x_n + e_n, \; n \in \mathbb{N},$$

where $\{e_n\}$ is the errors sequence. For $\{e_n\}$ summable, it was proved that [21] $\{x_n\}$ is strongly convergent if $s_n \to \infty$ and $\{\beta_n\} \subseteq (0,1)$ with $\sum_{n=0}^{\infty} \beta_n = \infty$ and $\lim_{n\to\infty} \beta_n = 0$.

Recently, Marino and Rugiano [9] introduced the following iteration process: for arbitrary chosen $x_0 \in C$ construct a sequence $\{x_n\}$ by

$$x_{n+1} = \beta_n f(x_n) + (1-\beta_n) T(\alpha_n x_n + (1-\alpha_n) x_{n+1}), \; n \in \mathbb{N},$$

where $\alpha_n, \beta_n \in (0,1)$ and f is a k-contraction mapping on H. They showed that this process converges strongly to the unique fixed point of the contraction $P_{\mathrm{Fix}(T)}$.

In 2014, Mongkolkeha, Cho and Kumam [13], defined the following iterative scheme, by $x_0 \in C$ and

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n U x_n, \\ y_n = (1-\beta_n) T x_n + \beta_n S z_n, \\ x_{n+1} = (1-\alpha_n) x_n + \alpha_n y_n, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$. They show that if $\liminf(1-\alpha_n)\alpha_n > 0$, $\liminf(1-\beta_n)\beta_n > 0$ and $\sum_{n\in\mathbb{N}} \gamma_n < \infty$ then $\{x_n\}$ converges weakly to an element of $\mathrm{Fix}(T) \cap \mathrm{Fix}(S)$.

In this paper, we introduce a composite iteration of resolvent average for a finite family of monotone operators as follows:

$$\begin{cases} x_1 \in H, \\ z_n = \gamma_n x_n + (1-\gamma_n) J_{R(\mathbf{A},\lambda)} x_n, \\ y_n = \beta_n x_n + (1-\beta_n) J_{R(\mathbf{A},\lambda)} z_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (\mathrm{Id} - \alpha_n B) y_n + e_n, \end{cases} \tag{1}$$

where B is a strongly monotone linear bounded self-adjoint operator and f is a k-contraction mapping on H. We prove, under certain appropriate assumptions on sequences $\{\alpha_n\}, \{\beta_n\} \subseteq (0,1)$, $\{\gamma_n\} \subseteq [0,1]$ and $\{e_n\}$, that $\{x_n\}$ converges strongly to a zero point of resolvent average of the family.

2. Preliminaries

Let K be a closed convex subset of H. Then for every point $x \in H$, there exists a unique *nearest point* in K, denoted by $P_K(x)$, such that

$$\|x - P_K(x)\| \le \|x - u\|, \; \forall u \in K.$$

The operator P_K is called the *metric projection* of H onto K. It is well known that $P_K(x)$ is nonexpansive. The metric projection $P_K(x)$ is characterized by $P_K(x) \in K$ and

$$\langle u - P_K(x), x - P_K(x) \rangle \le 0, \; \forall u \in K.$$

An operator $T : H \to H$ is said to be *firmly nonexpansive* if

$$\|Tx - Ty\|^2 + \|(\mathrm{Id} - T)x - (\mathrm{Id} - T)y\|^2 \le \|x - y\|^2, \; \forall x, y \in H.$$

A mapping $f : H \to H$ is said to be *k-contraction* on H if there exists a constant $k \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq k\|x - y\|, \ \forall x, y \in H.$$

A sequence of points $\{x_n\}$ in a Hilbert space H is said to *converge weakly* to a point x in H if

$$\langle x_n, y\rangle \to \langle x, y\rangle, \ \forall y \in H;$$

in symbols, $x_n \rightharpoonup x$.

The defining property of the *adjoint* of a bounded operator L on a Hilbert space, denoted by L^*, is that

$$\langle x, Ly\rangle = \langle L^* x, y\rangle, \ \forall x, y \in H.$$

A bounded linear operator $L : H \to H$ on a Hilbert space H is called *self-adjoint* if $L^* = L$.

An operator $B : H \multimap H$ is called *strongly monotone* with constant $\overline{\gamma} > 0$ if

$$\langle Bx - By, x - y\rangle \geq \overline{\gamma}\|x - y\|^2, \ \forall x, y \in H.$$

These basic definitions are also have presented in various parts of the book [2]. Now, we recall some properties of monotone operators.

Proposition 2.1. [2, Proposition 23.7] *Suppose that $A : H \multimap H$ is a set-valued mapping. Then*

(i) if A is monotone, then J_A is single-valued and firmly nonexpansive.
(ii) if A is maximal monotone, then J_A is single-valued and firmly nonexpansive and its domain is all of H.
(iii) $0 \in A(x)$ if and only if $x \in \mathrm{Fix}(J_A)$. Since the fixed point set of nonexpansive operators is closed and convex, the projection onto $Z = A^{-1}(0)$ is well defined whenever $Z \neq \emptyset$ (see [16]).

We recall (see [1]) the definition of the proximal average and resolvent average. To this end, we assume that $m \in \mathbb{N}$ and $I = \{1, 2, \ldots, m\}$. For every $i \in I$, let $A_i : H \multimap H$ be a set-valued mapping and let $\lambda_i > 0$ be such that $\sum_{i\in I} \lambda_i = 1$. We set $\boldsymbol{A} = (A_1, \ldots, A_m)$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$.

Definition 2.2. [1, Definition 1.4] *The $\boldsymbol{\lambda}$-weighted resolvent average of $\boldsymbol{A}$ is defined by*

$$R(\boldsymbol{A}, \boldsymbol{\lambda}) = \Big(\sum_{i\in I} \lambda_i (A_i + \mathrm{Id})^{-1}\Big)^{-1} - \mathrm{Id}. \tag{2}$$

The equation (2) is equivalent to the following equation (see [1]):

$$J_{R(\boldsymbol{A},\boldsymbol{\lambda})} = \sum_{i\in I} \lambda_i J_{A_i}. \tag{3}$$

Proposition 2.3. [1, Theorem 2.5] *Suppose that for each $i \in I$, $A_i : H \multimap H$ is monotone and $x \in H$. If $\bigcap_{i\in I} A_i^{-1}(\{0\}) \neq \emptyset$, then*

$$(R(\boldsymbol{A}, \boldsymbol{\lambda}))^{-1}(\{0\}) = \bigcap_{i\in I} A_i^{-1}(\{0\}).$$

Proposition 2.4. [1, Theorem 2.2] *Suppose that for each $i \in I$, $A_i : H \multimap H$ is a set-valued mapping. Then*

$$(R(\boldsymbol{A}, \boldsymbol{\lambda}))^{-1} = R(\boldsymbol{A}^{-1}, \boldsymbol{\lambda}).$$

Lemma 2.5. [1, Theorem 2.11] *Let $A_i : H \multimap H$ be monotone for each $i \in I$. Then $R(\boldsymbol{A}, \boldsymbol{\lambda})$ is monotone and*

$$\mathrm{dom}\, J_{R(\boldsymbol{A},\boldsymbol{\lambda})} = \bigcap_{i\in I} \mathrm{dom}\, J_{A_i}.$$

3. Main Results

In this section, we introduce a composite iteration for a finite family of monotone operators and its convergence analysis is given. First we present some useful lemmas.

Lemma 3.1. [11, Lemma 2.5] *Assume that B is a strongly monotone linear bounded self-adjoint operator on Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|B\|^{-1}$. Then $\|\mathrm{Id} - \rho B\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 3.2. Suppose that for each $i \in I$, $A_i : H \multimap H$ is a monotone operator. Then $(R(\boldsymbol{A}, \boldsymbol{\lambda}))^{-1}(0) = \mathrm{Fix}(J_{R(\boldsymbol{A},\boldsymbol{\lambda})})$

Lemma 3.3. *Let $\{A_i : H \multimap H\}_{i\in I}$ be a finite family of monotone operators with $(R(\boldsymbol{A}, \boldsymbol{\lambda}))^{-1}(\{0\}) \neq \emptyset$, where $\lambda_i > 0$ and $\sum_{i\in I} \lambda_i = 1$. Let B be a strongly monotone linear bounded self-adjoint operator with coefficient $\overline{\gamma} > 0$. Assume that f is a k-contraction mapping on H and $0 < \gamma < \overline{\gamma}/k$. Let $\{x_n\}$ be the sequence generated by (1). Assume that the following conditions hold:*

(i) $e_n \in H$ *and* $\sum_{n\in\mathbb{N}} \|e_n\| < \infty$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$.

Then $\{\|x_n - z\| : n \in \mathbb{N}\}$ is bounded for each $z \in (R(\boldsymbol{A}, \boldsymbol{\lambda}))^{-1}(\{0\})$. Consequently, $\{x_n\}$ and $\{\|J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n - x_n\| : n \in \mathbb{N}\}$ are bounded.

Proof. By using the Proposition 2.1, Lemma 3.2 and triangle inequality for any $z \in (R(\boldsymbol{A}, \boldsymbol{\lambda}))^{-1}(\{0\})$, we have:

$$\begin{aligned}
\|z_n - z\| &= \|\gamma_n x_n + (1 - \gamma_n) J_{R(\boldsymbol{A},\boldsymbol{\lambda})} x_n - z\| \\
&= \|\gamma_n (x_n - z) + (1 - \gamma_n)(J_{R(\boldsymbol{A},\boldsymbol{\lambda})} x_n - z)\| \\
&\leq \gamma_n \|x_n - z\| + (1 - \gamma_n)\|J_{R(\boldsymbol{A},\boldsymbol{\lambda})} x_n - J_{R(\boldsymbol{A},\boldsymbol{\lambda})} z\| \\
&\leq \gamma_n \|x_n - z\| + (1 - \gamma_n)\|x_n - z\| \\
&\leq \|x_n - z\|.
\end{aligned} \tag{4}$$

By our assumption and (4), we obtain:

$$\begin{aligned}
\|y_n - z\| &= \|\beta_n x_n + (1 - \beta_n) J_{R(\boldsymbol{A},\boldsymbol{\lambda})} z_n - z\| \\
&= \|\beta_n (x_n - z) + (1 - \beta_n)(J_{R(\boldsymbol{A},\boldsymbol{\lambda})} z_n - z)\| \\
&\leq \beta_n \|x_n - z\| + (1 - \beta_n)\|J_{R(\boldsymbol{A},\boldsymbol{\lambda})} z_n - J_{R(\boldsymbol{A},\boldsymbol{\lambda})} z\| \\
&\leq \beta_n \|x_n - z\| + (1 - \beta_n)\|z_n - z\| \\
&\leq \|x_n - z\|.
\end{aligned} \tag{5}$$

By the condition (ii), without loss of generality, we can assume that $\alpha_n < \|B\|^{-1}$ for all $n \in \mathbb{N}$. It follows from Lemma 3.1 that $\|\mathrm{Id} - \alpha_n B\| \leq 1 - \alpha_n \overline{\gamma}$. Hence, from triangle inequality and (5), we have

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\alpha_n \gamma f(x_n) + (\mathrm{Id} - \alpha_n B) y_n + e_n - z\| \\
&= \|\alpha_n \gamma f(x_n) + (\mathrm{Id} - \alpha_n B) y_n + e_n - z + \alpha_n Bz - \alpha_n Bz\| \\
&= \|\alpha_n (\gamma f(x_n) - Bz) + (\mathrm{Id} - \alpha_n B)(y_n - z) + e_n\| \\
&\leq \alpha_n \gamma \|f(x_n) - f(z)\| + \alpha_n \|\gamma f(z) - Bz\| + (1 - \alpha_n \overline{\gamma})\|y_n - z\| + \|e_n\| \\
&\leq \alpha_n k\gamma \|x_n - z\| + \alpha_n \|\gamma f(z) - Bz\| + (1 - \alpha_n \overline{\gamma})\|x_n - z\| + \|e_n\| \\
&\leq \Big(1 - \alpha_n(\overline{\gamma} - k\gamma)\Big)\|x_n - z\| + \alpha_n \|\gamma f(z) - Bz\| + \|e_n\| \\
&\leq \Big(1 - \alpha_n(\overline{\gamma} - k\gamma)\Big)\|x_n - z\| + \alpha_n(\overline{\gamma} - k\gamma)\frac{\|\gamma f(z) - Bz\|}{\overline{\gamma} - k\gamma} + \|e_n\| \\
&\leq \max\left\{\|x_n - z\|, \frac{\|\gamma f(z) - Bz\|}{\overline{\gamma} - k\gamma}\right\} + \|e_n\|.
\end{aligned}$$

This shows by induction that

$$\|x_{n+1} - z\| \leq \max\left\{\|x_1 - z\|, \frac{\|\gamma f(z) - Bz\|}{\overline{\gamma} - k\gamma}\right\} + \sum_{i=1}^{n} \|e_i\|.$$

Therefore, $\{\|x_n - z\| : n \in \mathbb{N}\}$ is bounded for each $z \in (R(\boldsymbol{A}, \boldsymbol{\lambda}))^{-1}(\{0\})$. Hence $\{x_n\}$ is bounded.

Finally, it follows from nonexpansivity of resolvent of $R(\boldsymbol{A}, \boldsymbol{\lambda})$ that

$$\|J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n - x_n\| \leq \|J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n - z\| + \|x_n - z\| \leq 2\|x_n - z\|.$$

Therefore, $\{\|J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n - x_n\| : n \in \mathbb{N}\}$ is bounded. □

Lemma 3.4. [18, Lemma 2.1] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in Banach space X and let $\{\beta_n\}$ be a sequence in $(0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and*

$$\limsup_{n\to\infty} \left(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\right) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 3.5. *Let $\{A_i : H \multimap H\}_{i\in I}$ be a finite family of monotone operators with $(R(\boldsymbol{A}, \boldsymbol{\lambda}))^{-1}(\{0\}) \neq \emptyset$, where $\lambda_i > 0$ and $\sum_{i\in I} \lambda_i = 1$. Let B be a strongly monotone linear bounded self-adjoint operator with coefficient $\overline{\gamma} > 0$. Assume that f is a k-contraction mapping on H and $0 < \gamma < \overline{\gamma}/k$. Let $\{x_n\}$ be the sequence generated by* (1). *Assume that the following conditions hold for all $n \in \mathbb{N}$:*

- *(i)* $e_n \in H$ *and* $\sum_{n\in\mathbb{N}} \|e_n\| < \infty$,
- *(ii)* $\lim_{n\to\infty} \alpha_n = 0$,
- *(iii)* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
- *(iv)* $\lim_{n\to\infty} \left|\gamma_{n+1} - \gamma_n\right| = 0$,
- *(v)* $\gamma_n - \beta_n > \epsilon$, *for some* $\epsilon \in (0, 1)$.

Then $\lim_{n\to\infty} \|x_n - J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n\| = 0$.

Proof. It follows from Lemma 3.3 that $\{x_n\}$ and $\{\|J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n - x_n\| : n \in \mathbb{N}\}$ are bounded. First, we claim that

$$\|x_{n+1} - x_n\| \to 0. \tag{6}$$

We observe from (1) that

$$\begin{cases} z_{n+1} = \gamma_{n+1}x_{n+1} + (1 - \gamma_{n+1})J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_{n+1}, \\ z_n = \gamma_n x_n + (1 - \gamma_n)J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n, \end{cases}$$

Then

$$z_{n+1} - z_n = (1 - \gamma_{n+1})(J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_{n+1} - J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n) + \gamma_{n+1}(x_{n+1} - x_n) + (\gamma_n - \gamma_{n+1})(J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n - x_n).$$

We obtain

$$\begin{aligned} \|z_{n+1} - z_n\| &\leq (1 - \gamma_{n+1})\|J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_{n+1} - J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n\| + \gamma_{n+1}\|x_{n+1} - x_n\| \\ &\quad + \left|\gamma_n - \gamma_{n+1}\right|\|J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n - x_n\| \\ &\leq \|x_{n+1} - x_n\| + \left|\gamma_n - \gamma_{n+1}\right|M, \end{aligned} \tag{7}$$

where $M := \sup\{\|J_{R(A,\lambda)}x_n - x_n\| : n \in \mathbb{N}\}$. Put

$$h_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n}.$$

Then

$$x_{n+1} = (1 - \beta_n)h_n + \beta_n x_n, \; n \in \mathbb{N}. \tag{8}$$

By using our assumption and (8), we have

$$\begin{aligned}
h_{n+1} - h_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \\
&= \frac{\alpha_{n+1}\gamma f(x_{n+1}) + (\mathrm{Id} - \alpha_{n+1}B)y_{n+1} + e_{n+1} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} \\
&\quad - \frac{\alpha_n\gamma f(x_n) + (\mathrm{Id} - \alpha_n B)y_n + e_n - \beta_n x_n}{1 - \beta_n} \\
&= \frac{\alpha_{n+1}\Big(\gamma f(x_{n+1}) - By_{n+1}\Big)}{1 - \beta_{n+1}} - \frac{\alpha_n\Big(\gamma f(x_n) - By_n\Big)}{1 - \beta_n} \\
&\quad + \frac{y_{n+1} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} - \frac{y_n - \beta_n x_n}{1 - \beta_n} + \frac{e_{n+1}}{1 - \beta_{n+1}} - \frac{e_n}{1 - \beta_n} \\
&= \frac{\alpha_{n+1}\Big(\gamma f(x_{n+1}) - By_{n+1}\Big)}{1 - \beta_{n+1}} - \frac{\alpha_n\Big(\gamma f(x_n) - By_n\Big)}{1 - \beta_n} \\
&\quad + J_{R(\mathbf{A},\lambda)}z_{n+1} - J_{R(\mathbf{A},\lambda)}z_n + \frac{e_{n+1}}{1 - \beta_{n+1}} - \frac{e_n}{1 - \beta_n}
\end{aligned}$$

Hence,

$$\begin{aligned}
\|h_{n+1} - h_n\| &\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|\gamma f(x_{n+1}) - By_{n+1}\| + \frac{\alpha_n}{1 - \beta_n}\|\gamma f(x_n) - By_n\| \\
&\quad + \|z_{n+1} - z_n\| + \frac{\|e_{n+1}\|}{1 - \beta_{n+1}} + \frac{\|e_n\|}{1 - \beta_n}.
\end{aligned} \tag{9}$$

Now, substitute (7) into (9) yields:

$$\begin{aligned}
\|h_{n+1} - h_n\| &\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|\gamma f(x_{n+1}) - By_{n+1}\| + \frac{\alpha_n}{1 - \beta_n}\|\gamma f(x_n) - By_n\| \\
&\quad + \|x_{n+1} - x_n\| + \Big|\gamma_{n+1} - \gamma_n\Big|M + \frac{\|e_{n+1}\|}{1 - \beta_{n+1}} + \frac{\|e_n\|}{1 - \beta_n}.
\end{aligned}$$

Then

$$\begin{aligned}
\|h_{n+1} - h_n\| - \|x_{n+1} - x_n\| &\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|\gamma f(x_{n+1}) - By_{n+1}\| + \frac{\alpha_n}{1 - \beta_n}\|\gamma f(x_n) - By_n\| \\
&\quad + \Big|\gamma_{n+1} - \gamma_n\Big|M + \frac{\|e_{n+1}\|}{1 - \beta_{n+1}} + \frac{\|e_n\|}{1 - \beta_n}.
\end{aligned}$$

By conditions (i)-(iv), we get:

$$\limsup_{n\to\infty}\Big(\|h_{n+1} - h_n\| - \|x_{n+1} - x_n\|\Big) \leq 0.$$

It follows from Lemma 3.4 that

$$\lim_{n\to\infty} \|h_n - x_n\| = 0. \tag{10}$$

From (8), we have:

$$x_{n+1} - x_n = (1-\beta_n)(h_n - x_n),$$

so (10) yields that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, i.e., (6) holds.

By assumption, we have $x_{n+1} - y_n = \alpha_n(\gamma f(x_n) - By_n) + e_n$. Therefore

$$\lim_{n\to\infty} (x_{n+1} - y_n) = 0. \tag{11}$$

Observing

$$\|y_n - x_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|,$$

using (6) and (11), we get $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

On the other hand, by assumption and nonexpansivity of resolvent of $R(\boldsymbol{A}, \boldsymbol{\lambda})$, we have

$$\begin{aligned}\|J_{R(A,\lambda)}x_n - x_n\| &\le \|x_n - y_n\| + \|y_n - J_{R(A,\lambda)}x_n\|\\ &\le \|x_n - y_n\| + \beta_n\|x_n - J_{R(A,\lambda)}x_n\| + \|J_{R(A,\lambda)}x_n - J_{R(A,\lambda)}z_n\|\\ &\le \|x_n - y_n\| + \beta_n\|x_n - J_{R(A,\lambda)}x_n\| + \|x_n - z_n\|\\ &\le \|x_n - y_n\| + \beta_n\|x_n - J_{R(A,\lambda)}x_n\| + (1-\gamma_n)\|x_n - J_{R(A,\lambda)}x_n\|,\end{aligned}$$

which implies $(\gamma_n - \beta_n)\|J_{R(A,\lambda)}x_n - x_n\| \le \|x_n - y_n\|$.

So, by condition (v), we obtain $\lim_{n\to\infty} \|J_{R(A,\lambda)}x_n - x_n\| = 0$. □

Lemma 3.6. [14] *There holds the following inequality:*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \ \forall x, y \in H.$$

Lemma 3.7. [5, Lemma 2.2] *For each $x_i \in H$, $a_i \in [0,1]$, $i = 1, 2$ with $\sum_{i=1}^{2} a_i = 1$, we have $\|a_1x_1 + a_2x_2\|^2 \le a_1\|x_1\|^2 + a_2\|x_2\|^2$.*

Lemma 3.8. *Let $x \in H$ and $\{\alpha_n\}$ be a sequence in H such that $\|\alpha_n\| \to 0$. Then there exists a constant $L > 0$ such that $\|x + \alpha_n\|^2 \le \|x\|^2 + L\|\alpha_n\|$.*

Proof. By Cauchy-Schwarz inequality and for $L \ge 2\|x\| + \sup_{n\in\mathbb{N}} \|\alpha_n\|$, we have:

$$\begin{aligned}\|x+\alpha_n\|^2 &= \|x\|^2 + 2\langle x, \alpha_n\rangle + \|\alpha_n\|^2\\ &\le \|x\|^2 + 2\|x\|\|\alpha_n\| + \|\alpha_n\|^2\\ &\le \|x\|^2 + \|\alpha_n\|(2\|x\| + \|\alpha_n\|)\\ &\le \|x\|^2 + L\|\alpha_n\|.\end{aligned}$$

We are done. □

Lemma 3.9. [21, Lemma 2.5] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n\delta_n + \beta_n, \ n \ge 0,$$

where $\{\gamma_n\}$,$\{\beta_n\}$ *and* $\{\delta_n\}$ *satisfy the conditions:*

(i) $\gamma_n \subseteq [0,1], \sum_{n=1}^{\infty} \gamma_n = \infty$,

(ii) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=1}^{\infty} |\gamma_n \delta_n| < \infty$,

(iii) $\beta_n \ge 0$ *for all* $n \ge 0$ *with* $\sum_{n=0}^{\infty} \beta_n < \infty$,

Then $\lim_{n\to\infty} a_n = 0$.

Theorem 3.10. *Let* $\{A_i : H \multimap H\}_{i\in I}$ *be a finite family of monotone operators with* $Z = (R(\boldsymbol{A}, \boldsymbol{\lambda}))^{-1}(\{0\}) \neq \emptyset$*, where* $\lambda_i > 0$ *and* $\sum_{i\in I} \lambda_i = 1$*. Let B be a strongly monotone linear bounded self-adjoint operator with coefficient* $\overline{\gamma} = \|B\| > 0$*. Assume that f is a k-contraction mapping on H and* $0 < \gamma < \overline{\gamma}/k$*. Let* $\{x_n\}$ *be the sequence generated by* (1)*. Assume that the following conditions hold for all* $n \in \mathbb{N}$*:*

(i) $e_n \in H$ *and* $\sum_{n\in\mathbb{N}} \|e_n\| < \infty$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n\in\mathbb{N}} \alpha_n = \infty$,

(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$,

(iv) $\lim_{n\to\infty} \left|\gamma_{n+1} - \gamma_n\right| = 0$,

(v) $\gamma_n - \beta_n > \epsilon$*, for some* $\epsilon \in (0,1)$.

Then $\{x_n\}$ *converges strongly to* $z = P_Z(\gamma f + (\mathrm{Id} - B))(z)$.

Proof. First, we show that there exists a unique $z \in Z$ such that $z = P_Z(\gamma f + (\mathrm{Id} - B))(z)$. Since, $(R(\boldsymbol{A}, \boldsymbol{\lambda}))^{-1}(\{0\})$ is nonempty, closed and convex, the projection P_Z is well defined. Since P_Z is nonexpansive and f is k-contraction, for each $x, y \in H$, we get:

$$\begin{aligned}
&\|P_Z(\gamma f + (\mathrm{Id} - B))(x) - P_Z(\gamma f + (\mathrm{Id} - B))(y)\| \\
&\qquad \le \|(\gamma f + (\mathrm{Id} - B))(x) - (\gamma f + (\mathrm{Id} - B))(y)\| \\
&\qquad \le \|\gamma f(x) - \gamma f(y)\| + \|\mathrm{Id} - B\|\|x - y\| \\
&\qquad \le \gamma k\|x - y\| + (1 - \overline{\gamma})\|x - y\| \\
&\qquad \le (1 - (\overline{\gamma} - \gamma k))\|x - y\|.
\end{aligned}$$

Banach's Contraction Principle guaranties that $P_Z(\gamma f + (\mathrm{Id} - B))$ has a unique fixed point. That is, there exists a unique element $z \in Z$ such that $z = P_Z(\gamma f + (\mathrm{Id} - B))(z)$. Now, consider the mapping $x \to t\gamma f(x) + (\mathrm{Id} - tB)J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x$.

For each $t \in (0,1)$, let ϕ_t on H be defined by

$$\phi_t(x) = t\gamma f(x) + (\mathrm{Id} - tB)J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x.$$

For every $x, y \in H$ and $t \in (0,1)$, we have:

$$\begin{aligned}
\|\phi_t(x) - \phi_t(y)\| &= \|(t\gamma f(x) + (\mathrm{Id} - tB)J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x) - (t\gamma f(y) + (\mathrm{Id} - tB)J_{R(\boldsymbol{A},\boldsymbol{\lambda})}y)\| \\
&\le t\gamma\|f(x) - f(y)\| + \|\mathrm{Id} - tB\|\|J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x - J_{R(\boldsymbol{A},\boldsymbol{\lambda})}y\| \\
&\le t\gamma k\|x - y\| + (1 - t\overline{\gamma})\|x - y\| \\
&\le (1 - t(\overline{\gamma} - \gamma k))\|x - y\|.
\end{aligned}$$

Then ϕ_t is contraction. Next, we show that

$$\limsup_{n\to\infty} \langle \gamma f(z) - Bz, x_n - z\rangle \le 0, \tag{12}$$

where $z = \lim_{t\to 0} x_t$ with x_t being fixed point of the contraction $x \mapsto t\gamma f(x) + (\mathrm{Id} - tB)J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x$. Since x_t solves the fixed point equation,

$$x_t = t\gamma f(x_t) + (\mathrm{Id} - tB)J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_t$$

By using Lemma 3.6 and Lemma 3.1, we obtain:

$$\begin{aligned}
\|x_t - x_n\|^2 &= \|(\mathrm{Id} - tB)(J_{R(A,\Lambda)}x_t - x_n) + t(\gamma f(x_t) - Bx_n)\|^2 \\
&\le (1-\overline{\gamma}t)^2\|J_{R(A,\Lambda)}x_t - x_n\|^2 + 2t\left\langle \gamma f(x_t) - Bx_n, x_t - x_n\right\rangle \\
&\le (1-\overline{\gamma}t)^2\Big(\|J_{R(A,\Lambda)}x_t - J_{R(A,\Lambda)}x_n\|^2 + \|J_{R(A,\Lambda)}x_n - x_n\|^2 \\
&\quad + 2\left\langle J_{R(A,\Lambda)}x_t - J_{R(A,\Lambda)}x_n, J_{R(A,\Lambda)}x_n - x_n\right\rangle\Big) + 2t\left\langle \gamma f(x_t) - Bx_n, x_t - x_n\right\rangle \\
&\le (1-\overline{\gamma}t)^2\Big(\|x_t - x_n\|^2 + \|J_{R(A,\Lambda)}x_n - x_n\|^2 + 2\|J_{R(A,\Lambda)}x_n - x_n\|\|x_n - x_t\|\Big) \\
&\quad + 2t\left\langle \gamma f(x_t) - Bx_t, x_t - x_n\right\rangle + 2t\left\langle Bx_t - Bx_n, x_t - x_n\right\rangle \\
&\le (1-\overline{\gamma}t)^2\Big(\|x_t - x_n\|^2 + \|J_{R(A,\Lambda)}x_n - x_n\|^2 + 2\|J_{R(A,\Lambda)}x_n - x_n\|\|x_n - x_t\|\Big) \\
&\quad + 2t\left\langle \gamma f(x_t) - Bx_t, x_t - x_n\right\rangle + 2t\|B\|\|x_n - x_t\|^2.
\end{aligned}$$

Therefore,

$$\left\langle Bx_t - \gamma f(x_t), x_t - x_n\right\rangle \le \frac{\overline{\gamma}^2 t}{2}\|x_t - x_n\|^2 + \frac{(1-\overline{\gamma}t)^2}{2t}\Big(\|J_{R(A,\Lambda)}x_n - x_n\|^2 + 2\|J_{R(A,\Lambda)}x_n - x_n\|\|x_n - x_t\|\Big).$$

By letting $n \to \infty$, we have:

$$\limsup_{n\to\infty}\left\langle Bx_t - \gamma f(x_t), x_t - x_n\right\rangle \le \limsup_{n\to\infty}\frac{\overline{\gamma}^2 t}{2}\|x_t - x_n\|^2.$$

Now, taking $t \to 0$, we obtain (12). Now, from assumption, Lemma 3.6 and Lemma 3.8, for some appropriate constant $L > 0$, we have:

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n\gamma f(x_n) + (\mathrm{Id} - \alpha_n B)y_n + e_n - z\|^2 \\
&= \|\alpha_n\gamma f(x_n) + (\mathrm{Id} - \alpha_n B)y_n + \alpha_n Bz - \alpha_n Bz + e_n - z\|^2 \\
&= \|(\mathrm{Id} - \alpha_n B)(y_n - z) + \alpha_n(\gamma f(x_n) - Bz) + e_n\|^2 \\
&\le \|(\mathrm{Id} - \alpha_n B)(y_n - z) + e_n\|^2 + 2\alpha_n\left\langle \gamma f(x_n) - Bz, x_{n+1} - z\right\rangle \\
&\le \|(\mathrm{Id} - \alpha_n B)(y_n - z)\|^2 + 2\alpha_n\left\langle \gamma f(x_n) - Bz, x_{n+1} - z\right\rangle + L\|e_n\| \\
&\le (1-\alpha_n\overline{\gamma})^2\|y_n - z\|^2 + 2\alpha_n\left\langle \gamma f(x_n) - Bz, x_{n+1} - z\right\rangle + L\|e_n\| \\
&= (1-\alpha_n\overline{\gamma})^2\|y_n - z\|^2 + 2\alpha_n\gamma\left\langle f(x_n) - f(z), x_{n+1} - z\right\rangle \\
&\quad + 2\alpha_n\left\langle \gamma f(z) - Bz, x_{n+1} - z\right\rangle + L\|e_n\| \\
&\le (1-\alpha_n\overline{\gamma})^2\|x_n - z\|^2 + 2\alpha_n\gamma k\|x_n - z\|\|x_{n+1} - z\| \\
&\quad + 2\alpha_n\left\langle \gamma f(z) - Bz, x_{n+1} - z\right\rangle + L\|e_n\| \\
&\le (1-\alpha_n\overline{\gamma})^2\|x_n - z\|^2 + \alpha_n\gamma k(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) \\
&\quad + 2\alpha_n\left\langle \gamma f(z) - Bz, x_{n+1} - z\right\rangle + L\|e_n\|,
\end{aligned}$$

This implies that

$$\|x_{n+1} - z\|^2 \le \frac{(1-\alpha_n\overline{\gamma})^2 + \alpha_n\gamma k}{1-\alpha_n\gamma k}\|x_n - z\|^2 + \frac{2\alpha_n}{1-\alpha_n\gamma k}\left\langle \gamma f(z) - Bz, x_{n+1} - z\right\rangle + \frac{L}{1-\alpha_n\gamma k}\|e_n\|$$

$$= \frac{(1-2\alpha_n\overline{\gamma}+\alpha_n\gamma k)}{1-\alpha_n\gamma k}\|x_n-z\|^2+\frac{\alpha_n^2\overline{\gamma}^2}{1-\alpha_n\gamma k}\|x_n-z\|^2$$
$$+\frac{2\alpha_n}{1-\alpha_n\gamma k}\langle \gamma f(z)-Bz, x_{n+1}-z\rangle+\frac{L}{1-\alpha_n\gamma k}\|e_n\|$$
$$\leq \Big(1-\frac{2\alpha_n(\overline{\gamma}-k\gamma)}{1-\alpha_n\gamma k}\Big)\|x_n-z\|^2$$
$$+\frac{2\alpha_n(\overline{\gamma}-k\gamma)}{1-\alpha_n\gamma k}\Big(\frac{1}{\overline{\gamma}-\gamma k}\langle \gamma f(z)-Bz, x_{n+1}-z\rangle+\frac{\alpha_n\overline{\gamma}^2}{2(\overline{\gamma}-\gamma k)}M\Big)+\frac{L}{1-\alpha_n\gamma k}\|e_n\|$$
$$\leq (1-\delta_n)\|x_n-z\|^2+\delta_n t_n+\eta_n,$$

where $M=\sup\{\|x_n-z\|^2 : n\in\mathbb{N}\}$, $\delta_n=\frac{2\alpha_n(\overline{\gamma}-k\gamma)}{1-\alpha_n\gamma k}$, $\eta_n=\frac{L}{1-\alpha_n\gamma k}\|e_n\|$ and $t_n=\frac{1}{\overline{\gamma}-\gamma k}\langle \gamma f(z)-Bz, x_{n+1}-z\rangle+\frac{\alpha_n\overline{\gamma}^2}{2(\overline{\gamma}-\gamma k)}M$. By assumption, $\lim_{n\to\infty}\delta_n=0$, $\sum_{n\in\mathbb{N}}\delta_n=\infty$, $\limsup_{n\to\infty}t_n\leq 0$ and $\sum_{n\in\mathbb{N}}\eta_n<\infty$. Hence, applying Lemma 3.9, we immediately deduce that $x_n\to z$ where $z=P_Z(\gamma f+(\mathrm{Id}-B))(z)$. □

Algorithm 1 Iterative algorithms for resolvent average

Input: $x_1\in H$, $\{\alpha_n\},\{\beta_n\}\subset(0,1)$, $\{\gamma_n\}\subset[0,1]$, $\{\lambda_i\}_{1\leq i\leq m}\subset(0,1)$, $\{e_n\}\in H$, $\mathbf{A}=(A_1,\ldots,A_m)$
Output: x_n
  for $i=1$ to m **do**
    $J_{A_i}(x_n):=(A_i+\mathrm{Id})^{-1}(x_n)$;
  end for
  Set $J_{R(\mathbf{A},\lambda)}(x_n):=\sum_{i=1}^{m}\lambda_i J_{A_i}(x_n)$;
  for $n=1$ to $\ldots$ **do**
    $z_n=\gamma_n x_n+(1-\gamma_n)J_{R(\mathbf{A},\lambda)}x_n$;
    $y_n=\beta_n x_n+(1-\beta_n)J_{R(\mathbf{A},\lambda)}z_n$;
    $x_{n+1}=\alpha_n\gamma f(x_n)+(\mathrm{Id}-\alpha_n B)y_n+e_n$;
  end for

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4. Numerical Examples

In this section, we evaluated strongly convergence of three step algorithm for weighted resolvent average of a finite family of monotone operators.

Example 4.1. *Let $A_1(x)=2x-1$ $A_2(x)=x$ and $A_3(x)=x+2$. Set $\boldsymbol{A}=(2x-1,x,x+2)$, $f(x)=\frac{x}{2}$ and for every $1\leq i\leq 3$, $\lambda_i=\frac{1}{3}$. Assume that $e_n=\left\{\frac{1}{n^n}\right\}$ is the sequence of errors and let $\alpha_n=\left\{\frac{1}{n^2}\right\}$, $\beta_n=\left\{\frac{1}{n+3}+\frac{1}{10}\right\}$ and $\gamma_n=\left\{\frac{1}{n+5}+\frac{4}{5}\right\}$. Let $B=\mathrm{Id}$ and $\gamma=1$. First note that $A_1^{-1}(x)=\frac{1}{2}(x+1)$, $A_2^{-1}(x)=x$ and $A_3^{-1}(x)=x-2$. So, $\boldsymbol{A}^{-1}=\left(\frac{1}{2}(x+1),x,x-2\right)$. Then by easy calculation, we get:*

$$J_{A_1^{-1}}(x_n)=(A_1^{-1}+\mathrm{Id})^{-1}(x_n)=\Big\{\frac{1}{3}(2x_n-1)\Big\}, J_{A_2^{-1}}(x_n)=\Big\{\frac{1}{2}x_n\Big\}, \text{ and } J_{A_3^{-1}}(x_n)=\Big\{\frac{1}{2}(x_n+2)\Big\}. \quad (13)$$

By using Proposition 2.4 and (13), *we obtain:*

$$\begin{aligned}(R(\boldsymbol{A},\boldsymbol{\lambda}))^{-1}(\{0\}) &= (R(\boldsymbol{A}^{-1},\boldsymbol{\lambda}))(\{0\}) = \Big(\Big(\sum_{i=1}^{m}\lambda_i J_{A_i^{-1}}\Big)^{-1} - \mathrm{Id}\Big)(\{0\})\\ &= \Big(\Big(\frac{1}{3}(A_1^{-1}+\mathrm{Id})^{-1} + \frac{1}{3}(A_2^{-1}+\mathrm{Id})^{-1} + \Big(\frac{1}{3}(A_3^{-1}+\mathrm{Id})^{-1}\Big)^{-1} - \mathrm{Id}\Big)(\{0\})\\ &= \Big\{x\in\mathbb{R} : 0 \in \Big(\frac{1}{3}(A_1^{-1}+\mathrm{Id})^{-1}(x) + \frac{1}{3}(A_2^{-1}+\mathrm{Id})^{-1}(x) + \frac{1}{3}(A_3^{-1}+\mathrm{Id})^{-1}(x)\Big)\Big\}\\ &= \Big\{x\in\mathbb{R} : 0 \in \Big(\frac{x}{6} + \frac{x+2}{6} + \frac{2x-1}{9}\Big)\Big\} = \{-0.4\}.\end{aligned}$$

Therefore, $Z = (R(\boldsymbol{A},\boldsymbol{\lambda}))^{-1}(\{0\}) = \Big\{-0.4\Big\}$. *Hence, we have:*

$$P_Z(f(z)) = P_{\{-0.4\}}(f(-0.4)) = P_{\{-0.4\}}(-0.2) = -0.4$$

Let $\{x_n\}$ *be the sequence generated by* (1) *for starting point* $x_1 \in \mathbb{R}$. *Clearly,*

$$J_{A_1}(x_n) = (A_1+\mathrm{Id})^{-1}(x_n) = \{y\in\mathbb{R} : x_n \in (A_1+\mathrm{Id})(y)\} = \Big\{\frac{1}{3}\Big(x_n+1\Big)\Big\}. \tag{14}$$

and similarly,

$$J_{A_2}(x_n) = \Big\{\frac{1}{2}x_n\Big\},\ \ J_{A_3}(x_n) = \Big\{\frac{1}{2}(x_n-2)\Big\}. \tag{15}$$

Substituting (14) *and* (15) *into* (3), *we obtain:*

$$J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n = \sum_{i=1}^{m}\lambda_i J_{A_i}x_n = \Big\{\frac{1}{9}(x_n+1) + \frac{1}{6}x_n + \frac{1}{6}(x_n-2)\Big\} = \Big\{\frac{1}{9}(4x_n-2)\Big\}.$$

Therefore,

$$\begin{cases} z_n = \Big(\frac{1}{n+5} + \frac{4}{5}\Big)x_n + 1/9\Big(\frac{1}{5} - \frac{1}{n+5}\Big)(4x_n-2),\\ y_n = \Big(\frac{1}{n+3} + \frac{1}{10}\Big)x_n + 1/9\Big(\frac{9}{10} - \frac{1}{n+3}\Big)(4z_n-2),\\ x_{n+1} = \frac{1}{2n^2}x_n + \Big(1-\frac{1}{n^2}\Big)y_n + \frac{1}{n^n}.\end{cases}$$

It follows from Theorem 3.10 that x_n *converges, say to* x. *Since* x_n *is convergent, by letting* $n\to\infty$ *in the above eqalities we obtain:*

$$\begin{cases} z = \frac{4}{5}x + \frac{1}{45}(4x-2),\\ y = \frac{1}{10}x + \frac{1}{10}(4z-2),\\ x = y.\end{cases}$$

Then, $x = -0.4$. *The numerical results with starting point* $x_1 = 0$, *which are shown in Table 1, shows that* $x_n \to -0.4$.

Table 1: Results for given starting point $x_1 = 0$ in Example 4.1

n	1	10	20	50	100	200	500	1000	···
x_n	0	-0.376042	-0.398733	-0.399836	-0.399961	-0.399990	-0.399998	-0.399999	···

Example 4.2. *Let* $\boldsymbol{A} = (x^3-1, x-1, (x+1)^3)$, $f(x) = \frac{4x}{5}$ *and* $\lambda_i = \frac{1}{3}$ *for every* $1\le i\le 3$. *Let* $\{e_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, B

and γ be the same as in Example 4.1. We have $\boldsymbol{A}^{-1} = \left((1+x)^{\frac{1}{3}}, 1+x, 1+x^{\frac{1}{3}}\right)$. Then

$$J_{A_1^{-1}}(x_n) = \left\{x_n + \frac{\left(\frac{2}{3}\right)^{\frac{1}{3}}}{h_1(x_n)} - \frac{h_1(x_n)}{(2)^{\frac{1}{3}}(3)^{\frac{2}{3}}}\right\},\ J_{A_2^{-1}}(x_n) = \left\{\frac{1}{2}(x_n - 1)\right\},\ J_{A_3^{-1}}(x_n) = \left\{x_n - \frac{\left(\frac{2}{3}\right)^{\frac{1}{3}}}{h_2(x_n)} + \frac{h_2(x_n)}{(2)^{\frac{1}{3}}(3)^{\frac{2}{3}}} - 1\right\}, \quad (16)$$

where $h_1(x_n) = \left(9 + 9x_n - \sqrt{3}\sqrt{31 + 54x_n + 27x_n^2}\right)^{\frac{1}{3}}$ and $h_2(x_n) = \left(9 - 9x_n + \sqrt{3}\sqrt{31 - 54x_n + 27x_n^2}\right)^{\frac{1}{3}}$. By using Proposition 2.4 and (16)*, we obtain $(R(\boldsymbol{A},\boldsymbol{\lambda}))^{-1}(\{0\}) = \{1\}$. Therefore, $Z = (R(\boldsymbol{A},\boldsymbol{\lambda}))^{-1}(\{0\}) = \{1\}$. Hence, we have:*

$$P_Z(f(z)) = P_{\{1\}}(f(1)) = P_{\{1\}}\left(\frac{4}{5}\right) = 1.$$

Let $\{x_n\}$ be the sequence generated by (1) *with starting point $x_1 \in \mathbb{R}$. We have:*

$$J_{A_1}(x_n) = \left\{\frac{h_1(x_n)}{(2)^{\frac{1}{3}}(3)^{\frac{2}{3}}} - \frac{(\frac{2}{3})^{\frac{1}{3}}}{h_1(x_n)}\right\},\ J_{A_2}(x_n) = \left\{\frac{1}{2}\left(1 + x_n\right)\right\},\ J_{A_3}(x_n) = \left\{\frac{(\frac{2}{3})^{\frac{1}{3}}}{h_2(x_n)} - \frac{h_2(x_n)}{(2)^{\frac{1}{3}}(3)^{\frac{2}{3}}} + 1\right\}. \quad (17)$$

Substituting (17) *into* (3)*, we obtain:*

$$J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n = \sum_{i=1}^{m} \lambda_i J_{A_i} x_n = \frac{1}{3}\left(\frac{3}{2} + \frac{1}{2}x_n + \frac{(\frac{2}{3})^{\frac{1}{3}}}{h_2(x_n)} - \frac{h_2(x_n)}{(2)^{\frac{1}{3}}(3)^{\frac{2}{3}}} + \frac{h_1(x_n)}{(2)^{\frac{1}{3}}(3)^{\frac{2}{3}}} - \frac{(\frac{2}{3})^{\frac{1}{3}}}{h_1(x_n)}\right). \quad (18)$$

Therefore,

$$\begin{cases} z_n = \left(\frac{1}{n+5} + \frac{4}{5}\right)x_n + \left(\frac{1}{5} - \frac{1}{n+5}\right)J_{R(\boldsymbol{A},\boldsymbol{\lambda})}x_n, \\ y_n = \left(\frac{1}{n+3} + \frac{1}{10}\right)x_n + \left(\frac{9}{10} - \frac{1}{n+3}\right)J_{R(\boldsymbol{A},\boldsymbol{\lambda})}z_n, \\ x_{n+1} = \frac{4}{5n^2}x_n + (1 - \frac{1}{n^2})y_n + \frac{1}{n^n}. \end{cases} \quad (19)$$

The numerical results which are shown in Table 2 shows that $x_n \to 1$.

Table 2: Results for given starting point $x_1 = 0$ in Example 4.2

n	1	10	20	50	100	200	500	1000	⋯
x_n	0	1.001696	0.998254	0.999780	0.999948	0.999987	0.999998	0.999999	⋯

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