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A New Convergence Inducing the SI-Topology

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Abstract. In their attempt to develop domain theory *in situ* T_0 spaces, Zhao and Ho introduced a new topology defined by irreducible sets of a resident topological space, called the SI-topology. Notably, the SI-topology of the Alexandroff topology of posets is exactly the Scott topology, and so the SI-topology can be seen as a generalisation of the Scott topology in the context of general T_0 spaces. It is well known that the convergence structure that induces the Scott topology is the Scott-convergence – also known as lim-inf convergence by some authors. Till now, it is not known which convergence structure induces the SI-topology of a given T_0 space. In this paper, we fill in this gap in the literature by providing a convergence structure, called the SI-convergence structure, that induces the SI-topology. Additionally, we introduce the notion of I-continuity that is closely related to the SI-convergence structure, but distinct from the existing notion of SI-continuity (introduced by Zhao and Ho earlier). For SI-continuity, we obtain here some equivalent conditions for it. Finally, we give some examples of non-Alexandroff SI-continuous spaces.

1. Introduction

Zhao and Ho defined SI-topology on T_0 spaces in an attempt to generalise the Scott topology on posets ([9, Definition 3.1]). The working principle which uses irreducible sets as the topological counterparts of directed sets is now called *Zhao-Ho replacement principle* in [1]. A subset U of a T_0 space X is SI-*open* if and only if (i) U is open in X and (ii) if F is irreducible in X then $\bigvee F \in U$ implies $F \cap U \neq \emptyset$ whenever $\bigvee F$ exists [9]. By observing the fact that Alexandroff-irreducible sets are exactly the directed sets, SI-topology appears to be a proper generalisation of the Scott topology.

It is well known that the Scott topology can be induced by a certain convergence structure which is defined via directed sets and order (see, e.g., [5]). In other words, given a poset P endowed with the Alexandroff topology, there is a convergence structure defined in it which induces the Scott topology. Then a natural question arises can the SI-topology be induced by some convergence structure defined in the underlying topological space? In this paper, we give a positive answer to this question. Moreover, we characterise those T_0 spaces in which the convergence structure inducing the SI-topology is topological.

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Besides generalising the Scott topology, Zhao and Ho also introduced the notion of SI-continuous spaces, as a topological parallel of continuous posets. However, their definition of an SI-continuous space makes use of directed sets – which is not consistent with the Zhao-Ho replacement principle. We shall remedy this unsatisfactory point by proving that the directed condition in the definition of SI-topology can be replaced interchangeably by irreducible condition.

It is true that if X is an SI-continuous space, then the SI-topology on X is just the Scott topology of some continuous poset (see [1, Remark 4.6]). However, this fact does not imply that the underlying topology of an SI-continuous poset is the Alexandroff topology of a certain poset. At the last part of this paper , we shall define a new topology on posets and deploy it to provide examples of an SI-continuous spaces which are not Alexandroff spaces.

2. Preliminaries

Throughout this paper, given a topological space X and $x \in X$, we denote by $\mathcal{O}(X)$ the topology on X and by $\mathcal{N}(x)$ the collection of all open sets containing x. Any order-theoretical notion on a T_0 space X refers to its specialisation order $\leq_{\mathcal{O}(X)}$ (or simply $\leq$). A nonempty subset F of a topological space X is *irreducible* if for every $U_1,\ U_2 \in \mathcal{O}(X)$, $F \cap U_1 \neq \emptyset$ and $F \cap U_2 \neq \emptyset$ imply $F \cap U_1 \cap U_2 \neq \emptyset$. The collection of all irreducible subsets of X is denoted by $\mathrm{Irr}(X)$. A topological space is an *Alexandroff space* if any intersection of open sets is again open, or equivalently, any point has a smallest open set containing it [3].

Let P be a poset. A subset of P is *Alexandroff open* if it is an upper set. An Alexandroff open set U is *Scott open* if it is inaccessible by directed suprema, i.e., for every directed set D whose supremum exists, if the supremum is in U then D meets U. The collection of all Alexandroff open (resp. Scott open) subsets of P forms a topology on P, called the *Alexandroff topology* (resp. the *Scott topology*) and denoted by $\alpha(P)$ (resp. $\sigma(P)$). The following property can be deduced easily.

Proposition 2.1. *A T_0 space X is an Alexandroff space if and only if $\mathcal{O}(X)$ is precisely the Alexandroff topology on the specialisation poset induced by X.*

A *convergence structure* in a set X is a class $\mathcal{C}$ of tuples $((x_i)_{i\in I}, x)$ where $(x_i)_{i\in I}$ is a net whose terms are elements of X and $x \in X$. A topology induced by a convergence class $\mathcal{C}$ in X is the collection $\tau_{\mathcal{C}}$ of all subsets U of X satisfying

$$((x_i)_{i\in I}, x) \in \mathcal{C} \text{ and } x \in U \Longrightarrow x_i \in U \text{ eventually.}$$

A convergence structure $\mathcal{C}$ in X is said to be *topological* if for every net $(x_i)_{i\in I}$ in X which topologically converges to x with respect to $\tau_{\mathcal{C}}$, it holds that $((x_i)_{i\in I}, x) \in \mathcal{C}$. Given a topological space X and a net $(x_i)_{i\in I}$ in X, we shall write $x_i \xrightarrow{\mathcal{O}(X)} x$ to denote $(x_i)_{i\in I}$ topologically converges to x in X.

For any other standard definitions and notations of topology and domain theory, we refer the reader to [5, 6, 8].

3. SI-Topology, SI-Convergence, and I-Continuity

In this section, we introduce the notion of SI-convergence which, we shall prove later, is the one that induces the SI-topology defined by Zhao and Ho. We first recall the definition of the SI-topology and some properties related to it.

Definition 3.1. (([9]) Let X be a T_0 space. A subset U of X is called SI-*open* if the following conditions are satisfied:

(1) U is an open set in X.

(2) For any $F \in \mathrm{Irr}(X)$, $\bigvee F \in U$ implies $F \cap U \neq \emptyset$ whenever $\bigvee F$ exists.

The set of all SI-open subsets of X is denoted by $\mathcal{O}_{\mathrm{SI}}(X)$.

Remark 3.2. For any T_0 space X, $\mathcal{O}_{\mathrm{SI}}(X)$ is a topology on X, called the SI-*topology*. The space $(X, \mathcal{O}_{\mathrm{SI}}(X))$ is then denoted by $\mathrm{SI}(X)$.

Proposition 3.3. ([9])

(1) *For a T_0 space X, $\mathcal{O}_{\mathrm{SI}}(X) \subseteq \mathcal{O}(X)$.*

(2) *If P is a poset and $X = (P, \alpha(P))$, then $\mathcal{O}_{\mathrm{SI}}(X)$ equals the Scott topology on P.*

(3) *The specialisation orders on X and $\mathrm{SI}(X)$ coincide.*

Proposition 3.4. *Let X be a T_0 space. Then $\mathrm{SI}(X)$ is a weak monotone convergence space, i.e., every monotone net having a supremum converges to that supremum [4, p. 459].*

Proof. Let $(x_i)_{i\in I}$ be a monotone net in $\mathrm{SI}(X)$ with supremum x and $U \in \mathcal{O}_{\mathrm{SI}}(X)$ such that $x \in U$. Then the set $D := \{x_i \mid i \in I\}$ is directed, hence irreducible, in X by Proposition 3.3(3). Since $\bigvee D = x \in U$, it follows that $D \cap U \neq \emptyset$. Then openness of U implies that $x_i \in U$ eventually. $\square$

Recall that given a poset P, a net $(x_i)_{i\in I}$ in P *Scott-converges* to $x \in P$ if and only if there exists a directed set D such that

(1) $\bigvee D$ exists with $x \leq \bigvee D$, and

(2) D is a set of eventual lower bounds of $(x_i)_{i\in I}$.

It is known that the Scott-convergence structure induces the Scott topology (see [5]). The following proposition shows that every set of eventual lower bounds of a net can be completely described by upper sets.

Proposition 3.5. *Let P be a poset, D a subset of P, and $(x_i)_{i\in I}$ a net in P. Then the following conditions are equivalent:*

(1) *D is a set of eventual lower bounds of $(x_i)_{i\in I}$.*

(2) *For every upper set U, $D \cap U \neq \emptyset$ implies $x_i \in U$ eventually.*

Proof. (1) $\Longrightarrow$ (2): Let $d \in D \cap U$. By assumption, $x_i \geq d$ eventually. Since $d \in U$ and U is upper, we have $x_i \in U$ eventually.

(2) $\Longrightarrow$ (1): Let $d \in D$. Then $D \cap {\uparrow} d \neq \emptyset$. By assumption, $x_i \in {\uparrow} d$ eventually, meaning that d is an eventual lower bound of $(x_i)_{i\in I}$. $\square$

By considering Proposition 3.5 and a poset P endowed with the Alexandroff topology on it, the definition of Scott-convergence can be rephrased in a topological way as follows: a net $(x_i)_{i\in I}$ in P Scott-converges to $x \in P$ if and only if there exists an irreducible set D in $(P, \alpha(P))$ such that

(1) $\bigvee D$ exists with $x \leq \bigvee D$, and

(2) for every $U \in \alpha(P)$, $D \cap U \neq \emptyset$ implies $x_i \in U$ eventually.

Lifting the above to the realm of T_0 spaces, we have the following definition.

Definition 3.6. Let X be a T_0 space. A net $(x_i)_{i\in I}$ in X is said to SI-*converge* to $x \in X$, denoted by $x_i \xrightarrow{\mathrm{SI}} x$ or $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}} x$, if there exists $F \in \mathrm{Irr}(X)$ such that

(i) $\bigvee F$ exists with $x \leq \bigvee F$, and

(ii) for every $U \in \mathcal{O}(X)$, $F \cap U \neq \emptyset$ implies $x_i \in U$ eventually.

We denote the topology induced by the SI-convergence structure by τ_{SI}, i.e., $V \in \tau_{\mathrm{SI}}$ if and only if for every net $(x_i)_{i\in I}$, $x_i \xrightarrow{\mathrm{SI}} x$ and $x \in V$ imply $x_i \in V$ eventually.

Remark 3.7. Let X be T_0 space and $(x_i)_{i\in I}$ be a net in X. If $x_i \xrightarrow{\mathrm{SI}} x$ then $x_i \xrightarrow{\mathrm{SI}} y$ for every $y \leq x$.

It is easy to verify that every constant net SI-converges to its constant term and if a net SI-converges to some point then any subnet also SI-converges to the same point.

Lemma 3.8. *For every $F \in \mathrm{Irr}(X)$ admitting a supremum, there exists a net $(x_i)_{i\in I_F}$ such that it* SI*-converges to $\bigvee F$ and all of its terms are in F.*

Proof. Let $x = \bigvee F$. Define $I_F = \{(e, O) \in F \times \mathcal{O}(X) \mid e \in O\}$ and equip I_F with $\leq$ defined as follows: $(e_1, O_1) \leq (e_2, O_2)$ if and only if $O_2 \subseteq O_1$. Irreducibility of F gives that I_F is directed. For every $(e, O) \in I_F$, we let $x_{(e,O)} = e$. If $U \in \mathcal{O}(X)$ is such that $F \cap U \neq \emptyset$, then there exists $d \in X$ such that $(d, U) \in I_F$. For every $(e, O) \geq (d, U)$ we have that $x_{(e,O)} = e \in U$. Hence $x_{(e,O)} \xrightarrow{\mathrm{SI}} x$. $\square$

The following theorem tells us that the topology induced by the SI-convergence structure in a T_0 space is precisely the SI-topology. This fact can be regarded as a topological parallel of the well-known fact in domain theory: Scott-convergence structure induces the Scott topology.

Theorem 3.9. *On a T_0 space X, the two topologies τ_{SI} and $\mathcal{O}_{SI}(X)$ coincide.*

Proof. Let $V \in \tau_{SI}$.

(1) Suppose that V is not open in X. Then there exists $x \in V$ such that for every $W \in \mathcal{N}(x) = \{W \in \mathcal{O}(X) \mid x \in W\}$, $W \nsubseteq V$. We equip $\mathcal{N}(x)$ with reverse inclusion order. We have that $\mathcal{N}(x)$ is a directed posets. For every $W \in \mathcal{N}(x)$, we let $x_W \in W \setminus V$ to form a net $(x_W)_{W\in\mathcal{N}(x)}$. It is clear that $\{x\} \in \mathrm{Irr}(X)$ and $x \leq \bigvee\{x\}$. Let $U \in \mathcal{O}(X)$ be such that $x \in U$. Then for every $W \in \mathcal{N}(x)$ such that $W \subseteq U$ it holds that $x_W \in U$. Hence $x_W \xrightarrow{\mathrm{SI}} x$. Since $V \in \tau_{SI}$, $x_W \in V$ for some $W \in \mathcal{N}(x)$, which is a contradiction. Therefore V is open in X.

(2) Let $F \in \mathrm{Irr}(X)$ such that $\bigvee F$ exists with $\bigvee F \in V$. By Lemma 3.8, there exists a net $(x_i)_{i\in I_F}$ such that it SI-converges to $\bigvee F$ and all of its terms are in F. Since $\bigvee F \in V$ and $V \in \tau_{\mathrm{SI}}$, it holds that $F \cap V \neq \emptyset$.

From (1) and (2) we have that $\tau_{\mathrm{SI}} \subseteq \mathcal{O}_{\mathrm{SI}}(X)$. Now let $V \in \mathcal{O}_{\mathrm{SI}}(X)$ and $(x_i)_{i\in I}$ be a net SI-converging to $x \in V$. By definition, there exists $F \in \mathrm{Irr}(X)$ such that $\bigvee F$ exists with $x \leq \bigvee F$ and for every $U \in \mathcal{O}(X)$, $F \cap U \neq \emptyset$ implies $x_i \in U$ eventually. Since $x \leq \bigvee F$ and $x \in V \in \mathcal{O}_{\mathrm{SI}}(X)$, $\bigvee F \in V$ holds, and hence $F \cap V \neq \emptyset$. Since $V \in \mathcal{O}(X)$, $x_i \in V$ holds eventually. Therefore $V \in \tau_{\mathrm{SI}}$. This completes the proof. $\square$

Proposition 3.10. *Let X and Y be T_0 spaces, and f be a continuous mapping from X to Y. Then the following conditions are equivalent:*

(1) *f is a continuous mapping from* $\mathrm{SI}(X)$ *to* $\mathrm{SI}(Y)$.

(2) *For every net $(x_i)_{i\in I}$ in X and $x \in X$, $x_i \xrightarrow{\mathrm{SI}} x$ in X implies $f(x_i) \xrightarrow{\mathrm{SI}} f(x)$ in Y.*

Proof. (1) $\Longrightarrow$ (2): It is a consequence of Theorem 3.9.

(2) $\Longrightarrow$ (1): Let $V \in \mathcal{O}_{\mathrm{SI}}(Y)$. Since f is continuous from X to Y, $f^{-1}(V) \in \mathcal{O}(X)$. Let F be irreducible in X such that $\bigvee F$ exists with $x := \bigvee F \in f^{-1}(V)$. Then $f(\bigvee F) \in V$. By Lemma 3.8, there exists a net $(x_i)_{i\in I_F}$ in X such that it SI-converges to x and all of its term are in F. The assumption then implies that the net $(f(x_i))_{i\in I_F}$ SI-converges to $f(x)$. It follows that the net $(f(x_i))_{i\in I_F}$ converges to $f(x)$ with respect to topology τ_{SI} on X, hence by Theorem 3.9, with respect to $\mathcal{O}_{\mathrm{SI}}(X)$. This implies $f(x_i) \in V$ eventually. Thus there exists $x_i \in F$ such that $x_i \in f^{-1}(V)$, implying that $F \cap f^{-1}(V) \neq \emptyset$. We conclude that $f^{-1}(V) \in \mathcal{O}_{\mathrm{SI}}(X)$, and therefore (1) holds. $\square$

It is known that a poset being continuous is a necessary and sufficient condition for Scott-convergence structure in it to be topological. Then it is natural to ask whether there exists a similar characterisation for SI-convergence case. The rest of this section shall be focused on the search of such characterisation. We first begin with a new notion of way-below relation.

Definition 3.11. Let X be a T_0 space. Define I-*way-below relation* $\ll_{\mathrm{I}}$ on X as follows: $x \ll_{\mathrm{I}} y$ if and only if for every irreducible set F in X with existing supremum, $y \le \bigvee F$ implies $x \in \mathrm{cl}(F)$.

One can see that when X is a poset P endowed with the Alexandroff topology, then the I-way-below relation is exactly the usual way-below relation on P.

Remark 3.12. Let X be a T_0 space and $u, x, y, z \in X$. Then

(i) $x \ll_{\mathrm{I}} y$ implies $x \le y$,

(ii) $u \le x \ll_{\mathrm{I}} y \le z$ implies $u \ll_{\mathrm{I}} z$,

(iii) $x \ll_{\mathrm{I}} y$ if and only if for every irreducible closed set F with existing supremum, $y \le \bigvee F$ implies $x \in F$.

In the following proposition, we have a connection between I-way-below relation and SI-convergence structure.

Proposition 3.13. *Let X be a T_0 space. Then $x \ll_{\mathrm{I}} y$ if and only if for every net $(x_i)_{i\in I}$, $x_i \xrightarrow{\mathrm{SI}} y$ implies $x_i \xrightarrow{\mathcal{O}(X)} x$.*

Proof. Necessity: Let $U \in \mathcal{N}(x)$ and $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}} y$. Then there exists $F \in \mathrm{Irr}(X)$ such that $\bigvee F$ exists with $y \le \bigvee F$ and $F \cap U \neq \emptyset$ implies $x_i \in U$ eventually. By assumption, it holds that $x \in \mathrm{cl}(F)$, hence $F \cap U \neq \emptyset$ and the result follows.

Sufficiency: Let $F \in \mathrm{Irr}(X)$ such that $\bigvee F$ exists with $y \le \bigvee F$. By Lemma 3.8, there exists a net $(x_i)_{i\in I_F}$ such that it SI-converges to $\bigvee F$ and all of its terms are in F. In virtue of Remark 3.7, we have $x_i \xrightarrow{\mathrm{SI}} y$, hence $x_i \xrightarrow{\mathcal{O}(X)} x$. Then for every $U \in \mathcal{N}(x)$, it holds that $x_i \in U$ eventually. Since $x_i \in F$ for every $i \in I_F$, this implies $x \in \mathrm{cl}(F)$. We conclude that $x \ll_I y$. □

Taking the special case when X is a poset P endowed with the Alexandroff topology, we have that x is way-below y in the poset P if and only if for every net Scott-converging to y, it holds that $x_i \in \uparrow x$ eventually which is precisely that given in [10, Lemma 1].

Definition 3.14. A T_0 space X is called I-*continuous* if for every $x \in X$ the set $\twoheaddownarrow_{\mathrm{I}} x := \{y \in X \mid y \ll_{\mathrm{I}} x\}$ is an irreducible set in X whose supremum is x.

Proposition 3.15. *A T_0 space X is* I-*continuous if and only if for every $x \in X$ there exists $F \in \mathrm{Irr}(X)$ with existing supremum such that $F \subseteq \twoheaddownarrow_{\mathrm{I}} x$ and $\bigvee F \ge x$.*

Proof. The necessity part is immediate from the definition of I-continuity. Now let $x \in X$ and assume there exists an irreducible subset F of $\twoheaddownarrow_{\mathrm{I}} x$ such that $\bigvee F$ exists with $\bigvee F \ge x$. Let U and V be open sets in X such that $\twoheaddownarrow_{\mathrm{I}} x \cap U \neq \emptyset$ and $\twoheaddownarrow_{\mathrm{I}} x \cap V \neq \emptyset$. There exist $u \in U$ and $v \in V$ such that $u \ll_{\mathrm{I}} x$ and $v \ll_{\mathrm{I}} x$. By the assumption, it holds that $u, v \in \mathrm{cl}(F)$. Hence $\mathrm{cl}(F) \cap U \neq \emptyset$ and $\mathrm{cl}(F) \cap V \neq \emptyset$, which imply $F \cap U \neq \emptyset$ and $F \cap V \neq \emptyset$. Irreducibility of F gives $F \cap U \cap V \neq \emptyset$. Since $F \subseteq \twoheaddownarrow_{\mathrm{I}} x$, it holds that $\twoheaddownarrow_{\mathrm{I}} x \cap U \cap V \neq \emptyset$, and hence $\twoheaddownarrow_I x$ is irreducible. The fact that $\bigvee F \ge x$ and $F \subseteq \twoheaddownarrow_I x$ implies that $\bigvee \twoheaddownarrow_{\mathrm{I}} x = x$. This completes the proof. □

Proposition 3.16. *Let X be an* I-*continuous space, $x, y \in X$, and $(x_i)_{i\in I}$ a net in X. Then the following statements are equivalent:*

(1) $x_i \xrightarrow{\mathrm{SI}} y$.

(2) *If $x \ll_{\mathrm{I}} y$ then $x_i \xrightarrow{\mathcal{O}(X)} x$.*

Proof. By Proposition 3.13, (1) implies (2) always holds even when X is not I-continuous. Now assume (2) and X is I-continuous. We have that $\twoheaddownarrow_{\mathrm{I}} y \in \mathrm{Irr}(X)$ and $y \le \bigvee \twoheaddownarrow_{\mathrm{I}} y$. Let $U \in \mathcal{O}(X)$ such that $\twoheaddownarrow_{\mathrm{I}} y \cap U \neq \emptyset$. Then there exists $x \in U$ such that $x \ll_{\mathrm{I}} y$. By assumption, $x_i \in U$ holds eventually. Hence $x_i \xrightarrow{\mathrm{SI}} y$. □

Corollary 3.17. *If X is* I*-continuous, then the* SI*-convergence structure in X satisfies the following condition:*

(Div) *If $(x_i)_{\in I}$ does not* SI*-converge to x then there exists a subnet $(y_j)_{j\in J}$ of $(x_i)_{i\in I}$ such that any subnet of $(y_j)_{j\in J}$ does not* SI*-converge to x.*

Proof. Suppose that $(x_i)_{\in I}$ does not SI-converge to x. Since $\Downarrow_{\mathrm{I}} x \in \mathrm{Irr}(X)$ and $x \leq \bigvee \Downarrow_{\mathrm{I}} x$, there exists $U \in \mathcal{O}(X)$ such that $\Downarrow_{\mathrm{I}} x \cap U \neq \emptyset$ and for all $i \in I$ there exists $j_i \in I$ such that $j_i \geq i$ and $x_{j_i} \notin U$. Thus the set $J := \{j \in I \mid x_j \notin U\}$ is cofinal in I. Now consider the subnet $(x_j)_{j\in J}$ of $(x_i)_{i\in I}$. Let $(z_k)_{k\in K}$ be any subnet of $(x_j)_{j\in J}$. Suppose to the contrary that $z_k \xrightarrow{\mathrm{SI}} x$. Notice that there exists $y \in U$ such that $y \ll_{\mathrm{I}} x$ as $\Downarrow_{\mathrm{I}} x \cap U \neq \emptyset$. By Proposition 3.16, it holds that $z_k \in U$ eventually, which is a contradiction. Therefore $(z_k)_{k\in K}$ does not SI-converge to x. □

If X is a poset P endowed with the Alexandroff topology, then it is clear that X being an I-continuous space is the same as P being a continuous poset. Given a continuous poset P, one can deduce that for every $x \in P$, the set of all elements in P which is way-above x, i.e., $\Uparrow x$, is Scott open. However, if one has an I-continuous space, the set $\Uparrow_{\mathrm{I}} x := \{y \in X \mid y \ll_{\mathrm{I}} x\}$ may not be SI-open, as shown in the following example.

Example 3.18. Let X be the natural numbers endowed with the cofinite topology. We can see that for every x in X, $x \ll_{\mathrm{I}} y$ if and only if $x = y$. Hence X is I-continuous. For every $x \in X$, it holds that $\Uparrow_{\mathrm{I}} x = \{x\}$ which is clearly not open in X, hence not open in $\mathrm{SI}(X)$.

The remaining of this section is devoted to the relation between SI-convergence on a space and the I-continuity of the space. We first define, for every T_0 space X, the following condition (I*).

(I*) Whenever X is I-continuous then the set $\Uparrow_{\mathrm{I}} x$ is always open in $\mathrm{SI}(X)$ for every $x \in X$.

Remark 3.19. Every T_0 Alexandroff space satisfies condition (I*).

Lemma 3.20. *Let X be a T_0 space satisfying condition*(I*)*. If X is* I*-continuous, then the* SI*-convergence structure in X is topological.*

Proof. It suffices to prove that if $(x_i)_{i\in I}$ topologically converges to x in $\mathrm{SI}(X)$ then $x_i \xrightarrow{\mathrm{SI}} x$. By I-continuity of X, we have that $F := \Downarrow_{\mathrm{I}} x \in \mathrm{Irr}(X)$ and $x \leq \bigvee F$. Now let $U \in \mathcal{O}(X)$ be such that $F \cap U \neq \emptyset$. Then there exists $u \in U$ such that $u \ll_{\mathrm{I}} x$. By the given assumption, we have that $\Uparrow_{\mathrm{I}} u$ is open in $\mathrm{SI}(X)$. Thus $x_i \in \Uparrow_{\mathrm{I}} u$ eventually. Since $\Uparrow_{\mathrm{I}} u \subseteq U$, $x_i \in U$ holds eventually. Hence $x_i \xrightarrow{\mathrm{SI}} x$, as desired. □

Lemma 3.21. *If the* SI*-convergence structure in X is topological, then X is* I*-continuous.*

Proof. Let $x \in X$ and $\mathcal{F}_x = \{F_i \mid i \in I\}$ be the collection of all irreducible subsets F_i of X such that $\bigvee F_i$ exists with $x \leq \bigvee F_i$. Define a preorder $\leq$ on I as follows: $i \leq j$ for all $i, j \in I$. For every $i \in I$, let $x_i = \bigvee F_i$ to form a net $(x_i)_{i\in I}$. It is immediate that we have $x_i \xrightarrow{\mathrm{SI}} x$. For every $i \in I$, by Lemma 3.8, there exists a net $(x_{i,j})_{j\in I_{F_i}}$ such that it SI-converges to x_i and $x_{i,j} \in F_i$ for every $j \in I_{F_i}$. Let $M := \Pi_{i\in I} I_{F_i}$ be equipped with pointwise order. Since SI-convergence is topological, the net $(x_{(i,f)})_{(i,f)\in I\times M}$ SI-converges to x, where $x_{(i,f)} = x_{i,f(i)}$ and the order on $I \times M$ is the pointwise order. Then there exists $F \in \mathrm{Irr}(X)$ such that

1. $\bigvee F$ exists with $x \leq \bigvee F$, and
2. for every $U \in \mathcal{O}(X)$, $F \cap U \neq \emptyset$ implies $x_{(i,f)} \in U$ eventually.

We will show that $F \subseteq \Downarrow_{\mathrm{I}} x$. Take any $e \in F$. Let E be irreducible in X such that $\bigvee E$ exists with $\bigvee E \geq x$ and U be open in X such that $e \in U$. Then there exists $i_0 \in I$ such that $E = F_{i_0}$. Since $F \cap U \neq \emptyset$ (by the presence of e), there exists $(i^*, f^*) \in I \times M$ such that $(i, f) \geq (i^*, f^*)$ implies $x_{(i,f)} \in U$ eventually. Since $(i_0, f^*) \geq (i^*, f^*)$, it holds that $x_{(i_0,f^*)} = x_{i_0,f^*(i_0)} \in U \cap F_{i_0} = U \cap E$, and hence $e \in \mathrm{cl}(E)$. Therefore $e \ll_{\mathrm{I}} x$. Since $\bigvee F \geq x$ and $F \subseteq \Downarrow_{\mathrm{I}} x$, in virtue of Proposition 3.15, we conclude that X is I-continuous. □

Combining Lemmas 3.20 and 3.21, we have the following theorem.

Theorem 3.22 (SI-**convergence Theorem).** *Let X be a T_0 space satisfying condition* (I*)*. Then the following two statements are equivalent:*

(1) *The* SI-*convergence structure in X is topological.*

(2) *X is I-continuous.*

Corollary 3.23 (Scott-convergence Theorem). *[10, Theorem 1] Let P be a poset. Then the Scott-convergence structure in P is topological if and only if P is continuous.*

4. SI-Continuous Spaces

In this section we revisit the notion of SI-continuity introduced in [9]. In particular, we prove some equivalent conditions for SI-continuity of a T_0 space. First recall the definition of SI-way-below relation and then continue with the definition of SI-continuous space.

Definition 4.1. ([9]) Let X be a T_0 space. For $x, y \in X$, define $x \ll_{\mathrm{SI}} y$, read as x is SI-*way-below* y, if for every irreducible set F, $y \leq \bigvee F$ implies $x \in \downarrow F$ whenever $\bigvee F$ exists.

Given any T_0 space X and subset A of X, it holds that $\downarrow A \subseteq \mathrm{cl}(A)$. Consequently, $x \ll_{\mathrm{SI}} y$ implies $x \ll_{\mathrm{I}} y$. However, the converse is not true in general, as witnessed by the following example.

Example 4.2. Let $\mathbb{J} = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$ be Johnstone's poset [7]. Now let $\mathbb{J}^* = \mathbb{J} \cup \{\top\}$ equipped with $\leq$ as on Johnstone's poset with an addition of a top element, which is greater than any element in $\mathbb{J}^*$. More precisely $\leq$ on $\mathbb{J}^*$ is defined as follows:

(i) $(m, n) \leq (p, q) \Leftrightarrow (m = p$ and $n \leq q)$ or $(q = \infty$ and $n \leq p)$, and

(ii) $x \leq \top$ for every $x \in \mathbb{J}^*$.

Let $X = \mathrm{SI}(\mathbb{J}^*, \sigma(\mathbb{J}^*))$. We then have that $\mathbb{J} \in \mathrm{Irr}(X)$ and $\mathrm{cl}(\mathbb{J}) = \mathbb{J}^*$. It can be verified that $\top \ll_{\mathrm{I}} \top$ but $\top \not\ll_{\mathrm{SI}} \top$.

Definition 4.3. ([9]) A T_0 space X is called SI-*continuous* if for every $x \in X$, the following conditions hold:

(SI1) $\twoheaduparrow_{\mathrm{SI}} x := \{y \in X \mid x \ll_{\mathrm{SI}} y\}$ is open in X.

(SI2) $\twoheaddownarrow_{\mathrm{SI}} x := \{y \in X \mid y \ll_{\mathrm{SI}} x\}$ is a directed subset of X with $\bigvee \twoheaddownarrow_{\mathrm{SI}} x = x$.

One may see that the definition of SI-continuity is somewhat not satisfying. This is merely because of condition (SI2), i.e., the set $\twoheaddownarrow_{SI} x$ needs to be directed, which is incoherent with the Zhao-Ho replacement principle. More precisely, when working directly in T_0 spaces, one should use irreducibility instead of directedness. In response to this, we define the notion of SI*-continuity by changing the "directed" condition in (SI2) to "irreducible" condition and later prove that SI-continuity and SI*-continuity are just the same notions.

Definition 4.4. A T_0 space X is called SI*-*continuous* if for every $x \in X$, the following conditions hold:

(SI*1) $\twoheaduparrow_{\mathrm{SI}} x := \{y \in X \mid x \ll_{\mathrm{SI}} y\}$ is open in X.

(SI*2) $\twoheaddownarrow_{\mathrm{SI}} x := \{y \in X \mid y \ll_{\mathrm{SI}} x\}$ is an irreducible subset of X with $\bigvee \twoheaddownarrow_{\mathrm{SI}} x = x$.

Remark 4.5. It is proven in [1] that condition (SI*2) is equivalent to the existence of an irreducible subset of $\twoheaddownarrow_{\mathrm{SI}} x$ whose supremum is x.

In the proof of [9, Theorem 6.4], it is proven that the relation $\ll_{\mathrm{SI}}$ on an SI-continuous space is interpolative, i.e., $x \ll_{\mathrm{SI}} y$ implies $x \ll_{\mathrm{SI}} z \ll_{\mathrm{SI}} y$ for some z. The same situation also happens on an SI*-continuous space.

Lemma 4.6. *Let X be an* SI**-continuous space and $F \in \mathrm{Irr}(X)$. Then $\Downarrow_{\mathrm{SI}} F := \bigcup\{\Downarrow_{\mathrm{SI}} d \mid d \in F\}$ is irreducible in X. In addition, if $\bigvee F$ exists, then $\bigvee \Downarrow_{\mathrm{SI}} F$ exists and $\bigvee \Downarrow_{\mathrm{SI}} F = \bigvee F$.*

Proof. Let U and V be open sets in X such that $(\Downarrow_{\mathrm{SI}} F) \cap U \neq \emptyset$ and $(\Downarrow_{\mathrm{SI}} F) \cap V \neq \emptyset$. Pick $a \in (\Downarrow_{\mathrm{SI}} F) \cap U$ and $b \in (\Downarrow_{\mathrm{SI}} F) \cap V$. Then we can find $d, e \in F$ such that $a \ll_{\mathrm{SI}} d$ and $a \in U$, $b \ll_{\mathrm{SI}} e$ and $b \in V$, implying $\Uparrow_{\mathrm{SI}} a \cap F \neq \emptyset$ and $\Uparrow_{SI} b \cap F \neq \emptyset$. As F is irreducible and $\Uparrow_{\mathrm{SI}} a$ and $\Uparrow_{\mathrm{SI}} b$ are open in X, we have $\Uparrow_{\mathrm{SI}} a \cap \Uparrow_{\mathrm{SI}} b \cap F \neq \emptyset$. Then there exists $z \in \Uparrow_{\mathrm{SI}} a \cap \Uparrow_{\mathrm{SI}} b \cap F$. It follows that $a \in \Downarrow_{\mathrm{SI}} z \cap U \neq \emptyset$ and $b \in \Downarrow_{\mathrm{SI}} z \cap V \neq \emptyset$. Note that $\Downarrow_{\mathrm{SI}} z$ is irreducible, thus $\Downarrow_{\mathrm{SI}} z \cap U \cap V \neq \emptyset$. Since $\Downarrow_{\mathrm{SI}} z \subseteq \Downarrow_{\mathrm{SI}} F$, we have $\Downarrow_{\mathrm{SI}} F \cap U \cap V \neq \emptyset$. Therefore $\Downarrow_{\mathrm{SI}} F$ is irreducible. The fact that $\bigvee \Downarrow_{\mathrm{SI}} F = \bigvee F$ follows from the fact that $\bigvee \Downarrow_{\mathrm{SI}} x = x$ for every $x \in X$. □

Corollary 4.7. *If X is an* SI**-continuous space, then the relation $\ll_{\mathrm{SI}}$ is interpolative.*

Proof. Suppose $x, y \in X$ satisfying $x \ll_{\mathrm{SI}} y$. By Lemma 4.6, we have that $F_y := \bigcup\{\Downarrow_{\mathrm{SI}} d \mid d \ll_{\mathrm{SI}} y\}$ is irreducible and $\bigvee F_y = \bigvee \Downarrow_{\mathrm{SI}} y = y$, implying $x \in \downarrow F_y = F_y$. Thus there exists $d \ll_{\mathrm{SI}} y$ such that $x \ll_{\mathrm{SI}} d$. □

One could notice that, in an SI-continuous space, the set F_y defined in the proof of Corollary 4.7 is directed (see also the proof of [9, Theorem 6.4]). This condition is too strong as in the definition of $\ll_{\mathrm{SI}}$ one only needs to consider irreducible sets. This allows us to retain the interpolation property of $\ll_{\mathrm{SI}}$ on an SI*-continuous space. In fact, in SI-continuous spaces, $\Downarrow_{\mathrm{SI}} x$ being irreducible can imply $\Downarrow_{\mathrm{SI}} x$ being directed. This surprising fact leads us to the equivalence between SI-continuity and SI*-continuity.

Theorem 4.8. *Let X be a T_0 space. Then X is* SI*-continuous if and only if it is* SI**-continuous.*

Proof. It suffices to show that if X is SI*-continuous, then for every $x \in X$ the statement (SI2) holds. Let $y, z \in \Downarrow_{\mathrm{SI}} x$. By Corollary 4.7, we have $\Uparrow_{\mathrm{SI}} y \cap \Downarrow_{\mathrm{SI}} x \neq \emptyset$ and $\Uparrow_{\mathrm{SI}} z \cap \Downarrow_{\mathrm{SI}} x \neq \emptyset$. Note that $\Downarrow_{\mathrm{SI}} x$ is irreducible and both $\Uparrow_{\mathrm{SI}} y$ and $\Uparrow_{\mathrm{SI}} z$ are open in X, implying $\Uparrow_{\mathrm{SI}} y \cap \Uparrow_{\mathrm{SI}} z \cap \Downarrow_{\mathrm{SI}} x \neq \emptyset$. Then there is $u \in \Downarrow_{\mathrm{SI}} x$ such that $y \leq u$ and $z \leq u$. Hence $\Downarrow_{\mathrm{SI}} x$ is directed. The conclusion $x = \bigvee \Downarrow_{\mathrm{SI}} x$ is trivial by the SI*-continuity of X □

Remark 4.9. At this juncture, we already have two notions of continuity of T_0 spaces, namely, I-continuity and SI-continuity (which equals SI*-continuity). It can be easily verified that on the I-continuous space given in Example 3.18, $\ll_{\mathrm{I}}$ is exactly $\ll_{\mathrm{SI}}$. Consequently the space is not SI-continuous. This shows that I-continuity and SI-continuity are generally distinct.

It is known that for a continuous poset P, the collection of all sets of the form $\Uparrow x$, $x \in P$, forms a base for the Scott topology on P. A similar property is also satisfied by SI-continuous spaces.

Proposition 4.10. *If X is an* SI*-continuous space then for every $x \in X$ the set $\Uparrow_{\mathrm{SI}} x$ is* SI*-open.*

Proof. Let F be an irreducible subset of X such that $\bigvee F$ exists with $\bigvee F \in \Uparrow_{\mathrm{SI}} x$. By Corollary 4.7, there exists $z \in X$ such that $x \ll_{\mathrm{SI}} z \ll_{\mathrm{SI}} \bigvee F$. It follows that $z \in \downarrow F$. Hence $F \cap \Uparrow_{\mathrm{SI}} x \neq \emptyset$. □

Corollary 4.11. *Let X be a T_0 space. Then X is* SI*-continuous if and only if for any $x \in X$, the following conditions hold:*

(i) $\Uparrow_{\mathrm{SI}} x$ *is* SI*-open.*

(ii) $\Downarrow_{\mathrm{SI}} x$ *is irreducible in X with $\bigvee \Downarrow_{\mathrm{SI}} x = x$.*

Proposition 4.12. *If X is* SI*-continuous, then the collection*

$$\{\Uparrow_{\mathrm{SI}} x \mid x \in X\}$$

forms a base for $\mathcal{O}_{\mathrm{SI}}(X)$.

Proof. Let U be SI-open and $x \in U$. By SI-continuity of X, $\Downarrow_{\mathrm{SI}} x$ is an irreducible set whose supremum is x. Since U is inaccesible by suprema of irreducible sets, there exists $u \in U$ such that $u \ll_{\mathrm{SI}} x$. Since U is an upper set, $x \in \Uparrow_{\mathrm{SI}} u \subseteq U$. This completes the proof as $\Uparrow_{\mathrm{SI}} u$ is SI-open from Corollary 4.11. $\square$

The following proposition provides a relation between SI-continuity and SI-convergence structure.

Proposition 4.13. *If X is an* SI-*continuous space, then the* SI-*convergence structure in X is topological.*

Proof. It suffices to prove that if $(x_i)_{i\in I}$ topologically converges to x in $\mathrm{SI}(X)$ then $x_i \xrightarrow{\mathrm{SI}} x$. By SI-continuity of X, we have that $F := \Downarrow_{SI} x \in \mathrm{Irr}(X)$ and $x \leq \bigvee F$. Now let $U \in \mathcal{O}(X)$ be such that $F \cap U \neq \emptyset$. Then there exists $u \in U$ such that $u \ll_{SI} x$. By Proposition 4.10, $\Uparrow_{\mathrm{SI}} u$ is open in $\mathrm{SI}(X)$. We then have that $x_i \in \Uparrow_{\mathrm{SI}} u$ eventually. Since $\Uparrow_{\mathrm{SI}} u \subseteq U$, we have $x_i \in U$ eventually. Hence $x_i \xrightarrow{\mathrm{SI}} x$, as desired. $\square$

Corollary 4.14. *Every* SI-*continuous space is* I-*continuous.*

Proof. It is immediate from Lemma 3.21 and Proposition 4.13. $\square$

The converse of Proposition 4.13 is not always true, as shown in the following example.

Example 4.15. Let X be the natural numbers endowed with cofinite topology. Then $\mathcal{O}(X) = \mathcal{O}_{\mathrm{SI}}(X)$. We claim that the SI-convergence structure in X is topological. Let $(x_i)_{i\in I}$ topologically converges to x in $\mathrm{SI}(X)$. Set $F = \{x\}$. We have $x \leq \bigvee F$. Let U be in $\mathcal{O}(X) = \mathcal{O}_{\mathrm{SI}}(X)$ such that $F \cap U \neq \emptyset$. Then U is an open set in $\mathrm{SI}(X)$ containing x. Hence x_i is in U eventually. We have that $x_i \xrightarrow{\mathrm{SI}} x$. Hence SI-convergence convergence in X is topological. We can see that $x \ll_{\mathrm{SI}} y$ if and only if $x = y$. Hence for every $x \in X$, $\Uparrow_{\mathrm{SI}} x = \{x\}$ which is clearly not SI-open. Therefore X is not SI-continuous.

Notice that, based on what we have already attained until this stage, the concepts of SI-topology and SI-continuity in the realm of T_0 spaces mimic the concepts of Scott topology and continuity in the realm of posets in some ways. Indeed, the SI-topology of an SI-continuous space is the Scott topology of a continuous poset, which is precisely the induced specialisation poset, as shown in the following.

Theorem 4.16. (see also [1, Remark 4.6]) *If X is an* SI-*continuous space, then:*

(1) $\left(X, \leq_{\mathcal{O}(X)}\right)$ *is a continuous poset, and*

(2) $\mathcal{O}_{\mathrm{SI}}(X) = \sigma\left(X, \leq_{\mathcal{O}(X)}\right)$.

Proof. By Proposition 3.4 and [9, Theorem 6.4], $\mathrm{SI}(X)$ is a weak monotone convergence C-space. Then (1) and (2) follow from [4, Theorem 4]. $\square$

While it is correct that the SI-continuity of a space implies the continuity of its induced specialisation poset, it is not true in general that the continuity of the specialisation poset induced by a given T_0 space implies the SI-continuity of the space. Indeed, one can easily find a continuous poset and an order-compatible topology on it which is strictly coarser than the Scott topology. By Theorem 4.16, this poset endowed with the topology is not SI-continuous.

Example 4.17. (1) Any infinite antichain A endowed with the *upper topology* $\nu(A)$ on A, i.e., that generated by sets of the form $A \setminus \downarrow x$ is not an SI-continuous space while the antichain itself is a continuous poset.

(2) The coproduct of infinitely many continuous posets is again continuous. Yet when we endow it with the upper topology, the resulting T_0 space is not SI-continuous.

Recall that the derivation of SI-topology from an existing topology is motivated by that of the Scott topology from the Alexandroff topology. From Theorem 4.16, we know that whenever a space is SI-continuous, its SI-topology is just the Scott topology. However, there is no information about the underlying topology. In fact, it is not necessarily the Alexandroff topology. In what follows, we shall put our attention on examples of non-Alexandroff SI-continuous spaces that we advertise in abstract. To provide those examples, we introduce the notion of a novel topology on posets called the Scott-max topology.

Definition 4.18. Let P be a poset. The set of all maximal elements in P is denoted by $\max(P)$. The *Scott-max topology* on P, denoted by $\xi(P)$, is defined to be the coarsest topology containing all Scott open subsets of P and all subsets of $\max(P)$.

Remark 4.19. Let P be a poset.

(1) The Scott-max topology on P is located in between the Scott topology and the Alexandroff topology on P. Hence it is an order compatible T_0 topology.

(2) If $\max(P) = \emptyset$ then the Scott-max topology on P is just the Scott topology.

The following lemma can be easily verified from the definition of Scott-max topology.

Lemma 4.20. *Let P be a poset and $U \subseteq P$. Then U is in $\xi(P)$ if and only if there exist $U^* \in \sigma(P)$ and $A \subseteq \max(P)$ such that $U = U^* \cup A$ and $U^* \cap A = \emptyset$.*

Proof. The sufficiency is clear by the definition. To prove the necessity, we first note that $\sigma(P) \cup \max(P)$ is closed under finite intersection. Thus it forms a base for $\xi(P)$. Now let U be in $\xi(P)$. Since $\sigma(P)$ and $\max(P)$ are closed under arbitrary union, we have that U is the union of some Scott open set V and some subset B of $\max(P)$. We then take $U^* = V$ and $A = B \setminus V$. Clearly, $U = U^* \cup A$ and $U^* \cap A = \emptyset$. □

Proposition 4.21. *Let P be a poset. Then $\alpha(P) = \xi(P)$ if and only if $x \ll x$ for every $x \in P \setminus \max(P)$.*

Proof. Let $\alpha(P) = \xi(P)$. Then for every $x \in P \setminus \max(P)$, the set $\uparrow x \in \xi(P)$. By Lemma 4.20, there exist $U^* \in \sigma(P)$ and $A \subseteq \max(P)$ such that $\uparrow x = U^* \cup A$ and $U^* \cap A = \emptyset$. Since $x \notin \max(P)$, it holds that $U^* = \uparrow x$, hence $x \ll x$.

Conversely, to show $\alpha(P) \subseteq \xi(P)$, it suffices to show that that $\uparrow x \in \xi(P)$ for all $x \in P$. If $x \in \max(P)$, then $\uparrow x = \{x\} \in \xi(P)$. If $x \notin \max(P)$, then, by assumption, for every directed set D such that $\bigvee D \in \uparrow x$ we have $D \cap \uparrow x \neq \emptyset$. Hence $\uparrow x \in \sigma(P) \subseteq \xi(P)$. □

Proposition 4.22. *Let P be a poset. Then the following conditions are equivalent:*

(1) $\sigma(P) = \xi(P)$.

(2) $x \ll x$ *for every* $x \in \max(P)$.

(3) $\max(P) \in \sigma(P)$.

Proof. (1) $\Longrightarrow$ (2): It is clear since, by assumption, $\{x\} \in \sigma(P)$ for every $x \in \max(P)$.

(2) $\Longrightarrow$ (3): It is clear that $\max(P)$ is an upper set. Now let D be a directed set admitting a supremum and $\bigvee D \in \max(P)$. By assumption $\bigvee D \ll \bigvee D$. This forces $\bigvee D \in D$, which implies $D \cap \max(P) \neq \emptyset$. Therefore $\max(P) \in \sigma(P)$.

(3) $\Longrightarrow$ (1): It suffices to show that $\{x\} \in \sigma(P)$ for every $x \in \max(P)$. Let $x \in \max(P)$. If D is a directed set whose supremum is x, then by assumption it holds that $D \cap \max(P) \neq \emptyset$, and hence $x \in D$. This completes the proof. □

By making use of the Scott-max topology, one can easily provide many examples of a space X such that X is SI-continuous but $\mathcal{O}(X)$ is not the Alexandroff topology on the specialisation poset induced by X, i.e., X is not an Alexandroff space. We first look at the following example.

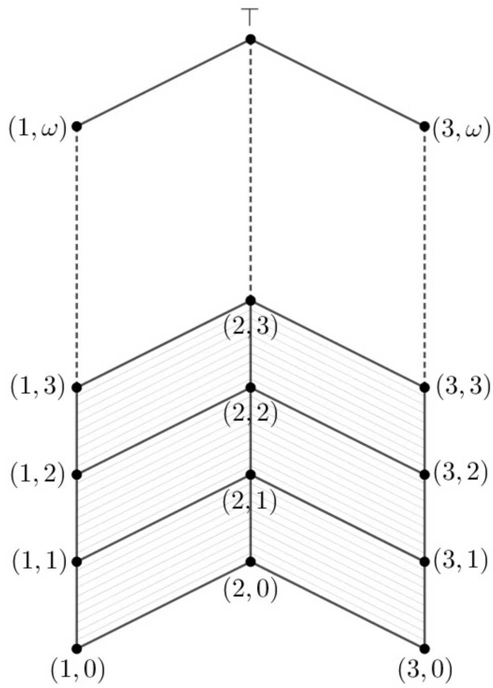

Figure 1: An illustration of the poset P

Example 4.23. Let $P = [0, \infty]$ be equipped with the usual order. By Propositions 4.21 and 4.22, we have that $\sigma(P) \subsetneq \xi(P) \subsetneq \alpha(P)$. We consider the T_0 space $X = (P, \xi(P))$. Every nonempty subset of P is directed, and hence irreducible sets in X are exactly directed subsets of P. Since P is a continuous poset, we have that X is *SI*-continuous.

At the end of this section, we provide another example of an SI-continuous space X such that X is not an Alexandroff space. Unlike the space given in Example 4.23, the space $(P, \xi(P))$ given in Example 4.24 satisfies the following property: the collection of all irreducible sets in $(P, \xi(P))$ and the collection of all directed sets in the specalisation poset induced by $(P, \xi(P))$ do not coincide.

Example 4.24. Let $I = \{1, 2, 3\}$, $\mathbb{R}^+$ be the set of all positive real numbers ordered with the usual order, and $\mathbb{R}^* = \mathbb{R}^+ \cup \{0, \omega\}$ in which ω is the top element. Define $P = I \times \mathbb{R}^*$ and a partial order $\leq$ in P as follows (see Figure 1):

(i) $x \leq (2, \omega) =: \top$ for every $x \in P$,

(ii) for every $i \in I$, $(i, r) \leq (i, s)$ if and only if $r \leq s$ in $\mathbb{R}^*$,

(iii) for every $r, s \in \mathbb{R}^*$, $(1, r)$ and $(3, s)$ are incomparable,

(iv) for every $i \in \{1, 3\}$ and $r, s \in \mathbb{R}^*$ such that $r \leq s$, $(i, r) \leq (2, s)$.

We have the following easy facts regarding P.

(1) Every nonempty subset of P has supremum.

(2) $\xi(P) \setminus \sigma(P) = \{\{\top\}\}$.

(3) For every $x \in P - \{(1, 0), (2, 0), (3, 0)\}$, it holds that $\uparrow x \in \alpha(P) - \xi(P)$.

From (2) and (3), we have $\sigma(P) \subsetneq \xi(P) \subsetneq \alpha(P)$. For every $i \in I$, define $R_i = \{(i, r) \mid r \in \mathbb{R}^+ \cup \{0\}\}$. The poset P is continuous from the fact that

(i) $\Downarrow\top = R_1 \cup R_2 \cup R_3$,

(ii) $\Downarrow(i, \omega) = R_i$ for every $i \in \{1, 3\}$,

(iii) $\Downarrow(i, r) = \downarrow(i, r) - \{(1, r), (2, r), (3, r)\}$ for every $i \in I$ and $r \in \mathbb{R}^+$, and

(iv) $\Downarrow(i, 0) = \{(i, 0)\}$ for every $i \in I$.

Claim: For every irreducible set F in $(P, \xi(P))$ whose supremum exists, there exists a directed set $D \subseteq F$ such that $\bigvee F = \bigvee D$.
Proof of Claim: We only need to consider the case when F is not directed. Clearly $\top \notin F$. We also have that $\bigvee F \notin R_1 \cup R_3$, otherwise $F \subseteq R_1$ or $F \subseteq R_3$ which implies F is directed. If $\bigvee F = \top$, then F contains a directed subset D of R_2 such that $\bigvee D = \top$. The remaining possible case is $\bigvee F =: (2, r) \in R_2$, where $r \in \mathbb{R} \cup \{0\}$. Suppose to the contrary that $F \cap R_2 = \emptyset$. Then, since F is not directed, there exist $a \in R_1$ and $b \in R_3$ such that $a, b \in F$. Consider the set $U_i = R_i \cup R_2 \cup \{(i, \omega), \top\}$, for $i \in \{1, 3\}$. It follows that $U_1, U_3 \in \xi(P)$ with $F \cap U_i \neq \emptyset$, for $i \in \{1, 3\}$, but $F \cap U_1 \cap U_3 = \emptyset$, which contradicts the fact that F is irreducible and therefore $F \cap R_2 \neq \emptyset$. Let $D = F \cap R_2$ and assume $\bigvee D = (2, s)$ for some $s \in \mathbb{R} \cup \{0\}$. Clearly, D is directed. We will show that $s = r$. Suppose to the contrary that $s < r$. Since $\bigvee F = (2, r)$, there exist $i_0 \in \{1, 3\}$ and $E \subseteq F \cap R_{i_0}$ such that $\bigvee E = (i_0, r)$. Define

$$U_1 = \uparrow(i_0, s) \cup \uparrow(2, s) \setminus \{(i_0, s), (2, s)\} \text{ and } U_2 = R_2 \cup \{\top\}.$$

Then $U_1, U_2 \in \xi(P)$ and $U_1 \cap U_2 = \uparrow(2, s) \setminus \{(2, s)\}$. We have that $F \cap U_1 \neq \emptyset$, $F \cap U_2 \neq \emptyset$, but $F \cap U_1 \cap U_2 = \emptyset$, which is a contradiction. Therefore $s = r$, which means $\bigvee D = \bigvee F$. We conclude that Claim is indeed correct.

As a corollary of the above claim, the relation $\ll_{\mathrm{SI}}$ on $(P, \xi(P))$ is exactly the way-below relation on P. This leads to the fact that $(P, \xi(P))$ is an SI-continuous space. Notice that if U is a nonempty Scott open set in P, then $\top \in U$. Consequently, as R_2 is a directed set whose supremum is $\top$, $R_2 \cap U \neq \emptyset$, and hence $(P \setminus \{\top\}) \cap U \neq \emptyset$. By the fact that $\xi(P) \setminus \sigma(P) = \{\{\top\}\}$, we have that $P \setminus \{\top\} \in \mathrm{Irr}(P, \xi(P))$. But, it is clear that $P \setminus \{\top\}$ is not directed. So we conclude that

$$\mathrm{Irr}(P, \xi(P)) \neq \mathrm{Dir}(P) := \{D \subseteq P \mid D \text{ is directed}\}.$$

5. Conclusion and Further Work

In this paper, we have defined the SI-convergence structure that induces the SI-topology first introduced in [9]. Additionally, we characterise those T_0 spaces X in which the SI-convergence structure is topological. Another contribution we have provided is the establishment of a more natural definition of SI-continuity and examples of non-Alexandroff SI-continuous spaces. Before we end this paper, let us pose some possible research directions.

1. Besides the Alexandroff and Scott topologies, there is another prominent topology on posets called the Lawson topology. This topology has some interesting properties, particularly when the underlying poset is quasicontinuous. It would be interesting to study a topological parallel of the Lawson topology and its connection with some notion of quasicontinuous spaces (one such notion is already studied in [2]).

2. A characterisation of T_0 spaces in which the SI-convergence structure is topological is provided in Theorem 3.22. However, this characterisation is not a complete characterisation since it only holds for a certain class of T_0 spaces, i.e., T_0 spaces satisfying condition (I*). This opens a direction of research: to find a characterisation better than that given in Theorem 3.22.

3. Although we can provide a space on which the I-way-below and SI-way-below relations do not coincide, we are currently unable to distinguish between I-continuous spaces and spaces satisfying condition (SI2). In fact, the space given in Example 4.2 is neither I-continuous nor SI-continuous. Moreover, we also have no information on the difference between the class of SI-continuous spaces and the class of I-continuous spaces satisfying condition (I^*). All of these facts could make one consider doing deeper investigation regarding *I*-continuity, SI-continuity, and connections between them.

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