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The Solvability of a Class of System of Nonlinear Integral Equations via Measure of Noncompactness

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Abstract. We propose a new notion of contraction mappings for two class of functions involving measure of noncompactness in Banach space. In this regard we present some theory and results on the existence of tripled fixed points and some basic Darbo's type fixed points for a class of operators in Banach spaces. Also as an application we discuss the existence of solutions for a general system of nonlinear functional integral equations which satisfy in new certain conditions . Further we give an example to verify the effectiveness and applicability of our results.

1. Introduction

Measures of noncompactness are very useful and powerful tools in functional analysis, for instance in the theory of operator equations in Banach spaces and in metric fixed point theory. They are also used in the studies of ordinary and partial differential equations, functional equations, integral and integro-differential equations, fractional partial differential equations, optimal control theory, and in the characterizations of compact operators between Banach spaces. In 1930, Kuratowski [24] introduce the first concept of measure of noncompactness (MNC). Later on, in 1955, G. Darbo [16] proved a fixed point theorem via the concept of Kuratowski MNC, which generalizes both the classical Schauder fixed point theorem and a special variant of Banach contraction principle. In 1957, the other measures of noncompactness were introduced by Goldenštein, Gohberg, and Markus [20], which was called the ball or Hausdorff MNC. There are some other definitions of measure of noncompactness which the authors were trying to introduce this definition in an axiomatic way. At first, it appeared in the paper of Sadovskii [30], but his axiomatics seems to be too general. In 1980 Banas [11] was introduced another axiomatic measure of noncompactness which was very useful in applications. Up to now several authors have presented some papers on the existence of solution for nonlinear integral equations which involves the use of measure of noncompactness and many other techniques, for instance see [1]-[6] and [7]-[31].

In this paper, we apply the method related to the technique of measures of noncompactness in order to extend the Darbo's fixed point theorem [16]. Our results are a generalization of the results of Roshan [29] from two dimension in to a three dimension version and the results of the paper Karakaya *et al*. [16] (with the approach that, the conditions of the related operators of integral equations are generalized. See

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Theorem 4.1) for proving some existence theorems of three dimension fixed points and tripled fixed points for a class of operators in Banach spaces. Moreover, as an application of this theorems, we study the problem of existence of solutions for the following class of system of nonlinear integral equations (which satisfy in new certain conditions).

$$\begin{cases} x(t) = A_1(t) + h_1(t, x(\xi_1(t)), y(\xi_1(t)), z(\xi_1(t))) + f_1\left(\begin{array}{c} t, x(\xi_1(t)), y(\xi_1(t)), z(\xi_1(t)), \\ \varphi(\int_0^{\beta_1(t)} g_1(t, s, x(\eta_1(s)), y(\eta_1(s)), z(\eta_1(s))) ds \end{array} \right) \\ y(t) = A_2(t) + h_2(t, y(\xi_2(t)), z(\xi_2(t)), x(\xi_2(t))) + f_2\left(\begin{array}{c} t, y(\xi_2(t)), z(\xi_2(t)), x(\xi_2(t)), \\ \varphi(\int_0^{\beta_2(t)} g_2(t, s, y(\eta_2(s)), z(\eta_2(s)), x(\eta_2(s))) ds \end{array} \right) \\ z(t) = A_3(t) + h_3(t, z(\xi_3(t)), x(\xi_3(t)), y(\xi_3(t))) + f_3\left(\begin{array}{c} t, z(\xi_3(t)), x(\xi_3(t)), y(\xi_3(t)), \\ \varphi(\int_0^{\beta_3(t)} g_3(t, s, z(\eta_3(s)), x(\eta_3(s)), y(\eta_3(s))) ds \end{array} \right) \end{cases}$$

2. Preliminaries

In this section, we recall notations, definitions and preliminary facts which are used throughout this paper. Denote by $\mathbb{R}$ the set of real numbers and put $\mathbb{R}_+ = [0, +\infty)$. Let $(E, \|.\|)$ be a real Banach space with zero element 0, and $\bar{B}(x, r)$ denotes the closed ball in E centered at x with radius r. The symbol $\bar{B}_r$ stand for the ball $\bar{B}(0, r)$. If X is a nonempty subset of E, we denote by $\overline{X}$, $ConvX$ the closure and the closed convex hull of X respectively. Moreover, we denote by $\mathcal{M}_E$ the family of nonempty bounded subsets of E and by $\mathcal{N}_E$ its subfamily consisting of all relatively compact subsets of E.

In this paper, we will use axiomatically defined measures of noncompactness as presented in the book [11].

Definition 2.1. *([11]) A mapping $\mu : \mathcal{M}_E \to [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following condition:*

M_1) The family $kerl\mu = \{X \in \mathcal{M}_E,\ \mu(X) = 0\}$ is nonempty and $ker\mu \subseteq \mathcal{N}_E$.

M_2) If $X \subseteq Y$ then $\mu(X) = \mu(Y)$.

M_3) $\mu\left(\overline{X}\right) = \mu(X)$.

M_4) $\mu(convX) = \mu(X)$.

M_5) $\mu(\lambda X + (1-\lambda) Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for any $\lambda \in [0, 1)$.

M_6) If (X_n) is a sequence of closed sets from $\mathcal{M}_E$ such that $X_{n+1} \subseteq X_n$, $(n \geq 1)$ and $lim_{n\to\infty}\mu(X_n) = 0$, then the intersection set $X_\infty = \cap_{n=1}^{\infty} X_n$ is nonempty.

The family $ker\mu$ described in (M_1) said to be the kernel of the measure of noncompactness μ. Observe that the intersection set X_∞ from (M_6) is a member of the family $ker\mu$. In fact, since $\mu(X_\infty) \leq \mu(x_n)$ for any n, we infer that $\mu(X_\infty) = 0$. This yields that $X_\infty \in ker\mu$.

Now we present the definition of a tripled fixed point for a bivariate vector function which we need in the proof of main results and a useful theorem in [11] related to the construction of a measure of noncompactness on finite product space.

Definition 2.2. *([15]) An element $(x, y, z) \in X \times X \times X$ is called the tripled fixed point of mapping $T : X \times X \times X \to X$ if*

$$\begin{cases} T(x, y, z) = x \\ T(y, x, y) = y \\ T(z, y, x) = z \end{cases} . \tag{1}$$

Theorem 2.3. *([11]) Suppose that $\mu_1, \mu_2, ..., \mu_n$ be the measures of noncompactness in the Banach spaces $E_1, E_2, ..., E_n$ respectively. Moreover, assume that the function $F :[0,\infty)^n \to [0,\infty)$ is convex and $F(x_1, x_2, ..., x_n) = 0$ if and only if $x_i = 0$ for $i = 1, 2, ..., n$. Then $\widetilde{\mu}(X) = F(\mu_1(X), \mu_2(X), ..., \mu_n(X))$ define a measure of noncompactness at $E_1 \times E_2 \times ...., \times E_n$ where X_i denotes the projection of X into E_i for $i = 1, 2, ..., n$.*

Theorem 2.4. *Suppose that μ_1, μ_2, μ_3 be the measures of noncompactness in the Banach spaces E_1, E_2, E_3 respectively. Moreover, assume that the function $F :[0,\infty)^3 \to [0,\infty)$ is convex and $F(x_1, x_2, x_3) = 0$ if and only if $x_i = 0$ for $i = 1, 2, 3$. Then $\widetilde{\mu}(X) = F(\mu_1(X), \mu_2(X), \mu_3(X))$ define a measure of noncompactness on $E_1 \times E_2 \times E_3$ where X_i denotes the projection of X into E_i for $i = 1, 2, 3$.*

Example 2.5. *Assume that μ be a measure of noncompactness on a Banach space E, consider $F(x, y, z) = x + y + z$ for every $(x, y, z) \in [0,\infty)^3$, then F is convex and if $F(x, y, z) = x + y + z = 0$ since $x \geq 0, y \geq 0, z \geq 0$ thus $x = y = z = 0$. So $\widetilde{\mu}(X) = \mu(X_1) + \mu(X_2) + \mu(X_3)$ is a measures of noncompactness on $E \times E \times E$. Which X_i denotes the projection of X into E_i for $i = 1, 2, 3$.*

Example 2.6. *Assume that μ be a measure of noncompactness on a Banach space E, consider $F(x, y, z) = \max\{x, y, z\}$ for every $(x, y, z) \in [0,\infty)^3$then F is convex and if $F(x, y, z) = \max\{x, y, z\} = 0$ since $x \geq 0, y \geq 0, z \geq 0$ thus $x = y = z = 0$ unto Theorem 2.3 $\widetilde{\mu}(X) = \max\{\mu(X_1), \mu(X_2), \mu(X_3)\}$ is a measures of noncompactness on $E \times E \times E$. Which X_i denotes the projection of X into E_i for $i = 1, 2, 3$.*

Theorem 2.7. *(Schauder[3]) Let Ω be a closed and convex subset of a Banach space E. Then every compact, continuous map $T : \Omega \to \Omega$ has at least one fixed point.*

Theorem 2.8. *(Darbo[8]) Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous mapping. Assume that there exists a constant $k \in [0, 1)$ such that $\mu(T(X)) \leq k\mu(X)$ for any $X \subset \Omega$. Then T has a fixed point.*

3. Main Results

In this section, we give and prove some theorems for the existence of tripled fixed point to a special class of operators. This basic result will be used in the next section.

First, we introduce the class Ψ of all functions $\psi : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ which have the following properties:

(k_1) $\psi(t_1 + t_2, s_1 + s_2, r_1 + r_2) \leq \psi(t_1, s_1, r_1) + \psi(t_2, s_2, r_2)$

(k_2) $\psi(t, s, r) = 0 \iff t = r = s = 0$

(k_3) ψ is a lower semicontinuous function on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ *i.e*, for every arbitrary sequences $\{a_n\}, \{b_n\}, \{c_n\}$ we have

$$\psi(\liminf_{n\to\infty} a_n, \liminf_{n\to\infty} b_n, \liminf_{n\to\infty} c_n) \leq \liminf_{n\to\infty} \psi(a_n, b_n, c_n).$$

For example the functions $\psi_1(t, s, r) = \ln(t + s + r + 1)$ and $\psi_2(t, s, r) = max\{t, s, r\}$ belong to Ψ.

Theorem 3.1. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E and let μ be a measure of noncompactness. Moreover, assume that $T : \Omega \times \Omega \times \Omega \to \Omega \times \Omega \times \Omega$ be a continuous function satisfying*

$$\phi(\widetilde{\mu}(T(X))) \leq \phi(\widetilde{\mu}(X)) - \psi(\widetilde{\mu}(X), \widetilde{\mu}(X), \widetilde{\mu}(X)) \tag{2}$$

for any nonempty subset X of $\Omega \times \Omega \times \Omega$, where $\widetilde{\mu}$ is defined by

$$\widetilde{\mu}(X) = F(\mu_1(X), \mu_2(X), \mu_3(X)),$$

and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous mapping and $\psi \in \Psi$. Then T has at least one fixed point in $\Omega \times \Omega \times \Omega$.

Proof. *By induction we construct the sequence* $\{\Omega_n \times \Omega_n \times \Omega_n\}_{n=1}^{\infty}$ *such that* $\Omega_0 \times \Omega_0 \times \Omega_0 = \Omega \times \Omega \times \Omega$ *and* $\Omega_n \times \Omega_n \times \Omega_n = convT(\Omega_{n-1} \times \Omega_{n-1} \times \Omega_{n-1})$ *for* $n = 1, 2, 3, ...$

Now we have

$$T(\Omega_0 \times \Omega_0 \times \Omega_0) = T(\Omega \times \Omega \times \Omega) \subset \Omega \times \Omega \times \Omega = \Omega_0 \times \Omega_0 \times \Omega_0,$$

and

$$\Omega_1 \times \Omega_1 \times \Omega_1 = convT(\Omega_0 \times \Omega_0 \times \Omega_0) \subseteq conv\, \Omega_0 \times \Omega_0 \times \Omega_0 = \Omega_0 \times \Omega_0 \times \Omega_0.$$

So

$$\Omega_2 \times \Omega_2 \times \Omega_2 = convT(\Omega_1 \times \Omega_1 \times \Omega_1) \subseteq convT(\Omega_0 \times \Omega_0 \times \Omega_0) = \Omega_1 \times \Omega_1 \times \Omega_1$$

Thus by continuing this process, we obtain

$$.... \subset \Omega_n \times \Omega_n \times \Omega_n \subset .... \subset \Omega_2 \times \Omega_2 \times \Omega_2 \subset \Omega_1 \times \Omega_1 \times \Omega_1.$$

If there exists an integer number $N > 0$ *such that* $\widetilde{\mu}(\Omega_N \times \Omega_N \times \Omega_N) = 0$ *then* $\Omega_N \times \Omega_N \times \Omega_N$ *is relatively compact and since*

$$T(\Omega_N \times \Omega_N \times \Omega_N) \subseteq convT(\Omega_N \times \Omega_N \times \Omega_N) = \Omega_{N+1} \times \Omega_{N+1} \times \Omega_{N+1} \subseteq \Omega_N \times \Omega_N \times \Omega_N,$$

therefore Theorem 2.7 implies that T has a fixed point. So we can assume that $\widetilde{\mu}(\Omega_n \times \Omega_n \times \Omega_n) > 0$ *for any* $n \geq 0$.

By our assumption, we get

$$\begin{aligned} \phi(\widetilde{\mu}(\Omega_{n+1} \times \Omega_{n+1} \times \Omega_{n+1})) &= \phi(\widetilde{\mu}(convT(\Omega_n \times \Omega_n \times \Omega_n))) \\ &= \phi(\widetilde{\mu}(\Omega_n \times \Omega_n \times \Omega_n)) \\ &\leq \phi(\widetilde{\mu}(\Omega_n \times \Omega_n \times \Omega_n)) \\ &\quad -\psi(\widetilde{\mu}(\Omega_n \times \Omega_n \times \Omega_n), \widetilde{\mu}(\Omega_n \times \Omega_n \times \Omega_n), \widetilde{\mu}(\Omega_n \times \Omega_n \times \Omega_n)). \end{aligned}$$

Since the sequence $\widetilde{\mu}(\Omega_n \times \Omega_n \times \Omega_n)$ *is nonincreasing and non-negative real numbers, thus there is an* $r \geq 0$ *so that* $\widetilde{\mu}(\Omega_n \times \Omega_n \times \Omega_n) \to r$ *as* $n \to \infty$.

Now from (2) we have

$$\begin{aligned} \phi(r) &= \limsup_{n\to\infty} \phi(\widetilde{\mu}(\Omega_{n+1} \times \Omega_{n+1} \times \Omega_{n+1})) \\ &\leq \limsup_{n\to\infty} \phi(\widetilde{\mu}(\Omega_n \times \Omega_n \times \Omega_n)) \\ &- \liminf_{n\to\infty} \psi(\widetilde{\mu}(\Omega_n \times \Omega_n \times \Omega_n), \widetilde{\mu}(\Omega_n \times \Omega_n \times \Omega_n), \widetilde{\mu}(\Omega_n \times \Omega_n \times \Omega_n)) \\ &\leq \limsup_{n\to\infty} \phi(\widetilde{\mu}(\Omega_n \times \Omega_n \times \Omega_n)) \\ &- \psi\left(\liminf_{n\to\infty} \widetilde{\mu}(\Omega_n \times \Omega_n \times \Omega_n), \liminf_{n\to\infty} \widetilde{\mu}(\Omega_n \times \Omega_n \times \Omega_n), \liminf_{n\to\infty} \widetilde{\mu}(\Omega_n \times \Omega_n \times \Omega_n)\right) \\ &= \phi(r) - \psi(r, r, r). \end{aligned}$$

Consequently $\psi(r, r, r) = 0$ *so* $r = 0$. *Therefore we deduce that* $\widetilde{\mu}(\Omega_n \times \Omega_n \times \Omega_n) \to 0$ *as* $n \to \infty$. *Since* $\Omega_{n+1} \times \Omega_{n+1} \times \Omega_{n+1} \subseteq \Omega_n \times \Omega_n \times \Omega_n$, *thus by axiom* (M_6) *of Definition 2.1 we derive that the set* $\Omega_\infty \times \Omega_\infty \times \Omega_\infty = \cap_{n=1}^{\infty} \Omega_n \times \Omega_n \times \Omega_n$ *is a nonempty convex closed set, invariant under the operator T and belongs to* $Ker\mu$. *Now by Theorem 2.7 T has at least one fixed point in* $\Omega_\infty \times \Omega_\infty \times \Omega_\infty$ *and hence in* $\Omega \times \Omega \times \Omega$. □

Theorem 3.2. *Let Ω be a closed, bounded, convex and nonempty subset of Banach space E. Moreover, assume that $T : \Omega \times \Omega \times \Omega \to \Omega$ be a continuous function where satisfying at the following condition.*

$$\phi\left(\mu\left(T\left(X_1 \times X_2 \times X_3\right)\right)\right) \leq \frac{1}{3}\phi\left(\mu\left(X_1\right) + \mu\left(X_2\right) + \mu\left(X_3\right)\right) - \psi\left(\mu\left(X_1\right), \mu\left(X_2\right), \mu\left(X_3\right)\right) \tag{3}$$

for every $X_1, X_2, X_3 \subseteq \Omega$ where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and linear function and for every $t, s \in \mathbb{R}_+$, and $\psi \in \Psi$. Then T has a tripled fixed point.

Proof. First note that Example 2.5 show that $\widetilde{\mu}\left(X\right) = \mu\left(X_1\right) + \mu\left(X_2\right) + \mu\left(X_3\right)$ is a measure of noncompactness in the space $E \times E \times E$. Where $X_i, i = 1, 2, 3$ denoted the natural projections of X into E. Now define $\widetilde{T}$ on the $\Omega \times \Omega \times \Omega$ by the formula $\widetilde{T}\ (x, y, z) = \left(T\left(x, y, z\right), T\left(y, z, x\right), T\left(z, x, y\right)\right)$, for every $(x, y, z) \in \Omega \times \Omega \times \Omega$. Since T is continuous so $\widetilde{T}$ is continuous on $\Omega \times \Omega \times \Omega$. We claim that $\widetilde{T}$ satisfies all the condition of Theorem 3.1. To prove this, let $X \subset \Omega \times \Omega \times \Omega$ be a nonempty subset. Then by (M_2) and (3) we get

$$\begin{aligned}
\phi\left(\widetilde{\mu}\left(\widetilde{T}\ (X)\right)\right) &\leq \psi\left(\widetilde{\mu}\left(T\left(X_1 \times X_2 \times X_3\right), T(X_2 \times X_3 \times X_1), T(X_3 \times X_1 \times X_2)\right)\right.\\
&= \phi\left(\mu\left(X_1 \times X_2 \times X_3\right) + \mu\left(X_2 \times X_3 \times X_1\right) + \mu\left(X_3 \times X_1 \times X_2\right)\right)\\
&= \phi\left(\mu\left(T\left(X_1 \times X_2 \times X_3\right)\right)\right) + \phi\left(\mu\left(T(X_2 \times X_3 \times X_1)\right)\right)\\
&+ \phi\left(\mu\left(T(X_3 \times X_1 \times X_2)\right)\right)\\
&\leq \frac{1}{3}\phi\left(\left(\mu\left(X_1\right) + \mu\left(X_2\right) + \mu\left(X_3\right)\right)\right) - \psi\left(\mu\left(X_1\right), \mu\left(X_2\right), \mu\left(X_3\right)\right)\\
&+ \frac{1}{3}\phi\left(\left(\mu\left(X_1\right) + \mu\left(X_2\right) + \mu\left(X_3\right)\right)\right) - \psi\left(\mu\left(X_2\right), \mu\left(X_3\right), \mu\left(X_1\right)\right)\\
&+ \frac{1}{3}\phi\left(\left(\mu\left(X_1\right) + \mu\left(X_2\right) + \mu\left(X_3\right)\right)\right) - \psi\left(\mu\left(X_3\right), \mu\left(X_1\right), \mu\left(X_2\right)\right)\\
&= \phi\left(\mu\left(X_1\right) + \mu\left(X_2\right) + \mu\left(X_3\right)\right)\\
&- \begin{bmatrix} \psi\left(\mu\left(X_1\right), \mu\left(X_2\right), \mu\left(X_3\right)\right) + \psi\left(\mu\left(X_2\right), \mu\left(X_3\right), \mu\left(X_1\right)\right) \\ +\psi\left(\mu\left(X_3\right), \mu\left(X_1\right), \mu\left(X_2\right)\right) \end{bmatrix}\\
&\leq \phi\left(\mu\left(X_1\right) + \mu\left(X_2\right) + \mu\left(X_3\right)\right)\\
&- \psi\begin{pmatrix} \mu\left(X_1\right) + \mu\left(X_2\right) + \mu\left(X_3\right), \mu\left(X_2\right) + \mu\left(X_3\right) + \mu\left(X_1\right), \\ \mu\left(X_3\right) + \mu\left(X_1\right) + \mu\left(X_2\right) \end{pmatrix}\\
&= \phi\left(\widetilde{\mu}\left(X\right)\right) - \psi\left(\widetilde{\mu}\left(X\right), \widetilde{\mu}\left(X\right), \widetilde{\mu}\left(X\right)\right).
\end{aligned}$$

So we get

$$\phi\left(\widetilde{\mu}\left(\widetilde{T}\ (X)\right)\right) \leq \phi\left(\widetilde{\mu}\left(X\right)\right) - \psi\left(\widetilde{\mu}\left(X\right), \widetilde{\mu}\left(X\right), \widetilde{\mu}\left(X\right)\right)$$

hence, by using Theorem 3.1 T has at least one tripled fixed point. □

Corollary 3.3. *Let Ω be a nonempty, closed, bounded and convex subset of Banach space E and μ be a measure of noncompactness. Moreover, assume that*
$T : \Omega \times \Omega \times \Omega \to \Omega$ is a continuous function such that there exist nonnegative constant k with $0 < k < \frac{1}{3}$. If

$$\mu\left(T(X_1 \times X_2 \times X_3)\right) \leq k\mu\left(X_1 \times X_2 \times X_3\right),$$

for every $X_1, X_2, X_3 \subseteq \Omega$. Then T has at least one tripled fixed point.

Proof. Taking $\phi\left(t\right) = t$, $t > 0$ and $\psi\left(t, s, r\right) = \frac{1-3k}{3}\left(s + t + r\right)$ in Theorem 3.2 we obtain the desired result. □

Corollary 3.4. *Let Ω be a nonempty, closed, bounded and convex subset of Banach space E and μ be an arbitrary measure of noncompactness. Moreover, assume that*
$T : \Omega \times \Omega \times \Omega \to \Omega$ is a continuous function such that there exist nonnegative constants k_1, k_2, k_3 such that $k_1 + k_2 + k_3 < 1$. If

$$\mu\left(T(X_1 \times X_2 \times X_3)\right) \leq \frac{k_1}{3}\mu\left(X_1\right) + \frac{k_2}{3}\mu\left(X_2\right) + \frac{k_3}{3}\mu\left(X_3\right)$$

for every $X_1, X_2, X_3 \subseteq \Omega$. Then T has at least one tripled fixed point.

Proof. Taking $\phi\left(t\right) = t$, $t > 0$ and $\psi\left(t, s, r\right) = \left(\frac{1-k_1}{3}\right)t + \left(\frac{1-k_2}{3}\right)s + \left(\frac{1-k_3}{3}\right)r$ in Theorem 3.2 we conclude that T has at least one tripled fixed point in $\Omega \times \Omega \times \Omega$. □

Corollary 3.5. *Let Ω be a nonempty, closed, bounded and convex subset of Banach space E and μ be a arbitrary measure of noncompactness. Moreover, assume that*
$T : \Omega \times \Omega \times \Omega \to \Omega$ is a continuous function and there exists a nonnegative constant k with $0 < k < 1$ such that

$$\mu\left(T(X_1 \times X_2 \times X_3)\right) \leq k \max\left\{\mu\left(X_1\right), \mu\left(X_2\right), \mu\left(X_3\right)\right\}$$

for every $X_1, X_2, X_3 \subseteq \Omega$.
Then T has a tripled fixed point i.e.

$$\begin{cases} T\left(x, y, z\right) = x \\ T\left(y, x, y\right) = y \\ T\left(z, y, x\right) = z \end{cases}.$$

Proof. Taking $\phi\left(t\right) = t$, $t > 0$ and $\psi\left(t, s, r\right) = (1 - k) \max\left\{t, s, r\right\}$ in Theorem 3.2 we conclude that T has at least one tripled fixed point in $\Omega \times \Omega \times \Omega$. □

Corollary 3.6. *Let Ω be a nonempty, closed, bounded and convex subset of Banach space E and μ be a arbitrary measure of noncompactness. Moreover, assume that $T : \Omega \times \Omega \times \Omega \to \Omega$ is a continuous function such that there exists a nonnegative constants k_1, k_2, k_3 such that $k_1 + k_2 + k_3 < 1$. If*

$$\mu\left(T(X_1 \times X_2 \times X_3)\right) \leq k_1\mu\left(X_1\right) + k_2\mu\left(X_2\right) + k_3\mu\left(X_3\right),$$

for every $X_1, X_2, X_3 \subseteq \Omega$. Then T has at least a tripled fixed point.

Proof. It should be noted that

$$\begin{aligned}
&\mu\left(T\left(X_1 \times X_2 \times X_3\right)\right) \\
&\leq k_1\mu\left(X_1\right) + k_2\mu\left(X_2\right) + k_3\mu\left(X_3\right) \\
&\leq k_1 \max\left\{\mu\left(X_1\right), \mu\left(X_2\right), \mu\left(X_3\right)\right\} + k_2 \max\left\{\mu\left(X_1\right), \mu\left(X_2\right), \mu\left(X_3\right)\right\} \\
&+ k_3 \max\left\{\mu\left(X_1\right), \mu\left(X_2\right), \mu\left(X_3\right)\right\} \\
&= (k_1 + k_2 + k_3) \max\left\{\mu\left(X_1\right), \mu\left(X_2\right), \mu\left(X_3\right)\right\} \\
&= k \max\left\{\mu\left(X_1\right), \mu\left(X_2\right), \mu\left(X_3\right)\right\}
\end{aligned}$$

where $k = k_1 + k_2 + k_3 < 1$. Now from Corollary (3.5), T has at least one tripled fixed point. □

Corollary 3.7. *Let Ω be a nonempty, closed, bounded and convex subset of Banach space E and μ be an arbitrary measure of noncompactness. Moreover, assume that*
$T : \Omega \times \Omega \times \Omega \to \Omega$ is a continuous function such that

$$\mu\left(T\left(X_1 \times X_2 \times X_3\right)\right) \leq \frac{\mu\left(X_1\right) + \mu\left(X_2\right) + \mu\left(X_3\right)}{3} - \ln\left(\mu\left(X_1\right) + \mu\left(X_2\right) + \mu\left(X_3\right) + 1\right)$$

for every $X_1, X_2, X_3 \subseteq \Omega$. Then T has at least one tripled fixed point in $\Omega \times \Omega \times \Omega$.

Proof. *Taking $\phi(t) = t$, $t > 0$ and $\psi(s,t,r) = ln(s+t+r+1)$ and using Theorem 3.2 we conclude that T has one tripled fixed point in $\Omega \times \Omega \times \Omega$.* $\square$

In this part of the paper we will introduce another class of functions and in this direction, we present some tripled fixed point theorem.

First, we consider the usual order relation"$\preccurlyeq$" on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ as follows :

$$(s_1, t_1, r_1) \preccurlyeq (s_2, t_2, r_2) \Longleftrightarrow s_1 \le s_2, t_1 \le t_2, r_1 \le r_2$$

for every $s_1, t_1, r_1, s_2, t_2, r_2, s_3, t_3, r_3 \in \mathbb{R}_+$.

Now we denote by Φ, the class of all functions $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ with the following properties:

φ_1) ϕ is continuous and nondecreasing function on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$.

φ_2) $\phi(t,t,t) < t$ for every $t > 0$.

φ_3) $\frac{1}{3}\left[\phi(s_1,t_1,r_1) + \phi(s_2,t_2,r_2) + \phi(s_3,t_3,r_3)\right] \le \phi\left(\frac{s_1+s_2+s_3}{3}, \frac{t_1+t_2+t_3}{3}, \frac{r_1+r_2+r_3}{3}\right)$

for every $s_1, t_1, r_1, s_2, t_2, r_2, s_3, t_3, r_3 \in \mathbb{R}_+$.

For example the functions $\phi_1(s,t,r) = \ln\left(1 + \frac{s+t+r}{3}\right)$ and $\phi_2(s,t,r) = k_1 t + k_2 s + k_3 r$ where $k_1, k_2, k_3 \in \mathbb{R}_+$ and $k_1 + k_2 + k_3 < 1$ belong to Φ.

Theorem 3.8. *Let Ω be a closed, bounded, convex and nonempty subset of Banach space E. Moreover, assume that T: $\Omega \times \Omega \times \Omega \to \Omega \times \Omega \times \Omega$ be a continuous function where satisfying at the following condition*

$$\widetilde{\mu}(T(X)) \le \phi(\widetilde{\mu}(X), \widetilde{\mu}(X), \widetilde{\mu}(X))$$

for every nonempty subset X of $\Omega \times \Omega \times \Omega$ and also $\widetilde{\mu}$ as $\widetilde{\mu}(X) = F(\mu(X_1), \mu(X_2), \mu(X_3))$ and $\phi \in \Phi$. Then T has at least one fixed point in $\Omega \times \Omega \times \Omega$.

Proof. *By induction we construct the sequence $\{\Omega_n \times \Omega_n \times \Omega_n\}_{n=1}^{\infty}$ such that $\Omega_0 \times \Omega_0 \times \Omega_0 = \Omega \times \Omega \times \Omega$ and $\Omega_n \times \Omega_n \times \Omega_n = convT(\Omega_{n-1} \times \Omega_{n-1} \times \Omega_{n-1})$ for $n = 1, 2, 3, ...$*

Similar to the proof of Theorem 3.1 we obtain

$$... \subset \Omega_n \times \Omega_n \times \Omega_n \subset .... \subset \Omega_2 \times \Omega_2 \times \Omega_2 \subset \Omega_1 \times \Omega_1 \times \Omega_1.$$

If there exists an integer number $N > 0$ such that $\widetilde{\mu}(\Omega_N \times \Omega_N \times \Omega_N) = 0$ then $\Omega_N \times \Omega_N \times \Omega_N$ is relatively compact and since

$$\ldots T(\Omega_N \times \Omega_N \times \Omega_N) \subseteq convT(\Omega_N \times \Omega_N \times \Omega_N) = \Omega_{N+1} \times \Omega_{N+1} \times \Omega_{N+1} \subseteq \Omega_N \times \Omega_N \times \Omega_N,$$

therefore Theorem 2.7 implies that T has a fixed point. So we can assume that $\widetilde{\mu}(\Omega_N \times \Omega_N \times \Omega_N) > 0$ for any $n \ge 0$. By our assumption, we get

$$\begin{aligned}
\widetilde{\mu}(\Omega_{n+1} \times \Omega_{n+1} \times \Omega_{n+1}) &= \widetilde{\mu}(convT(\Omega_n \times \Omega_n \times \Omega_n)) \\
&= \widetilde{\mu}(\Omega_n \times \Omega_n \times \Omega_n) \\
&\le \phi\left(\begin{array}{c} \widetilde{\mu}(\Omega_n \times \Omega_n \times \Omega_n), \widetilde{\mu}(\Omega_n \times \Omega_n \times \Omega_n), \\ \widetilde{\mu}(\Omega_n \times \Omega_n \times \Omega_n) \end{array}\right).
\end{aligned}$$

Since the sequence $\widetilde{\mu}(\Omega_n \times \Omega_n \times \Omega_n)$ is nonincreasing and nonnegative real numbers, thus, there is an $r \ge 0$ so that $\widetilde{\mu}(\Omega_n \times \Omega_n \times \Omega_n) \to r$ as $n \to \infty$. We claim that $r = 0$. On the contrary if $r > 0$ then we obtain

$$\begin{aligned} r &= \lim_{n\to\infty}\widetilde{\mu}(\Omega_{n+1}\times\Omega_{n+1}\times\Omega_{n+1}) \\ &\le \phi\left(\lim_{n\to\infty}\widetilde{\mu}\left(\Omega_n\times\Omega_n\times\Omega_n\right),\lim_{n\to\infty}\widetilde{\mu}\left(\Omega_n\times\Omega_n\times\Omega_n\right),\lim_{n\to\infty}\widetilde{\mu}\left(\Omega_n\times\Omega_n\times\Omega_n\right)\right) \\ &= \phi\left(r,r,r\right) < r. \end{aligned}$$

Which is a contradiction. Hence $\widetilde{\mu}\left(\Omega_n\times\Omega_n\times\Omega_n\right)=0$.

Then $\Omega_n\times\Omega_n\times\Omega_n$ *is relatively compact. On the other hand,* $\Omega_{n+1}\times\Omega_{n+1}\times\Omega_{n+1}\subseteq\Omega_n\times\Omega_n\times\Omega_n$ *thus by axiom* (M_6) *of Definition 2.1 we derive that the set* $\Omega_\infty\times\Omega_\infty\times\Omega_\infty=\cap_{n=1}^{\infty}\Omega_n\times\Omega_n\times\Omega_n$ *is a nonempty closed convex set, invariant under the operator T and belongs to Ker*μ*. Now by Theorem 2.7 T has at least one fixed point in* $\Omega_\infty\times\Omega_\infty\times\Omega_\infty$ *and hence in* $\Omega\times\Omega\times\Omega$. □

Theorem 3.9. *Let* Ω *be a closed, bounded, convex and nonempty subset of Banach space E and* μ *be an arbitrary measure of noncompactness. Moreover, assume that* T_i: $\Omega\times\Omega\times\Omega\to\Omega$, $i=1,2,3$ *are continuous functions where satisfying at the following condition:*

$$\begin{cases} \mu\left(T_1\left(X_1\times X_2\times X_3\right)\right)\le\phi\left(\mu\left(X_1\right),\mu\left(X_2\right),\mu\left(X_3\right)\right) \\ \mu\left(T_2\left(X_2\times X_3\times X_1\right)\right)\le\phi\left(\mu\left(X_2\right),\mu\left(X_3\right),\mu\left(X_1\right)\right) \\ \mu\left(T_3\left(X_3\times X_1\times X_2\right)\right)\le\phi\left(\mu\left(X_3\right),\mu\left(X_1\right),\mu\left(X_2\right)\right) \end{cases}$$

for all $X_1,X_2,X_3\subseteq\Omega$, *where* $\phi\in\Phi$. *Then there exist* x^*,y^*,z^* *such that*

$$\begin{cases} T_1(x^*,y^*,z^*) &= x^* \\ T_2(y^*,z^*,x^*) &= y^* \\ T_3(z^*,x^*,y^*) &= z^* \end{cases}.$$

Proof. *First* $\widetilde{\mu}\left(X\right)=\mu\left(X_1\right)+\mu\left(X_2\right)+\mu\left(X_3\right)$ *is a measure of noncompactness in* $E\times E\times E$ *which* X_i, $i=1,2,3$ *is natural projection of X into E. Now we define* $\widetilde{T}:\Omega\times\Omega\times\Omega\to\Omega\times\Omega\times\Omega$ *with the following:*

$$\widetilde{T}\left(x,y,z\right)=\left(T_1\left(x,y,z\right),T_2(y,z,x),T_3\left(z,x,y\right)\right)$$

for every $(x,y,z)\in\Omega\times\Omega\times\Omega$. *It is easy to see that* $\widetilde{T}$ *is continuous on* $\Omega\times\Omega\times\Omega$. *We claim that* $\widetilde{T}$ *satisfying in all condition of the Theorem 3.8. For this, assume that* $X\subset\Omega\times\Omega\times\Omega$ *be a nonempty subset. Then the condition* (M_2) *of Definition 2.1 and Theorem 3.8 imply that :*

$$\begin{aligned} \widetilde{\mu}\left(\widetilde{T}\left(X\right)\right) &\le \widetilde{\mu}\left(T_1\left(X_1\times X_2\times X_3\right),T_2(X_2\times X_3\times X_1),T_3(X_3\times X_1\times X_2)\right) \\ &= \mu\left(T_1\left(X_1\times X_2\times X_3\right)\right)+\mu(T_2(X_2\times X_3\times X_1))+\mu(T_3(X_3\times X_1\times X_2)) \\ &\le \phi\left(\mu\left(X_1\right),\mu\left(X_2\right),\mu\left(X_3\right)\right)+\phi\left(\mu\left(X_2\right),\mu\left(X_3\right),\mu\left(X_1\right)\right) \\ &+ \phi\left(\mu\left(X_3\right),\mu\left(X_1\right),\mu\left(X_2\right)\right) \\ &\le 3\phi\left(\frac{\mu\left(X_1\right)+\mu\left(X_2\right)+\mu\left(X_3\right)}{3},\frac{\mu\left(X_1\right)+\mu\left(X_2\right)+\mu\left(X_3\right)}{3},\frac{\mu\left(X_1\right)+\mu\left(X_2\right)+\mu\left(X_3\right)}{3}\right). \end{aligned}$$

Now suffice we chose that $\widetilde{\mu}'=\frac{1}{3}\widetilde{\mu}$, *we get*

$$\widetilde{\mu}'\left(\widetilde{T}\left(X\right)\right)\le\phi\left(\widetilde{\mu}'\left(X\right),\widetilde{\mu}'\left(X\right),\widetilde{\mu}'\left(X\right)\right).$$

Since $\widetilde{\mu}'$ *is a measure of noncompactness, so by Theorem 3.8* $\widetilde{T}$ *has at least a fixed point i.e. there exist* $(x^*,y^*,z^*)\in\Omega\times\Omega\times\Omega$ *such that*

$$\begin{cases} T_1(x^*,y^*,z^*) &= x^* \\ T_2(y^*,z^*,x^*) &= y^* \\ T_3(z^*,x^*,y^*) &= z^* \end{cases}.$$

□

Corollary 3.10. *Let Ω be a nonempty, closed, bounded and convex subset of Banach space E, and μ be an arbitrary measure of noncompactness. Moreover, assume that T_i: $\Omega \times \Omega \times \Omega \to \Omega$, $i = 1, 2, 3$ are continuous functions where satisfying at the following condition:*

$$\begin{aligned} \mu\left(T_1(X_1 \times X_2 \times X_3)\right) &\leq k_1\mu\left(X_1\right) + k_2\mu\left(X_2\right) + k_3\mu\left(X_3\right), \\ \mu\left(T_2(X_2 \times X_3 \times X_1)\right) &\leq k_1\mu\left(X_2\right) + k_2\mu\left(X_3\right) + k_3\mu\left(X_1\right), \\ \mu\left(T_3(X_3 \times X_1 \times X_2)\right) &\leq k_1\mu\left(X_3\right) + k_2\mu\left(X_1\right) + k_3\mu\left(X_2\right), \end{aligned}$$

for each $X_1, X_2, X_3 \subseteq \Omega$, where k_1, k_2, k_3 are nonnegative constants such that $k_1 + k_2 + k_3 < 1$. Then there exist $(x^*, y^*, z^*) \in \Omega \times \Omega \times \Omega$ such that

$$\begin{cases} T_1(x^*, y^*, z^*) &= x^* \\ T_2(y^*, z^*, x^*) &= y^* \\ T_3(z^*, x^*, y^*) &= z^* \end{cases}.$$

Proof. *Taking $\phi(s, t, r) = k_1 t + k_2 s + k_3 r$ in Theorem 3.9, we obtain the desired conclusion.* □

Corollary 3.11. *Let Ω be a nonempty, closed, bounded and convex subset of Banach space E, and μ be a arbitrary measure of noncompactness. Moreover, assume that T_i: $\Omega \times \Omega \times \Omega \to \Omega$, $i = 1, 2, 3$ are continuous functions where satisfying at the following condition:*

$$\begin{aligned} \mu\left(T_1(X_1 \times X_2 \times X_3)\right) &\leq \ln\left(1 + \frac{\mu\left(X_1\right) + \mu\left(X_2\right) + \mu\left(X_3\right)}{3}\right), \\ \mu\left(T_2(X_2 \times X_3 \times X_1)\right) &\leq \ln\left(1 + \frac{\mu\left(X_1\right) + \mu\left(X_2\right) + \mu\left(X_3\right)}{3}\right), \\ \mu\left(T_3(X_3 \times X_1 \times X_2)\right) &\leq \ln\left(1 + \frac{\mu\left(X_1\right) + \mu\left(X_2\right) + \mu\left(X_3\right)}{3}\right). \end{aligned}$$

for every $X_1, X_2, X_3 \subseteq \Omega$. Then there exist $(x^*, y^*, z^*) \in \Omega \times \Omega \times \Omega$ such that

$$\begin{cases} T_1(x^*, y^*, z^*) &= x^* \\ T_2(y^*, z^*, x^*) &= y^* \\ T_3(z^*, x^*, y^*) &= z^* \end{cases}.$$

Proof. *Taking$\phi(s, t, r) = \ln\left(1 + \frac{s+t+r}{3}\right)$ in Theorem 3.9, we obtain the desired conclusion.* □

Now, we present a three-dimension version of Corollary 3.5 in Aghajani et al. [5]

Corollary 3.12. *Let Ω be a nonempty, closed, bounded and convex subset of Banach space E and let $F_i : \Omega \times \Omega \times \Omega \longrightarrow E$ for $i = 1, 2, 3$ are operators such that*

$$\begin{aligned} \left\|F_1(x, y, z) - F_1(u, v, w)\right\| &\leq \phi\left(\|x - u\|, \left\|y - v\right\|, \|z - w\|\right), \\ \left\|F_2(y, z, x) - F_2(v, w, u)\right\| &\leq \phi\left(\left\|y - v\right\|, \|z - w\|, \|x - u\|\right), \\ \left\|F_2(z, x, y) - F_2(w, u, v)\right\| &\leq \phi\left(\|z - w\|, \|x - u\|, \left\|y - v\right\|\right). \end{aligned}$$

Assume that $G_i : \Omega \times \Omega \times \Omega \longrightarrow E$ be continuous and compact operators and the operators $T_i : \Omega \times \Omega \times \Omega \longrightarrow \Omega$ for $i = 1, 2, 3$ defined as the following

$$\begin{aligned} \left\|T_1(x, y, z) - T_1(u, v, w)\right\| &\leq \left\|F_1(x, y, z) - F_1(u, v, w)\right\| + \psi\left(\left\|G_1(x, y, z) - G_1(u, v, w)\right\|\right) \\ \left\|T_2(y, z, x) - T_2(v, w, u)\right\| &\leq \left\|F_2(y, z, x) - F_2(v, w, u)\right\| + \psi\left(\left\|G_2(y, z, x) - G_2(v, w, u)\right\|\right) \\ \left\|T_2(z, x, y) - T_2(w, u, v)\right\| &\leq \left\|F_2(z, x, y) - F_2(w, u, v)\right\| + \psi\left(\left\|G_2(z, x, y) - G_2(w, u, v)\right\|\right) \end{aligned}$$

where $\phi \in \Phi$ and $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous and nondecreasing function and $\psi(0) = 0$.
Then there exist $(x^*, y^*, z^*) \in \Omega \times \Omega \times \Omega$ such that

$$\begin{cases} T_1(x^*, y^*, z^*) &= x^* \\ T_2(y^*, z^*, x^*) &= y^* \\ T_3(z^*, x^*, y^*) &= z^* \end{cases}.$$

Proof. Assume that X_1, X_2, X_3 are the subset of Ω. From the definition of Kuratowski measure of noncompactness for every $\epsilon > 0$, conclusion that there exist $A_1, A_2, ..., A_n, B_1, B_2, ..., B_n$ and $C_1, C_2, ..., C_n$ such that

$$\begin{aligned} X_1 \times X_2 \times X_3 &\subset \cup_{k=1}^{n} A_k \\ X_2 \times X_3 \times X_1 &\subset \cup_{k=1}^{n} B_k \\ X_3 \times X_1 \times X_2 &\subset \cup_{k=1}^{n} C_k \end{aligned}$$

and

$$\begin{aligned} diam\,(F_1\,(A_k)) &\leq \alpha\,(F_1\,(X_1 \times X_2 \times X_3)) + \epsilon, \\ diam\,(G_1\,(A_k)) &< \epsilon, \end{aligned} \tag{4}$$

$$\begin{aligned} diam\,(F_2\,(B_k)) &\leq \alpha\,(F_2\,(X_1 \times X_2 \times X_3)) + \epsilon, \\ diam\,(G_2\,(B_k)) &< \epsilon, \end{aligned} \tag{5}$$

$$\begin{aligned} diam\,(F_3\,(C_k)) &\leq \alpha\,(F\,(X_1 \times X_2 \times X_3)) + \epsilon, \\ diam\,(G_3\,(C_k)) &< \epsilon. \end{aligned} \tag{6}$$

Let $k \in \{1, 2, \ldots n\}$ be arbitrary. Then for every $a_1, a_2 \in A_k, b_1, b_2 \in B_k$ and $c_1, c_2 \in C_k$ we have

$$\begin{aligned} \|T_1\,(a_1) - T_1\,(a_2)\| &\leq \|F_1\,(a_1) - F_1\,(a_2)\| + \psi\,(\|G_1\,(a_1) - G_1\,(a_2)\|)\,, \\ \|T_2\,(b_1) - T_2\,(b_2)\| &\leq \|F_2\,(b_1) - F_2\,(b_2)\| + \psi\,(\|G_2\,(b_1) - G_2\,(b_2)\|)\,, \\ \|T_3\,(c_1) - T_3\,(c_2)\| &\leq \|F_3\,(c_1) - F_3\,(c_2)\| + \psi\,(\|G_3\,(c_1) - G_3\,(c_2)\|)\,, \end{aligned}$$

therefore from the properties ψ we obtain

$$\begin{aligned} diam\,(T_1\,(A_k)) &\leq diam\,(F_1\,(A_k)) + \psi\,(diam\,(G_1\,(A_k)))\,, \\ diam\,(T_2\,(B_k)) &\leq diam\,(F_2\,(B_k)) + \psi\,(diam\,(G_2\,(B_k)))\,, \\ diam\,(T_3\,(C_k)) &\leq diam\,(F3\,(C_k)) + \psi\,(diam\,(G_3\,(C_k)))\,. \end{aligned}$$

Hence from (4), (5) and (6)

$$\begin{aligned} diam\,(T_1\,(A_k)) &\leq \alpha\,(F_1\,(X_1 \times X_2 \times X_3)) + \epsilon + \psi\,(\epsilon)\,, \\ diam\,(T_2\,(B_k)) &\leq \alpha\,(F_2\,(X_2 \times X_3 \times X_1)) + \epsilon + \psi\,(\epsilon)\,, \\ diam\,(T_2\,(C_k)) &\leq \alpha\,(F_3\,(X_3 \times X_1 \times X_2)) + \epsilon + \psi\,(\epsilon)\,. \end{aligned}$$

Since ϵ is arbitrary and ϕ is a continuous and nondecreasing function thus

$$\begin{aligned} \alpha\,(T_1\,(X_1 \times X_2 \times X_3)) &\leq \alpha\,(F_1\,(X_1 \times X_2 \times X_3))\,, \\ \alpha\,(T_2\,(X_2 \times X_3 \times X_1)) &\leq \alpha\,(F_2\,(X_2 \times X_3 \times X_1))\,, \\ \alpha\,(T_3\,(X_3 \times X_1 \times X_2)) &\leq \alpha\,(F_3\,(X_3 \times X_1 \times X_2))\,. \end{aligned} \tag{7}$$

Now we show that F satisfying in the following condition

$$\begin{aligned} \alpha\,(F_1\,(X_1 \times X_2 \times X_3)) &\leq \phi\,(\mu\,(X_1)\,, \mu\,(X_2)\,, \mu\,(X_3))\,, \\ \alpha\,(F_2\,(X_2 \times X_3 \times X_1)) &\leq \phi\,(\mu\,(X_2)\,, \mu\,(X_3)\,, \mu\,(X_1))\,, \\ \alpha\,(F_3\,(X_3 \times X_1 \times X_2)) &\leq \phi\,(\mu\,(X_3)\,, \mu\,(X_1)\,, \mu\,(X_2))\,. \end{aligned} \tag{8}$$

For this, let $x, u \in X_1$, $y, v \in X_2$, $z, w \in X_3$ then we have

$$\begin{aligned}\left\|F_1(x,y,z) - F_1(u,v,w)\right\| &\leq \phi\left(\|x-u\|, \left\|y-v\right\|, \|z-w\|\right)\\ &\leq \phi\left(diam\left(X_1\right), diam\left(X_2\right), diam\left(X_3\right)\right),\end{aligned}$$

$$\begin{aligned}\left\|F_2(y,z,x) - F_2(v,w,u)\right\| &\leq \phi\left(\left\|y-v\right\|, \|z-w\|, \|x-u\|\right)\\ &\leq \phi\left(diam\left(X_2\right), diam\left(X_3\right), diam\left(X_1\right)\right),\end{aligned}$$

$$\begin{aligned}\left\|F_3(z,x,y) - F_3(w,u,v)\right\| &\leq \phi\left(\|z-w\|, \|x-u\|, \left\|y-v\right\|\right)\\ &\leq \phi\left(diam\left(X_3\right), diam\left(X_1\right), diam\left(X_2\right)\right),\end{aligned}$$

and therefore

$$\begin{aligned}diam\left\{F_1\left(X_1 \times X_2 \times X_3\right)\right\} &\leq \phi\left(diam\left(X_1\right), diam\left(X_2\right), diam\left(X_3\right)\right),\\ diam\left\{F_2\left(X_2 \times X_3 \times X_1\right)\right\} &\leq \phi\left(diam\left(X_2\right), diam\left(X_3\right), diam\left(X_1\right)\right),\\ diam\left\{F_3\left(X_3 \times X_1 \times X_2\right)\right\} &\leq \phi\left(diam\left(X_3\right), diam\left(X_1\right), diam\left(X_2\right)\right),\end{aligned}$$

By definition of Kuratowski measure of noncompactness we have:

$$\begin{aligned}\alpha\left(F_1\left(X_1 \times X_2 \times X_3\right)\right) &\leq \phi\left(\mu\left(X_1\right), \mu\left(X_2\right), \mu\left(X_3\right)\right),\\ \alpha\left(F_2\left(X_2 \times X_3 \times X_1\right)\right) &\leq \phi\left(\mu\left(X_2\right), \mu\left(X_3\right), \mu\left(X_1\right)\right),\\ \alpha\left(F_3\left(X_3 \times X_1 \times X_2\right)\right) &\leq \phi\left(\mu\left(X_3\right), \mu\left(X_1\right), \mu\left(X_2\right)\right).\end{aligned}$$

Now from (7) and (8)

$$\begin{aligned}\alpha\left(T_1\left(X_1 \times X_2 \times X_3\right)\right) &\leq \phi\left(\mu\left(X_1\right), \mu\left(X_2\right), \mu\left(X_3\right)\right),\\ \alpha\left(T_2\left(X_2 \times X_3 \times X_1\right)\right) &\leq \phi\left(\mu\left(X_2\right), \mu\left(X_3\right), \mu\left(X_1\right)\right),\\ \alpha\left(T_3\left(X_3 \times X_1 \times X_2\right)\right) &\leq \phi\left(\mu\left(X_3\right), \mu\left(X_1\right), \mu\left(X_2\right)\right).\end{aligned}$$

Since each T_i is continuous operator for $i = 1, 2, 3$, so by the Theorem 3.9 the proof is complete. □

4. Applications and Examples

Now we are going to describe some measure of noncompactness in the function space $BC(\mathbb{R}_+)$ discussed previously. Let us briefly recall that $BC(\mathbb{R}_+)$ denotes the space of all real functions defined, continuous and bounded on $\mathbb{R}_+$ with the standard supremum norm, *i.e.*

$$\|x\| = \sup\left\{|x(t)| : t \geq 0\right\}.$$

We will use a measure of noncompactness in the space $BC(\mathbb{R}_+)$. In order to define this measure let us fix a nonempty bounded subset of $BC(\mathbb{R}_+)$, this means that $X \in \mathcal{M}_{BC(\mathbb{R}_+)}$. Fix numbers $\epsilon > 0$, $T > 0$ and a function $x \in X$. Let us define the following quantity denote by $\omega^T(x, \epsilon)$ the modulus of continuity of x on the interval $[0, T]$, *i.e.*

$$\omega^T(x,\epsilon) = \sup\left\{|x(t) - x(s)| : t, s \in [0, T], |t-s| \leq \epsilon\right\}.$$

Moreover, let us put $\omega^T(X, \epsilon) = \sup\left\{\omega^T(x,\epsilon) : x \in X\right\}$ is the modulus of quantity of the set X. Since the function $\epsilon \longrightarrow \omega^T(X,\epsilon)$ is nondecreasing, we infer that there exists a finite limit $\lim_{\epsilon\to 0}\omega^T(X,\epsilon)$. We denote this limit by $\omega_0^T(X)$, *i.e.*, we put

$$\omega_0^T(X) = \lim_{\epsilon\to 0}\omega^T(X,\epsilon).$$

Next, let us define the quantity $\omega_0(X)$ by putting $\omega_0(X) = \lim_{T\to\infty} \omega_0^T(X)$. If t is a fixed number from $\mathbb{R}_+$, let us denote

$$X(t) = \{x(t) : x \in X\}.$$

Notice that the quantity $\omega_0(X)$ is not a measure of noncompactness in the space $BC(\mathbb{R}_+)$. To show this fact, let us take the set $X=\{x_n : n = 1, 2, ...\}$, where $x_n : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is the function defined in the following way

$$x_n(t) = \begin{cases} \sin \pi(t + n - 1) & \text{for } t \in [n-1, n] \\ 0 & \text{otherwise.} \end{cases}$$

Obviously $X \in \mathcal{M}_{BC(\mathbb{R}_+)}$. Moreover, it is easily seen that $\omega_0(X) = 0$, but X is not relatively compact in $BC(\mathbb{R}_+)$ since $\|x_n - x_m\| = 1$ for $m \neq n, n, m = 1, 2, \ldots$ (See [10], page 6).

Finally, consider the function μ defined on $M_{BC(\mathbb{R}_+)}$ by formula

$$\mu(X) = \omega_0(X) + \lim_{t\longrightarrow\infty} \sup diam X(t) \tag{9}$$

where

$$diam X(t) = \sup\left\{\left|x(t) - y(t)\right| : x, y \in X\right\}.$$

It is shown that the function $\mu(X)$ defines a sublinear measure of noncompactness in the sense of accepted Definition 2.1.

Now we present an application and an example and resolve the following system of nonlinear integral equations:

$$\begin{cases} x(t) = A_1(t) + h_1(t, x(\xi_1(t)), y(\xi_1(t)), z(\xi_1(t))) + f_1\begin{pmatrix} t, x(\xi_1(t)), y(\xi_1(t)), z(\xi_1(t)), \\ \varphi(\int_0^{\beta_1(t)} g_1(t, s, x(\eta_1(s)), y(\eta_1(s)), z(\eta_1(s))) ds \end{pmatrix} \\ y(t) = A_2(t) + h_2(t, y(\xi_2(t)), z(\xi_2(t)), x(\xi_2(t))) + f_2\begin{pmatrix} t, y(\xi_2(t)), z(\xi_2(t)), x(\xi_2(t)), \\ \varphi(\int_0^{\beta_2(t)} g_2(t, s, y(\eta_2(s)), z(\eta_2(s)), x(\eta_2(s))) ds \end{pmatrix} \\ z(t) = A_3(t) + h_3(t, z(\xi_3(t)), x(\xi_3(t)), y(\xi_3(t))) + f_3\begin{pmatrix} t, z(\xi_3(t)), x(\xi_3(t)), y(\xi_3(t)), \\ \varphi(\int_0^{\beta_3(t)} g_3(t, s, z(\eta_3(s)), x(\eta_3(s)), y(\eta_3(s))) ds \end{pmatrix} \end{cases}. \tag{10}$$

For this consider the following assumptions:

(*i*) the functions $A_i(t) : \mathbb{R}_+ \longrightarrow \mathbb{R}$ are continuous and bounded with $M_i = \sup\{|A_i(t)| : t \in \mathbb{R}_+\}$.

(*ii*) the functions $\xi_i, \beta_i, \eta_i : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are continuous and $\xi_i(t) \longrightarrow \infty$ as $t \longrightarrow \infty$.

(*iii*) the function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is continuous and there are positive constants α, δ such that

$$\left|\varphi(t_1) - \varphi(t_2)\right| \leq \delta |t_1 - t_2|^\alpha$$

for any $t_1, t_2 \in \mathbb{R}_+$ and moreover, $\varphi(0) = 0$.

(*iv*) the functions defined by $t \longrightarrow \left|f_i(t, 0, 0, 0, 0)\right|$ and $t \longrightarrow |h_i(t, 0, 0, 0)|$ are bounded on $\mathbb{R}_+$, *i.e.*

$$\begin{aligned} M_i' &= \sup\left\{\left|f_i(t, 0, 0, 0, 0)\right|, t \in \mathbb{R}_+\right\} < \infty, \\ M_i'' &= sup\{|h_i(t, 0, 0, 0)|, t \in \mathbb{R}_+\} < \infty. \end{aligned}$$

(v) the functions $f_i : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ and $h_i : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous and

there is a function $\phi \in \Phi$, and there are three nondecreasing continuous functions $\theta_i : \mathbb{R}_+ \longrightarrow \mathbb{R}$ with $\theta_i(0) = 0$ such that

$$\left|h_i(t,x,y,z) - h_i(t,u,v,w)\right| \leq \frac{1}{2}\phi\left(|x-u|, \left|y-v\right|, |z-w|\right)$$

and

$$\left|f_i(t,x,y,z,p) - f_i(t,u,v,w,q)\right| \leq \frac{1}{2}\phi\left(|x-u|, \left|y-v\right|, |z-w|\right) + \theta_i\left(\left|p-q\right|\right)$$

for any $t \geq 0$, and for all $x, y, z, u, v, w \in \mathbb{R}_+$.

(vi) the functions $g_i : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous function such that

$$\lim_{t \longrightarrow \infty} \int_0^{\beta_i(t)} \left|g_i(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s))) - g_i(t,s,u(\eta(s)),v(\eta(s)),w(\eta(s)))\right| ds = 0 \tag{11}$$

uniformly with respect to $x, y, z, u, v, w \in BC(\mathbb{R}_+)$, where

$$M_i^{'''} = \sup\left\{\left|\int_0^{\beta_i(t)} g_i(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))\,ds\right|^{\alpha}, t \in \mathbb{R}_+, x, y, z \in BC(\mathbb{R}_+)\right\}.$$

(vii) there exists a positive solution ρ of the inequality

$$M_i + \phi_i(r,r,r) + M_i^{'} + M_i^{''} + \theta_i\left(\delta_i M_i^{'''}\right) < \rho. \tag{12}$$

Theorem 4.1. *Suppose that the conditions (i)-(vii) holds. Then E.q.(10) has at least one solution in the space $BC(\mathbb{R}_+) \times BC(\mathbb{R}_+) \times BC(\mathbb{R}_+)$.*

Proof. *Consider the following three operators*

$$T_1(x,y,z) = A_1(t) + h_1(t, x(\xi_1(t)), y(\xi_1(t)), z(\xi_1(t))) + f_1\left(\begin{array}{c} t, x(\xi_1(t)), y(\xi_1(t)), z(\xi_1(t)), \\ \varphi(\int_0^{\beta_1(t)} g_1(t,s,x(\eta_1(s)), y(\eta_1(s)), z(\eta_1(s)))\,ds \end{array}\right),$$

$$T_2(x,y,z) = A_2(t) + h_2(t, y(\xi_2(t)), z(\xi_2(t)), x(\xi_2(t))) + f_2\left(\begin{array}{c} t, y(\xi_2(t)), z(\xi_2(t)), x(\xi_2(t)), \\ \varphi(\int_0^{\beta_2(t)} g_2(t,s,y(\eta_2(s)), z(\eta_2(s)), x(\eta_2(s)))\,ds \end{array}\right),$$

$$T_3(x,y,z) = A_3(t) + h_3(t, z(\xi_3(t)), x(\xi_3(t)), y(\xi_3(t))) + f_3\left(\begin{array}{c} t, z(\xi_3(t)), x(\xi_3(t)), y(\xi_3(t)), \\ \varphi(\int_0^{\beta_3(t)} g_3(t,s,z(\eta_3(s)), x(\eta_3(s)), y(\eta_3(s)))\,ds \end{array}\right).$$

Since the proof is similar for all of three operators T_1, T_2, and T_3, so we present it for one of the operators e.g. T_1.

First, since A_1, f_1, h_1 are continuous. Then the operator T_1 is continuous. Moreover for $x, y, z \in BC(\mathbb{R}_+)$

$$\begin{aligned}
\left|T_1(x,y,z)\right| &= \left|\begin{array}{c} A_1(t) + h_1(t, x(\xi_1(t)), y(\xi_1(t)), z(\xi_1(t))) \\ +f_1\left(\begin{array}{c} t, x(\xi_1(t)), y(\xi_1(t)), z(\xi_1(t)), \\ \varphi(\int_0^{\beta_1(t)} g_1(t,s,x(\eta_1(s)), y(\eta_1(s)), z(\eta_1(s)))\,ds \end{array}\right) \end{array}\right| \\
&\le |A_1(t)| + \left|h_1(t, x(\xi_1(t)), y(\xi_1(t)), z(\xi_1(t))) - h_1(t,0,0,0)\right| \\
&+ \left(\left|f_1\left(\begin{array}{c} t, x(\xi_1(t)), y(\xi_1(t)), z(\xi_1(t)), \\ \varphi\left(\int_0^{\beta_1(t)} g_i(t,s,x(\eta_1(s)), y(\eta_1(s)), z(\eta_1(s)))\,ds\right) \end{array}\right) - f_1(t,0,0,0,0)\right|\right) \\
&+ |h_1(t,0,0,0)| + \left|f_1(t,0,0,0,0)\right| \\
&\le M_1 + \frac{1}{2}\phi\left(|x(\xi_1(t))|, \left|y(\xi_1(t))\right|, |z(\xi_1(t))|\right) \\
&+ \frac{1}{2}\phi\left(|x(\xi_1(t))|, \left|y(\xi_1(t))\right|, |z(\xi_1(t))|\right) \\
&+ \theta_1\left(\left|\varphi\left(\int_0^{\beta_1(t)} g_1(t,s,x(\eta_1(s)), y(\eta_1(s)), z(\eta_1(s)))\,ds\right)\right|\right) + M_1' + M_1'' \\
&\le M_1 + M_1' + M_1'' + \phi\left(\|x\|, \left\|y\right\|, \|z\|\right) \\
&+ \theta_1\left(\left|\varphi\left(\int_0^{\beta_1(t)} g_1(t,s,x(\eta_1(s)), y(\eta_1(s)), z(\eta_1(s)))\,ds\right)\right|^{\alpha}\right) \\
&\le M_1 + M_1' + M_1'' + \phi\left(\|x\|, \left\|y\right\|, \|z\|\right) + \theta_1\left(\delta_1 M_1'''\right) \\
&< \rho.
\end{aligned}$$

Hence $T_1\left(\bar{B}_\rho \times \bar{B}_\rho \times \bar{B}_\rho\right) \subseteq \bar{B}$, which implies that T_1 is well defined.
Now we prove that T_1 is continuous on $\bar{B}_\rho \times \bar{B}_\rho \times \bar{B}$. For this taking $(x,y,z) \in \bar{B}_\rho \times \bar{B}_\rho \times \bar{B}_\rho$ and $\epsilon > 0$ arbitrary. Moreover, consider $(u,v,w) \in \bar{B}_\rho \times \bar{B}_\rho \times \bar{B}_\rho$ with $\left\|(x,y,z) - (u,v,w)\right\|_{BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)} < \frac{\epsilon}{2}$.
Now we have

$$\begin{aligned}
\left|T_1(x,y,z) - T_1(u,v,w)\right| &\le \left|\begin{array}{c} h_1(t, x(\xi_1(t)), y(\xi_1(t)), z(\xi_1(t))) \\ -h_1(t, u(\xi_1(t)), v(\xi_1(t)), w(\xi_1(t))) \end{array}\right| \\
&+ \left|\begin{array}{c} f_1\left(\begin{array}{c} t, x(\xi_1(t)), y(\xi_1(t)), z(\xi_1(t)), \\ \varphi(\int_0^{\beta_1(t)} g_1(t,s,x(\eta_1(t)), y(\eta_1(t)), z(\eta_1(t)))\,ds \end{array}\right) \\ -f_1\left(\begin{array}{c} t, u(\xi_1(t)), v(\xi_1(t)), w(\xi_1(t)), \\ \varphi(\int_0^{\beta_1(t)} g_1(t,s,u(\eta_1(t)), v(\eta_1(t)), w(\eta_1(t)))\,ds \end{array}\right) \end{array}\right| \\
&\le \frac{1}{2}\phi\left(\begin{array}{c} |x(\xi_1(t)) - u(\xi_1(t))|, \left|y(\xi_1(t)) - v(\xi_1(t))\right|, \\ |z(\xi_1(t)) - w(\xi_1(t))| \end{array}\right) \\
&+ \frac{1}{2}\phi\left(\begin{array}{c} |x(\xi_1(t)) - u(\xi_1(t))|, \left|y(\xi_1(t)) - v(\xi_1(t))\right|, \\ |z(\xi_1(t)) - w(\xi_1(t))| \end{array}\right) \\
&+ \theta_1\left(\left|\begin{array}{c} \varphi_1\left(\int_0^{\beta_1(t)} g_1(t,s,x(\eta_1(s)), y(\eta_1(s)), z(\eta_1(s)))\,ds\right) \\ -\varphi\left(\int_0^{\beta_1(t)} g_1(t,s,u(\eta_1(s)), v(\eta_1(s)), w(\eta_1(s)))\,ds\right) \end{array}\right|\right) \\
&\le \phi\left(\|x-u\|, \left\|y-v\right\|, \|z-w\|\right) \\
&+ \theta_1\left(\delta\left|\int_0^{\beta_1(t)}\left(g_1\left(\begin{array}{c} t,s,x(\eta_1(s)), y(\eta_1(s)), z(\eta_1(s)) \\ -g_1(t,s,u(\eta_1(s)), v(\eta_1(s)), w(\eta_1(s))) \end{array}\right)\right)ds\right|^{\alpha}\right).
\end{aligned}$$

In addition from (vi), there exists $L>0$ such that if $t>L$ then

$$\theta_1\left(\delta\left|\int_0^{\beta_1(t)}\left(g_1\left(\begin{array}{c} t,s,x(\eta_1(s)),y(\eta_1(s)),z(\eta_1(s)) \\ -g_1(t,s,u(\eta_1(s)),v(\eta_1(s)),w(\eta_1(s))) \end{array}\right)\right)ds\right|^{\alpha}\right)\leq\frac{\epsilon}{2}$$

for any $x,y,z,u,v,w\in BC(\mathbb{R}_+)$.
Now we consider two cases

C_1) *If $t>L$, then we get*

$$\left\|T_1(x,y,z)-T_1(u,v,w)\right\|\leq\phi\left(\frac{\epsilon}{2},\frac{\epsilon}{2},\frac{\epsilon}{2}\right)+\frac{\epsilon}{2}=\epsilon.$$

C_2) *If $t\in[0,L]$, then by the argument similar to those given in*

$$\begin{aligned}\left|T_1(x,y,z)-T_1(u,v,w)\right| &\leq\phi_1\left(\frac{\epsilon}{2},\frac{\epsilon}{2},\frac{\epsilon}{2}\right)\\ &+\theta_1\left(\delta\left|\int_0^{\beta_1(t)}\left(g_1\left(\begin{array}{c} t,s,x(\eta_1(s)),y(\eta_1(s)),z(\eta_1(s)) \\ -g_1(t,s,u(\eta_1(s)),v(\eta_1(s)),w(\eta_1(s))) \end{array}\right)\right)ds\right|^{\alpha}\right)\\ &<\frac{\epsilon}{2}+\theta_1\left(\delta_1\left(\beta_1^L\omega(\epsilon)\right)^{\alpha}\right),\end{aligned}$$

where

$$\omega(\epsilon)=\sup\left\{\begin{array}{c}\left|g_1(t,s,x,y,z)-g_1(t,s,u,v,w)\right|:t\in[0,L],s\in\left[0,\beta_1^L\right],\\ x,y,z,u,v,w\in[-\rho,\rho],\left\|(x,y,z)-(u,v,w)\right\|<\frac{\epsilon}{2}\end{array}\right\}$$

and $\beta_1^L=\sup\{\beta_1(t):t\in[0,L]\}$.

By using the continuity of g_1 on $[0,L]\times\left[0,\beta_1^L\right]\times[-\rho,\rho]\times[-\rho,\rho]\times[-\rho,\rho]$ we have $\omega(\epsilon)\longrightarrow 0$ as $\epsilon\longrightarrow 0$ and by continuity of θ_1 we get $\theta_1\left(\delta_1\left(\beta_1^L\omega(\epsilon)\right)^{\alpha}\right)\longrightarrow 0$ as $\epsilon\longrightarrow 0$.

Hence T_1 is continuous on $BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)$.

Now we only need to show that T_1 satisfied the conditions of Theorem 3.9. To prove that let $L,\epsilon\in\mathbb{R}_+$ and $X_1\times X_2\times X_3$ be an arbitrary nonempty subset of $\bar{B}_\rho$ and take $t_1,t_2\in[0,L]$, such that $|t_1-t_2|\leq\epsilon$.

Without loss of generality, we may assume that $\beta_1(t_1)<\beta_1(t_2)$ and we assume that $(x,y,z)\in X_1\times X_2\times X_3$.

$$\begin{aligned}&\left|T_1(x,y,z)(t_2)-T_2(x,y,z)(t_1)\right|\\ &\leq|A_1(t_2)-A_1(t_1)|\\ &+\left|h_1(t_2,x(\xi(t_2)),y(\xi(t_2)),z(\xi(t_2)))-h_1(t_1,x(\xi(t_1)),y(\xi(t_1)),z(\xi(t_1)))\right|\\ &+\left|\begin{array}{l} f_1\left(t_2,x(\xi(t_2)),y(\xi(t_2)),z(\xi(t_2)),\varphi\left(\int_0^{\beta_1(t_2)}g_1(t_2,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))ds\right)\right)\\ -f_1\left(t_1,x(\xi(t_1)),y(\xi(t_1)),z(\xi(t_1)),\varphi\left(\int_0^{\beta_1(t_1)}g_1(t_1,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))ds\right)\right)\end{array}\right|\end{aligned}$$

$$\begin{aligned}
&\leq |A_1(t_2) - A_1(t_1)| + \left| \begin{array}{l} h_1(t_2, x(\xi(t_2)), y(\xi(t_2)), z(\xi(t_2))) \\ -h_1(t_2, x(\xi(t_1)), y(\xi(t_1)), z(\xi(t_1))) \end{array} \right| \\
&+ \left| \begin{array}{l} h_1(t_2, x(\xi(t_1)), y(\xi(t_1)), z(\xi(t_1))) \\ -h_1(t_2, x(\xi(t_1)), y(\xi(t_1)), z(\xi(t_1))) \end{array} \right| \\
&+ \left| \begin{array}{l} f_1\left(t_2, x(\xi(t_2)), y(\xi(t_2)), z(\xi(t_2)), \varphi\left(\int_0^{\beta_1(t_2)} g_i(t_2, s, x(\eta(s)), y(\eta(s)), z(\eta(s)))\,ds\right)\right) \\ -f_1\left(t_2, x(\xi(t_1)), y(\xi(t_1)), z(\xi(t_1)), \varphi\left(\int_0^{\beta_1(t_2)} g_1(t_2, s, x(\eta(s)), y(\eta(s)), z(\eta(s)))\,ds\right)\right) \end{array} \right| \\
&+ \left| \begin{array}{l} f_1\left(t_2, x(\xi(t_1)), y(\xi(t_1)), z(\xi(t_1)), \varphi\left(\int_0^{\beta_1(t_2)} g_1(t_2, s, x(\eta(s)), y(\eta(s)), z(\eta(s)))\,ds\right)\right) \\ -f_1\left(t_1, x(\xi(t_1)), y(\xi(t_1)), z(\xi(t_1)), \varphi\left(\int_0^{\beta_1(t_2)} g_1(t_2, s, x(\eta(s)), y(\eta(s)), z(\eta(s)))\,ds\right)\right) \end{array} \right| \\
&+ \left| \begin{array}{l} f_1\left(t_1, x(\xi(t_1)), y(\xi(t_1)), z(\xi(t_1)), \varphi\left(\int_0^{\beta_1(t_2)} gi(t_2, s, x(\eta(s)), y(\eta(s)), z(\eta(s)))\,ds\right)\right) \\ -f_1\left(t_1, x(\xi(t_1)), y(\xi(t_1)), z(\xi(t_1)), \varphi\left(\int_0^{\beta_1(t_2)} g_1(t_1, s, x(\eta(s)), y(\eta(s)), z(\eta(s)))\,ds\right)\right) \end{array} \right| \\
&+ \left| \begin{array}{l} f_1\left(t_1, x(\xi(t_1)), y(\xi(t_1)), z(\xi(t_1)), \varphi\left(\int_0^{\beta_1(t_2)} g_1(t_2, s, x(\eta(s)), y(\eta(s)), z(\eta(s)))\,ds\right)\right) \\ -f_1\left(t_1, x(\xi(t_1)), y(\xi(t_1)), z(\xi(t_1)), \varphi\left(\int_0^{\beta_1(t_1)} g_1(t_1, s, x(\eta(s)), y(\eta(s)), z(\eta(s)))\,ds\right)\right) \end{array} \right| \\
&\leq \omega^T(A_1, \epsilon) + \frac{1}{2}\phi\left(|x(\xi_1(t_2)) - x(\xi_1(t_1))|, \left|y(\xi_1(t_2)) - y(\xi_1(t_1))\right|, |z(\xi_1(t_2)) - z(\xi_1(t_1))|\right) \\
&+ \omega_\rho^T(h_1, \epsilon) + \frac{1}{2}\phi\left(|x(\xi_1(t_2)) - x(\xi_1(t_1))|, \left|y(\xi_1(t_2)) - y(\xi_1(t_1))\right|, |z(\xi_1(t_2)) - z(\xi_1(t_1))|\right) \\
&+ \omega_{\rho,k}^T(f_1, \epsilon) + \theta_1\left(\left| \begin{array}{l} \varphi\left(\int_0^{\beta_1(t_2)} g_1(t_2, s, x(\eta_1(s)), y(\eta_1(s)), z(\eta_1(s)))\,ds\right) \\ -\varphi\left(\int_0^{\beta_1(t_2)} g_1(t_1, s, x(\eta_1(s)), y(\eta_1(s)), z(\eta_1(s)))\,ds\right) \end{array} \right|\right) \\
&+ \theta_1\left(\left|\int_{\beta_1(t_1)}^{\beta_1(t_2)} g_1(t_1, s, x(\eta_1(s)), y(\eta_1(s)), z(\eta_1(s)))\right|\right) \\
&\leq \omega^T(A_1, \epsilon) + \omega_\rho^T(h_1, \epsilon) + \phi\left(\omega^T\left(x, \omega^T(\xi_1, \epsilon)\right), \omega^T\left(y, \omega^T(\xi_1, \epsilon)\right), \omega^T\left(z, \omega^T(\xi_1, \epsilon)\right)\right) \\
&+ \omega_{\rho,k}^T(f_1, \epsilon) + \theta_1\left(\left(\beta_1^L \omega_\rho^T(g_1, \epsilon)\right)^\alpha\right) + \theta_1\left(\left(k\omega^T(\beta_1, \epsilon)\right)^{\alpha_1}\right),
\end{aligned}$$

where

$$\begin{aligned}
\omega^T(A_1, \epsilon) &= \sup\{|A_1(t_1) - A_1(t_2)| : t_1, t_2 \in [0, L], |t_1 - t_2| \leq \epsilon\} \\
\omega_\rho^T(h_1, \epsilon) &= \sup\left\{\left|h_1(t_2, x, y, z) - h_1(t_1, x, y, z)\right| : t_1, t_2 \in [0, L], |t_1 - t_2| \leq \epsilon, x, y, z \in [-\rho, \rho]\right\}, \\
\omega^T(\xi_1, \epsilon) &= \sup\{|\xi_1(t_1) - \xi_1(t_2)| : t_1, t_2 \in [0, L], |t_1 - t_2| \leq \epsilon\}, \\
\omega^T\left(x, \omega^T(\xi_1, \epsilon)\right) &= \sup\left\{|x(t_1) - x(t_2)| : t_1, t_2 \in [0, L], |t_1 - t_2| \leq \omega^T(\xi_1, \epsilon)\right\}, \\
k &= \beta_L \sup\left\{\left|g_1(t, x, y, z)\right| : t \in [0, L], s \in [0, \beta_L], x, y, z \in [-\rho, \rho]\right\}, \\
\omega_{\rho,k}^T(f_1, \epsilon) &= \sup\left\{\left|f_1(t_2, x, y, z, p) - f_1(t_1, x, y, z, p)\right| : t_1, t_2 \in [0, L], x, y, z \in [-\rho, \rho], p \in [-\delta k^{\alpha_1}, \delta k^{\alpha_1}]\right\}, \\
\omega_\rho^T(g_1, \epsilon) &= \sup\left\{\left|g_1(t_1, s, x, y, z) - g_1(t_2, s, x, y, z)\right| : t_1, t_2 \in [0, L], |t_1 - t_2| \leq \epsilon, s \in [0, \beta_L], x, y, z \in [-\rho, \rho]\right\}, \\
\omega^T(\beta_1, \epsilon) &= \sup\left\{\left|\beta_1(t_1) - \beta_1(t_2)\right| : t_1, t_2 \in [0, L], |t_1 - t_2| \leq \epsilon\right\}.
\end{aligned}$$

Since (x, y, z) is an arbitrary element of $X_1 \times X_2 \times X_3$, we obtain

$$\omega^T(T_1(X_1\times X_2\times X_3),\epsilon)\leq \omega^T(A_1,\epsilon)+\omega_\rho^T(h_1,\epsilon)+\phi\left(\omega^T\left(x,\omega^T(\xi_1,\epsilon)\right),\omega^T\left(y,\omega^T(\xi_1,\epsilon)\right),\omega^T\left(z,\omega^T(\xi_1,\epsilon)\right)\right)$$
$$+\omega_{\rho,k}^T(f_1,\epsilon)+\theta_1\left(\left(\beta_1^L\omega_\rho^T(g_1,\epsilon)\right)^\alpha\right)+\theta_1\left(\left(k\omega^T(\beta_1,\epsilon)\right)^\alpha\right). \tag{13}$$

Since f,g, and h are uniformly continuous on $[0,L]\times[-\rho,\rho]\times[-\rho,\rho]\times[-\rho,\rho]\times[-\delta k^\alpha,\delta k^\alpha]$, $[0,L]\times\left[0,\beta^L\right]\times[-\rho,\rho]\times[-\rho,\rho]\times[-\rho,\rho]$,
$[0,L]\times[-\rho,\rho]\times[-\rho,\rho]\times[-\rho,\rho]$ *respectively, we obtain* $\omega_{\rho,k}^T(f,\epsilon)\longrightarrow 0$, $\omega_\rho^T(g,\epsilon)\longrightarrow 0$, $\omega_\rho^T(h,\epsilon)\longrightarrow 0$ *as* $\epsilon\longrightarrow 0$.

Moreover, because the functions ξ_1,β_1 and A_1 are uniformly continuous on $[0,L]$, *we have that* $\omega^T(\xi_1,\epsilon)\longrightarrow 0$, $\omega^T(\beta_1,\epsilon)\longrightarrow 0$, $\omega^T(A_1,\epsilon)\longrightarrow 0$ *as* $\epsilon\longrightarrow 0$.

By the assumption (v), since θ are nondecreasing continuous functions with $\theta(0)=0$ and k is finite, therefore we have

$$\theta\left(\left(\beta^L\omega_\rho^T(g,\epsilon)\right)^\alpha\right)+\theta\left(\left(k\omega^T(\beta,\epsilon)\right)^\alpha\right)\longrightarrow 0$$

as $\epsilon\longrightarrow 0$.

Now taking the limit from (13), we derive that

$$\omega_0^T(T_1(X_1\times X_2\times X_3))\leq\phi\left(\omega_0^T(X_1),\omega_0^T(X_2),\omega_0^T(X_3)\right) \tag{14}$$

as $\epsilon\longrightarrow 0$.

When letting $T\longrightarrow\infty$ in (14) we get

$$\omega_0(T_1(X_1\times X_2\times X_3))\leq\phi(\omega_0(X_1),\omega_0(X_2),\omega_0(X_3)). \tag{15}$$

By the same method, one can show that

$$\begin{aligned}\omega_0(T_2(X_2\times X_3\times X_1)) &\leq \phi(\omega_0(X_2),\omega_0(X_3),\omega_0(X_1)),\\ \omega_0(T_3(X_3\times X_1\times X_2)) &\leq \phi(\omega_0(X_3),\omega_0(X_1),\omega_0(X_2)).\end{aligned} \tag{16}$$

On the other hand

$$\begin{aligned}&\left|T_1(x,y,z)(t)-T_1(u,v,w)(t)\right|\\ &\leq\left|h_1(t,x(\xi_1(t)),y(\xi_1(t)),z(\xi_1(t)))-h_1(t,u(\xi_1(t)),v(\xi_1(t)),w(\xi_1(t)))\right|\\ &+\left|\begin{array}{l} f_1\left(t,x(\xi_1(t)),y(\xi_1(t)),z(\xi_1(t)),\varphi\left(\int_0^{\beta_1(t)}g_1(t,s,x(\eta_1(s)),y(\eta_1(s)),z(\eta_1(s)))\,ds\right)\right)\\ -f_1\left(t,u(\xi_1(t)),v(\xi_1(t)),w(\xi_1(t)),\varphi\left(\int_0^{\beta_1(t)}g_1(t,s,u(\eta_1(s)),v(\eta_1(s)),w(\eta_1(s)))\,ds\right)\right)\end{array}\right|\\ &\leq\frac{1}{2}\phi\left(|x(\xi_1(t))-u(\xi_1(t))|,\left|y(\xi_1(t))-v(\xi_1(t))\right|,|z(\xi_1(t))-w(\xi_1(t))|\right)\\ &+\frac{1}{2}\phi\left(|x(\xi_1(t))-u(\xi_1(t))|,\left|y(\xi_1(t))-v(\xi_1(t))\right|,|z(\xi_1(t))-w(\xi_1(t))|\right)\\ &+\theta_1\left(\left|\begin{array}{l}\varphi\left(\int_0^{\beta_1(t)}g_1(t,s,x(\eta_1(s)),y(\eta_1(s)),z(\eta_1(s)))\,ds\right)\\ -\varphi\left(\int_0^{\beta_1(t)}g_1(t,s,u(\eta_1(s)),v(\eta_1(s)),w(\eta_1(s)))\,ds\right)\end{array}\right|\right)\\ &\leq\phi(diamX_1(\xi_1(t)),diamX_2(\xi_1(t)),diamX_3(\xi_1(t)))\\ &+\theta_1\left(\delta_1\left|\int_0^{\beta_1(t)}(g_1(t,s,x(\eta_1(s)),y(\eta_1(s)),z(\eta_1(s))-g_1(t,s,u(\eta_1(s)),v(\eta_1(s)),w(\eta_1(s)))))\,ds\right|^\alpha\right).\end{aligned} \tag{17}$$

Since $(x,y,z),(u,v,w)$ *and* t *are arbitrary in (17), we get*

$$\begin{aligned} diamT_1(X_1\times X_2\times X_3) &\leq \phi\left(diamX_1(\xi_1(t)), diamX_2(\xi_1(t)), diamX_3(\xi_1(t))\right)\\ &+\theta_1\left(\delta_1\left|\int_0^{\beta_1(t)}\left(g_1\left(\begin{array}{c} t,s,x(\eta_1(s)),y(\eta_1(s)),z(\eta_1(s))\\ -g_1(t,s,u(\eta_1(s)),v(\eta_1(s)),w(\eta_1(s)))\end{array}\right)\right)ds\right|^{\alpha}\right). \end{aligned} \tag{18}$$

Thus by (ii) *and* $\xi_1(t)\longrightarrow\infty$ *as* $t\longrightarrow\infty$ *in the inequality (18), then using (11) we obtain*

$$\limsup_{t\longrightarrow\infty} diamT_1(X_1\times X_2\times X_3)\leq\phi\left(\limsup_{t\longrightarrow\infty} diamX_1(\xi_1(t)),\limsup_{t\longrightarrow\infty} diamX_2(\xi_1(t)),\limsup_{t\longrightarrow\infty} diamX_3(\xi_1(t))\right). \tag{19}$$

By the same method, one can show that

$$\begin{aligned} &\limsup_{t\longrightarrow\infty} diamT_2(X_2\times X_3\times X_1)\leq\phi\left(\limsup_{t\longrightarrow\infty} diamX_2(\xi_2(t)),\limsup_{t\longrightarrow\infty} diamX_3(\xi_2(t)),\limsup_{t\longrightarrow\infty} diamX_1(\xi_2(t))\right),\\ &\limsup_{t\longrightarrow\infty} diamT_3(X_3\times X_1\times X_2)\leq\phi\left(\limsup_{t\longrightarrow\infty} diamX_3(\xi_3(t)),\limsup_{t\longrightarrow\infty} diamX_1(\xi_3(t)),\limsup_{t\longrightarrow\infty} diamX_2(\xi_1(t))\right). \end{aligned} \tag{20}$$

If we blending (15), (19) we conclude that

$$\begin{aligned} &\omega_0(T_1(X_1\times X_2\times X_3))+\limsup_{t\longrightarrow\infty} diamT_1(X_1\times X_2\times X_3)\\ &\leq\phi(\omega_0(X_1),\omega_0(X_2),\omega_0(X_3))+\phi\left(\limsup_{t\longrightarrow\infty} diamX_1(\xi_1(t)),\limsup_{t\longrightarrow\infty} diamX_2(\xi_1(t)),\limsup_{t\longrightarrow\infty} diamX_3(\xi_1(t))\right)\\ &\leq 3\phi\left(\frac{\omega_0(X_1)+\lim\limits_{t\longrightarrow\infty}\sup diamX_1(t)}{3},\frac{\omega_0(X_2)+\lim\limits_{t\longrightarrow\infty}\sup diamX_2(t)}{3},\frac{\omega_0(X_3)+\lim\limits_{t\longrightarrow\infty}\sup diamX_3(t)}{3}\right). \end{aligned}$$

So

$$\frac{1}{3}\mu(T_1(X_1\times X_2\times X_3))\leq\phi\left(\frac{\mu(X_1)}{3},\frac{\mu(X_2)}{3},\frac{\mu(X_3)}{3}\right).$$

Taking $\mu'=\frac{1}{3}\mu$, *we obtain*

$$\mu'(T_1(X_1\times X_2\times X_3))\leq\phi(\mu'(X_1),\mu'(X_2),\mu'(X_3)).$$

Where μ' *is the measure of noncompactness defined in (9).*
By the same method, from (16) and (20) we can show that

$$\begin{aligned} \mu'(T_2(X_2\times X_3\times X_1)) &\leq \phi(\mu'(X_2),\mu'(X_3),\mu'(X_1)),\\ \mu'(T_3(X_3\times X_1\times X_2)) &\leq \phi(\mu'(X_3),\mu'(X_1),\mu'(X_2)) \end{aligned}$$

Thus by Theorem 3.9, E.q. (10) has at least one solution in the space $BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)$. □

Example 4.2. *Let the system of integral equation*

$$\begin{cases} x(t) = \frac{3}{4}e^{-t^2} + \left(\frac{e^{-t^2}}{4} + \frac{t^2}{1+t^2}\right)\ln\left(1 + \left|x\left(t^2\right)\right|\right) + \left(\frac{e^{-t}}{6} + \frac{t^4}{2+t^4}\right)\ln\left(1 + \left|y\left(t^2\right)\right|\right) + \left(\frac{e^{-t^2}}{3} + \frac{t^2}{3+4t^2}\right)\ln\left(1 + \left|z\left(t^2\right)\right|\right) \\ + \ln\left(1 + \int_0^t \frac{\ln\left(1+\sqrt[3]{s}\left|x(t)\right|\right)\ln\left(1+\sqrt[3]{s}\left|y(t)\right|\right)\ln\left(1+\sqrt[3]{s}\left|z(t)\right|\right) + s\left(1+x^2(t)\right)\left(1+y^2(t)\right)\left(1+z^2(t)\right)}{e^{t^2}\left(1+x^2(t)\right)\left(1+y^2(t)\right)\left(1+z^2(t)\right)}ds\right) \\ + \ln\left(1+\frac{p}{3}\right) \\ y(t) = \frac{t^4}{2+t^4} + \frac{e^{-t^2}}{2} + \left(\frac{e^{-t}}{3} + \frac{t}{3+t}\right)\ln\left(1 + \left|y\left(\sqrt{t}\right)\right|\right) + \left(\frac{e^{-t^2}}{5} + \frac{e^{-t}}{2}\right)\ln\left(1 + \left|z\left(\sqrt{t}\right)\right|\right) + \left(\frac{e^{-t^2}}{10} + \frac{t^4}{3+t^4}\right)\ln\left(1 + \left|x\left(\sqrt{t}\right)\right|\right) \\ + \ln\left(1 + \int_0^{\sqrt{t}} \frac{\ln\left(1+\sqrt[5]{s^3}\left|y(t)\right|\right)\ln\left(1+\sqrt[5]{s}\left|z(t)\right|\right)\ln\left(1+\sqrt[5]{s}\left|x(t)\right|\right) + s\left(1+y^3(t)\right)\left(1+z^3(t)\right)\left(1+x^3(t)\right)}{e^{t}\left(1+y^3(t)\right)\left(1+z^3(t)\right)\left(1+x^3(t)\right)}ds\right) \\ + \ln\left(1+\frac{p}{3}\right) \\ z(t) = \frac{3}{4}e^{-t^2} + \frac{t^4}{5(2+t^4)} + \left(\frac{e^{-t^2}}{4} + \frac{1}{1+t^2}\right)\ln\left(1 + 2\left|z(t)\right|\right) + \frac{5e^{-t}}{8}\ln\left(1 + 4\left|x(t)\right|\right) + \left(\frac{t^2}{1+4t^2} + \frac{e^{-t}}{3}\right)\ln\left(1 + 3\left|y(t)\right|\right) \\ + \ln\left(1 + \int_0^{t^2} \frac{\ln\left(1+\sqrt[7]{s^2}\left|z(t)\right|\right)\ln\left(1+\sqrt[7]{s^3}\left|x(t)\right|\right)\ln\left(1+\sqrt[7]{s^2}\left|y(t)\right|\right) + s^2\left(1+z^4(t)\right)\left(1+x^4(t)\right)\left(1+y^4(t)\right)}{e^{t}\left(1+z^4(t)\right)\left(1+x^4(t)\right)\left(1+y^4(t)\right)}ds\right) \\ + \ln\left(1+\frac{p}{3}\right). \end{cases} \tag{21}$$

Where

$$\begin{aligned} A_1(t) &= \frac{3}{4}e^{-t^2}, h_1(t,x,y,z) = \frac{1}{4}e^{-t} + \frac{e^{-t^2}}{4}\ln\left(1+\left|x(t)\right|\right) + \frac{e^{-t}}{6}\ln\left(1+\left|y(t)\right|\right) + \frac{t^2}{3+4t^2}\ln\left(1+\left|z(t)\right|\right), \\ f_1(t,x,y,z,p) &= \frac{3}{10}e^{-t} + \frac{t^2}{1+t^2}\ln\left(1+\left|x(t)\right|\right) + \frac{t^4}{2+t^4}\ln\left(1+\left|y(t)\right|\right) + \frac{e^{-t^2}}{3}\ln\left(1+\left|z(t)\right|\right) + \ln\left(1+\frac{p}{3}\right), \\ g_1(t,s,x,y,z) &= \frac{\ln\left(1+\sqrt[3]{s}\left|x(t)\right|\right)\ln\left(1+\sqrt[3]{s}\left|y(t)\right|\right)\ln\left(1+\sqrt[3]{s}\left|z(t)\right|\right) + s\left(1+x^2(t)\right)\left(1+y^2(t)\right)\left(1+z^2(t)\right)}{e^{t^2}\left(1+x^2(t)\right)\left(1+y^2(t)\right)\left(1+z^2(t)\right)}, \\ \xi_1(t) &= t^2, \eta_1(t) = \sqrt{t}, \beta_1(t) = t, \varphi(x) = \ln\left(1+\frac{|x|}{3}\right), \phi(t,s,r) = \ln\left(1+\frac{t+s+r}{3}\right), \theta(t) = \frac{t}{3}. \end{aligned}$$

Also

$$\begin{aligned} A_2(t) &= \frac{t^4}{2+t^4}, h_2(t,x,y,z) = \frac{1}{2}e^{-t^2} + \frac{e^{-t}}{3}\ln\left(1+\left|x(t)\right|\right) + \frac{e^{-t^2}}{5}\ln\left(1+\left|y(t)\right|\right) \\ &+ \frac{t^2+1}{3+t^2}\ln\left(1+\left|z(t)\right|\right), \\ f_2(t,x,y,z,p) &= \frac{3}{10}e^{-t} + \frac{t^2}{1+3t^2} + \frac{t}{1+t}\ln\left(1+\left|x(t)\right|\right) + \frac{e^{-t^3}}{2}\ln\left(1+\left|y(t)\right|\right) + \frac{e^{-t^2}}{5}\ln\left(1+\left|z(t)\right|\right) \\ &+ \ln\left(1+\frac{p}{3}\right), \\ g_2(t,s,x,y,z) &= \frac{\ln\left(1+\sqrt[5]{s^3}\left|x(t)\right|\right)\ \ln\left(1+\sqrt[5]{s}\left|y(t)\right|\right)\ \ln\left(1+\sqrt[5]{s}\left|z(t)\right|\right) + s\ \left(1+x^3(t)\right)\left(1+y^3(t)\right)\left(1+z^3(t)\right)}{e^{t^2}\left(1+x^3(t)\right)\left(1+y^3(t)\right)\left(1+z^3(t)\right)}, \\ \xi_2(t) &= \sqrt{t}, \eta_2(t) = t, \beta_2(t) = \sqrt{t}, \phi(t,s,r) = \ln\left(1+\frac{t+s+r}{3}\right), \theta(t) = \frac{t}{3}, \\ \varphi(x) &= \ln\left(1+\frac{|x|}{3}\right). \end{aligned}$$

And so

$$A_3(t) = \frac{1}{7}e^{-t^2}, h_3(t,x,y,z) = \frac{t^4}{5(2+t^4)} + \frac{e^{-t^2}}{4}\ln(1+2|x(t)|) + \frac{e^{-t}}{8}\ln\left(1+4|y(t)|\right) + \frac{t^2}{4t^2+1}\ln(1+3|z(t)|),$$

$$f_3(t,x,y,z,p) = \frac{3}{14}e^{-t^2} + \frac{1}{1+t^2}\ln(1+2|x(t)|) + \frac{e^{-t}}{2}\ln\left(1+4|y(t)|\right) + \left(\frac{e^{-t}}{3}+1\right)\ln(1+3|z(t)|) + \ln\left(1+\frac{p}{3}\right),$$

$$g_3(t,s,x,y,z) = \frac{\ln\left(1+\sqrt[7]{s^2}|x(t)|\right)\ \ln\left(1+\sqrt[7]{s^3}|y(t)|\right)\ \ln\left(1+\sqrt[7]{s^2}|z(t)|\right) + s^2\ \left(1+x^4(t)\right)\left(1+y^4(t)\right)\left(1+z^4(t)\right)}{e^{t^3}(1+x^4(t))(1+y^4(t))(1+z^4(t))},$$

$$\xi_3(t) = t, \eta_3(t) = t^2, \beta_3(t) = t^2, \phi(t,s,r) = \ln\left(1+\frac{t+s+r}{3}\right), \theta(t) = \frac{t}{3}, \varphi(x) = \ln\left(1+\frac{|x|}{3}\right).$$

Now we show that all the condition of Theorem 4.1 are satisfied for E.q. (21).

(i) *the function* $A_1(t) = \frac{e^{-t^2}}{3}$ *clearly continuous and bounded with* $M_1 = \sup\{|A_1(t)| : t \in \mathbb{R}_+\} = \frac{3}{4}$.

(ii) *the functions* $\xi_1(t) = t^2$,$\eta_1(t) = \sqrt{t}$, $\beta_1(t) = t$:$\mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *are continuous and* $\lim_{t\longrightarrow\infty}\xi_1(t) = \lim_{t\longrightarrow\infty} t = \infty$.

(iii) *the function* $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ *with* $\varphi(x) = \ln\left(1+\frac{|x|}{2}\right)$ *is continuous*

$$\begin{aligned}\left|\varphi(t_1) - \varphi(t_2)\right| &= \left|\ln\left(1+\frac{|t_1|}{2}\right) - \ln\left(1+\frac{|t_2|}{2}\right)\right| = \left|\ln\frac{1+\frac{|t_1|}{2}}{1+\frac{|t_2|}{2}}\right| = \left|\ln\frac{2+t_1}{2+t_2}\right| \\ &= \ln\left(1+\frac{|t_1|-|t_2|}{2+t_2}\right) \le \ln(1+|t_1-t_2|) \le |t_1-t_2|\end{aligned}$$

with $\alpha = \delta = 1$. *For any* $t_1, t_2 \in \mathbb{R}_+$, *and moreover,* $\varphi(0) = \ln(1) = 0$.

(iv) *the functions defined by* $t \longrightarrow \left|f_1(t,0,0,0,0)\right|$ *and* $t \longrightarrow |h_1(t,0,0,0)|$ *are bounded on* $\mathbb{R}_+$, *i.e.*

$$\begin{aligned}M_1' &= \sup\{f_1(t,0,0,0,0) : t \in \mathbb{R}_+\} = \frac{3e^{-t}}{10} = \frac{3}{10} < \infty, \\ M_1'' &= \sup\{h_1(t,0,0,0) : t_2 \in \mathbb{R}_+\} = \frac{e^{-t}}{4} = \frac{1}{4} < \infty.\end{aligned}$$

(v) *the functions* f_1 *and* h_1 *are continuous.*

Now assume that $t \in \mathbb{R}_+$ *and* $x, y, z, p, u, v, w, q \in \mathbb{R}$ *with* $|x| \ge |u|$, $|y| \ge |v|$, *and* $|z| \ge |w|$.*Then by using the Mean Value Theorem for the function* $\varphi(x) = \ln\left(1+\frac{|x|}{2}\right)$ *and the fact that* $\phi(t,s,r) = \ln\left(1+\frac{t+s+r}{3}\right) \in \Phi$, *we can get following results:*

D_1)

$$
\begin{aligned}
&\left|h_1(t,x,y,z)-h_1(t,u,v,w)\right| \\
&= \left| \begin{array}{c} \frac{e^{-t}}{4}+\frac{e^{-t^2}}{4}\ln(1+|x(t)|)+\frac{e^{-t}}{6}\ln\left(1+\left|y(t)\right|\right)+\frac{t^2}{3+4t^2}\ln(1+|z(t)|) \\ -\frac{e^{-t}}{4}-\frac{e^{-t^2}}{4}\ln(1+|u(t)|)-\frac{e^{-t}}{6}\ln(1+|v(t)|)-\frac{t^2}{3+4t^2}\ln(1+|w(t)|) \end{array} \right| \\
&\le \left|\frac{e^{-t^2}}{4}\left(\ln(1+|x(t)|)-\ln(1+|u(t)|)\right)\right|+\left|\frac{e^{-t}}{6}\left(\ln\left(1+\left|y(t)\right|\right)-\ln(1+|v(t)|)\right)\right| \\
&+\left|\frac{t^2}{3+4t^2}\left(\ln(1+|z(t)|)-\ln(1+|w(t)|)\right)\right| \\
&= \frac{e^{-t}}{4}\left|\ln\left(\frac{1+|x(t)|}{1+|u(t)|}\right)\right|+\frac{e^{-t}}{6}\left|\ln\left(\frac{1+\left|y(t)\right|}{1+|v(t)|}\right)\right|+\frac{t^2}{3+4t^2}\left|\ln\left(\frac{1+|z(t)|}{1+|w(t)|}\right)\right| \\
&\le \frac{1}{4}\left|\ln\left(1+\frac{|x(t)|-|u(t)|}{1+|u(t)|}\right)\right|+\frac{1}{4}\left|\ln\left(1+\frac{\left|y(t)\right|-|v(t)|}{1+|v(t)|}\right)\right|+\frac{1}{4}\left|\ln\left(1+\frac{|z(t)|-|w(t)|}{1+|w(t)|}\right)\right| \\
&\le \frac{1}{4}\ln(1+|x-u|)+\frac{1}{4}\ln\left(1+\left|y-v\right|\right)+\frac{1}{4}\ln(1+|z-w|) \\
&\le \frac{1}{2}\ln\left(1+\frac{|x-u|+\left|y-v\right|+|z-w|}{3}\right)=\frac{1}{2}\phi\left(|x-u|,\left|y-v\right|,|z-w|\right).
\end{aligned}
$$

So we have

$$
\left|h_1(t,x,y,z)-h_1(t,u,v,w)\right| \le \frac{1}{2}\phi\left(|x-u|,\left|y-v\right|,|z-w|\right). \tag{22}
$$

D_2)

$$
\begin{aligned}
&\left|f_1(t,x,y,z,p)-f_1(t,u,v,w,q)\right| \\
&= \left| \begin{array}{c} \frac{3e^{-t}}{10}+\frac{t^2}{1+t^2}\ln(1+|x(t)|)+\frac{t^4}{2+t^4}\ln\left(1+\left|y(t)\right|\right) \\ +\frac{e^{-t^2}}{3}\ln(1+|z(t)|)+\ln\left(1+\frac{|p|}{2}\right) \\ -\left(\begin{array}{c} \frac{3e^{-t}}{10}+\frac{t^2}{1+t^2}\ln(1+|u(t)|)+\frac{t^4}{2+t^4}\ln(1+|v(t)|) \\ +\frac{e^{-t^2}}{3}\ln(1+|w(t)|)+\ln\left(1+\frac{|q|}{2}\right) \end{array} \right) \end{array} \right| \\
&\le \left|\frac{t^2}{1+t^2}\left(\ln(1+|x(t)|)-\ln(1+|u(t)|)\right)\right|+\left|\frac{t^4}{2+t^4}\left(\ln\left(1+\left|y(t)\right|\right)-\ln(1+|v(t)|)\right)\right| \\
&+\left|\frac{e^{-t^2}}{3}\left(\ln(1+|z(t)|)-\ln(1+|w(t)|)\right)\right|+\left|\ln\left(1+\frac{|p|}{2}\right)-\ln\left(1+\frac{|q|}{2}\right)\right| \\
&\le \left|\ln\left(\frac{1+|x(t)|}{1+|u(t)|}\right)\right|+\left|\ln\left(\frac{1+\left|y(t)\right|}{1+|v(t)|}\right)\right|+\frac{1}{3}\left|\ln\left(\frac{1+|z(t)|}{1+|w(t)|}\right)\right|+\frac{1}{2}\left|\ln\left(\frac{\left(1+\frac{|p|}{2}\right)}{\left(1+\frac{|q|}{2}\right)}\right)\right| \\
&\le \frac{1}{4}\ln(1+|x-u|)+\frac{1}{4}\ln\left(1+\left|y-v\right|\right)+\frac{1}{4}\ln(1+|z-w|)+\frac{1}{2}\left|p-q\right| \\
&\le \frac{1}{2}\phi\left(|x-u|,\left|y-v\right|,|z-w|\right)+\theta\left(\left|p-q\right|\right).
\end{aligned}
$$

So we have

$$\left|f_1(t,x,y,z,p)-f_1(t,u,v,w,q)\right| \le \frac{1}{2}\phi\left(|x-u|,\left|y-v\right|,|z-w|\right)+\theta\left(\left|p-q\right|\right), \tag{23}$$

the case $|u| \ge |x|$, $|v| \ge \left|y\right|$, *and* $|w| \ge |z|$ *can be done in the same manner for 22 and 23.*

(vi) *Clearly, g is continuous, Moreover, for each* $t,s \in \mathbb{R}_+$, *and* $x,y,z,u,v,w \in \mathbb{R}$ *we have*

$$\begin{aligned}
&\left|g_1(t,s,x,y,z)-g_1(t,s,u,v,w)\right| \\
=&\left|\begin{array}{c}\frac{\ln\left(1+\sqrt[3]{s}|x(t)|\right)\ln\left(1+\sqrt[3]{s}\left|y(t)\right|\right)\ln\left(1+\sqrt[3]{s}|z(t)|\right)+s\left(1+z^2(t)\right)\left(1+x^2(t)\right)\left(1+y^2(t)\right)}{e^t(1+x^2(t))\left(1+y^2(t)\right)(1+z^2(t))} \\ -\frac{\ln\left(1+\sqrt[3]{s}|u(t)|\right)\ln\left(1+\sqrt[3]{s}|v(t)|\right)\ln\left(1+\sqrt[3]{s}|w(t)|\right)+s\left(1+u^2(t)\right)\left(1+v^2(t)\right)\left(1+w^2(t)\right)}{e^t(1+u^2(t))(1+v^2(t))(1+w^2(t))}\end{array}\right| \\
=&\left|\begin{array}{c}\frac{\ln\left(1+\sqrt[3]{s}|x(t)|\right)\ln\left(1+\sqrt[3]{s}\left|y(t)\right|\right)\ln\left(1+\sqrt[3]{s}|z(t)|\right)}{e^t(1+x^2(t))\left(1+y^2(t)\right)(1+z^2(t))}+\frac{s}{e^t} \\ -\frac{\ln\left(1+\sqrt[3]{s}|u(t)|\right)\ln\left(1+\sqrt[3]{s}|v(t)|\right)\ln\left(1+\sqrt[3]{s}|w(t)|\right)}{e^t(1+u^2(t))(1+v^2(t))(1+w^2(t))}-\frac{s}{e^t}\end{array}\right| \\
\le& \frac{2s}{e^t}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\lim_{t\longrightarrow\infty}\int_0^t \left|g_1(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))-g_1(t,s,u(\eta(s)),v(\eta(s)),w(\eta(s)))\right| ds \\
\le& \lim_{t\longrightarrow\infty}\int_0^t \frac{2s}{e^t}ds=\lim_{t\longrightarrow\infty}\frac{t}{e^t}=0,
\end{aligned}$$

uniformly with respect to $x,y,z,u,v,w \in BC(\mathbb{R}_+)$.

Moreover, we have

$$\begin{aligned}
&\left|\int_0^{\beta(t)} g_1(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))\,ds\right| \\
\le& \int_0^t \left|g_1(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))\right| ds \\
\le& \int_0^t \frac{2s}{e^t}ds=\frac{t}{e^t},
\end{aligned}$$

for any $t,s \in \mathbb{R}_+$, *and* $x,y,z \in \mathbb{R}$.

Thus

$$\begin{aligned}
M_1^{'''} &= \sup\left\{\left|\int_0^{\beta(t)} g_1(t,s,x(\eta(s)),y(\eta(s)),z(\eta(s)))\,ds\right| : t,s\in\mathbb{R}_+, x,y,z\in BC(\mathbb{R}_+)\right\} \\
&\le \sup\left\{\frac{t}{e^t} : t\ge 0\right\}=r_0<\infty.
\end{aligned} \tag{24}$$

(vii) *By choosing* $M_1^{'''}=r_0$ *from (24) along with* $M_1=\frac{3}{4}, M_1^{'}=\frac{3}{10}, M_1^{''}=\frac{1}{4}$, *and* $\delta=1$ *in the inequality (12), we obtain the inequality* $\frac{2}{5}+\ln(1+r)+r_0<r$, *which* $\rho=3$ *is a solution. Consequently, all conditions of Theorem 4.1 are satisfied for the first equation in E.q. (21) and the rest of them, can be proven equivalently.*

Accordingly, the system of integral equations of (21) has at least one solution in the space $BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)\times BC(\mathbb{R}_+)$.

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