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Three Limit Representations of the Core-EP Inverse

Mengmeng Zhou[a], Jianlong Chen[a], Tingting Li[a], Dingguo Wang[b]

[a] *School of Mathematics, Southeast University, Nanjing, Jiangsu 210096, China*
[b] *School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong, 273165, China*

Abstract. In this paper, we present three limit representations of the core-EP inverse. The first approach is based on the full-rank decomposition of a given matrix. The second and third approaches, which depend on the explicit expression of the core-EP inverse, are established. The corresponding limit representations of the dual core-EP inverse are also given. In particular, limit representations of the core and dual core inverse are derived.

1. Introduction

In 1974, limit representation of the Drazin inverse was established by Meyer [12]. In 1986, Kalaba and Rasakhoo [8] introduced limit representation of the Moore-Penrose inverse. In 1994, An alternative limit representation of the Drazin inverse was given by Ji [7]. It is well known that the six kinds of classical generalized inverses: the Moore-Penrose inverse, the weighted Moore-Penrose inverse, the group inverse, the Drazin inverse, the Bott-Duffin inverse and the generalized Bott-Duffin inverse can be presented as particular generalized inverse $A^{(2)}_{T,S}$ with prescribed range and null space (see, for example, [2, 3, 5, 14]).

Let T be a subspace of $\mathbb{C}^n$ and let S be a subspace of $\mathbb{C}^m$. The generalized inverse $A^{(2)}_{T,S}$ [2] of matrix $A \in \mathbb{C}^{m\times n}$ is the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying

$$XAX = X, \mathcal{R}(X) = T, \mathcal{N}(X) = S.$$

In 1998, Wei [18] established a unified limit representation of the generalized inverse $A^{(2)}_{T,S}$. Let $A \in \mathbb{C}^{m\times n}$ be matrix of rank r, let T be a subspace of $\mathbb{C}^n$ of dimension $s \le r$, and let S be a subspace of $\mathbb{C}^m$ of dimension $m - s$. Suppose that $G \in \mathbb{C}^{n\times m}$ such that $R(G) = T$ and $N(G) = S$. If $A^{(2)}_{T,S}$ exists, then

$$A^{(2)}_{T,S} = \lim_{\lambda\to 0}(GA - \lambda I)^{-1}G = \lim_{\lambda\to 0} G(AG - \lambda I)^{-1},$$

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Email addresses: mmz9209@163.com (Mengmeng Zhou), jlchen@seu.edu.cn (Jianlong Chen), littnanjing@163.com (Tingting Li), dgwang@qfnu.edu.cn (Dingguo Wang)

where $R(G)$ and $N(G)$ denote the range space and null space of G, respectively.

In 1999, Stanimirović [15] introduced a more general limit formula. Suppose that M and N are two arbitrary $p \times q$ matrices, then

$$\lim_{\lambda \to 0}(M^*N + \lambda I)^{-1}M^* = \lim_{\lambda \to 0} M^*(NM^* + \lambda I)^{-1}. \tag{1}$$

The limit representation of generalized inverse $A^{(2)}_{T,S}$ is a special case of the above general formula.

In 2012, Liu et al. [10] introduced limit representation of the generalized inverse $A^{(2)}_{R(B),N(C)}$ in Banach space. Let $A \in \mathbb{C}^{m\times n}$ be matrix of rank r. Let T be a subspace of $\mathbb{C}^n$ of dimension $s \leq r$, and let S be a subspace of $\mathbb{C}^m$ of dimension $m - s$. Suppose $B \in \mathbb{C}^{n\times s}$ and $C \in \mathbb{C}^{s\times m}$ such that $R(B) = T$ and $N(C) = S$. If $A^{(2)}_{T,S}$ exists, then

$$A^{(2)}_{T,S} = \lim_{\lambda \to 0} B(CAB + \lambda I)^{-1}C.$$

In 2010, the core and dual core inverse were introduced by Baksalary and Trenkler for square matrices of index at most 1 in [1]. In 2014, the core inverse was extended to the core-EP inverse defined by Manjunatha Prasad and Mohana [11]. The core-EP inverse coincides with the core inverse if the index of a given matrix is 1. In this paper, the dual conception of the core-EP inverse was called the dual core-EP inverse. The characterizations of the core-EP and core inverse were investigated in complex matrices and rings (see, for example, [4, 6, 9, 13, 16, 17, 19]).

From the above mentioned limit representations of the generalized inverse, we know that limit representation of the core-EP inverse similar to the form of (1) has not been investigated in the literature.

The purpose of this paper is to establish three limit representations of the core-EP inverse. The first approach is based on the full-rank decomposition of a given matrix. The second and third approaches, which depend on the explicit expression of the core-EP inverse, are represented. The corresponding limit representations of the dual core-EP inverse are also given. In particular, limit representations of the core and dual core inverse are derived.

2. Preliminaries

In this section, we give some auxiliary definitions and lemmas.

For arbitrary matrix $A \in \mathbb{C}^{m\times n}$, the symbol $\mathbb{C}^{m\times n}$ denotes the set of all complex $m \times n$ matrices. A^* and $\mathrm{rk}(A)$ denote the conjugate transpose and rank of A, respectively. I is the identity matrix of an appropriate order. If k is the smallest nonnegative integer such that $\mathrm{rk}(A^k) = \mathrm{rk}(A^{k+1})$, then k is called the index of A and denoted by $\mathrm{ind}(A)$.

Definition 2.1. [2] *Let $A \in \mathbb{C}^{m\times n}$. The unique matrix $A^\dagger \in \mathbb{C}^{n\times m}$ is called the Moore-Penrose inverse of A if it satisfies*

$$AA^\dagger A = A,\ A^\dagger AA^\dagger = A^\dagger,\ (AA^\dagger)^* = AA^\dagger,\ (A^\dagger A)^* = A^\dagger A.$$

Definition 2.2. [2] *Let $A \in \mathbb{C}^{n\times n}$. The unique matrix $A^D \in \mathbb{C}^{n\times n}$ is called the Drazin inverse of A if it satisfies*

$$A^{k+1}A^D = A^k,\ A^DAA^D = A^D,\ AA^D = A^DA,$$

where $k = \mathrm{ind}(A)$. When $k = 1$, the Drazin inverse reduced to the group inverse and it is denoted by $A^\#$.

Definition 2.3. [1] *A matrix $A^{\circledast} \in \mathbb{C}^{n\times n}$ is called the core inverse of $A \in \mathbb{C}^{n\times n}$ if it satisfies*

$$AA^{\circledast} = P_A \text{ and } R(A^{\circledast}) \subseteq R(A).$$

Dually, a matrix $A_{\circledast} \in \mathbb{C}^{n\times n}$ is called the dual core inverse of $A \in \mathbb{C}^{n\times n}$ if it satisfies

$$A_{\circledast}A = P_{A^*} \text{ and } R(A_{\circledast}) \subseteq R(A^*).$$

Definition 2.4. [11] *A matrix $X \in \mathbb{C}^{n\times n}$, denoted by $A^{\oplus}$, is called the core-EP inverse of $A \in \mathbb{C}^{n\times n}$ if it satisfies*

$$XAX = X \text{ and } R(X) = R(X^*) = R(A^D).$$

Dually, a matrix $X \in \mathbb{C}^{n\times n}$, denoted by $A_{\oplus}$, is called the dual core-EP inverse of $A \in \mathbb{C}^{n\times n}$ if it satisfies

$$XAX = X \text{ and } R(X) = R(X^*) = R((A^*)^D).$$

The core-EP inverse was extended from complex matrices to rings by Gao and Chen in [6].

Lemma 2.5. [6] *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$ and let m be a positive integer with $m \geq k$. Then $A^{\oplus} = A^D A^m (A^m)^{\dagger}$.*

Clearly, If $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$, then it has a unique core-EP inverse. So according to Lemma 2.5, we have

$$A^{\oplus} = A^D A^k (A^k)^{\dagger} = A^D A^{k+1} (A^{k+1})^{\dagger} = A^k (A^{k+1})^{\dagger}. \tag{2}$$

As for an arbitrary matrix $A \in \mathbb{C}^{n\times n}$. If A is nilpotent, then $A^D = 0$. In this case, $A^{\oplus} = A_{\oplus} = 0$. This case is considered to be trivial. So we restrict the matrix A to be non-nilpotent in this paper.

Lemma 2.6. [2] *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. If $A = B_1G_1$ is a full-rank decomposition and $G_iB_i = B_{i+1}G_{i+1}$ are also full-rank decompositions, $i = 1, 2, ..., k-1$. Then the following statements hold:*

(1) *G_kB_k is invertible.*

(2) $A^k = B_1B_2 \cdots B_kG_k \cdots G_2G_1$.

(3) $A^D = B_1B_2 \cdots B_k(G_kB_k)^{-k-1}G_k \cdots G_1$.

In particular, for $k = 1$, then G_1B_1 is invertible and

$$A^{\#} = B_1(G_1B_1)^{-2}G_1.$$

According to [2], it is also known that $A^{\dagger} = G_1^*(G_1G_1^*)^{-1}(B_1^*B_1)^{-1}B_1^*$.

Lemma 2.7. [17] *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = 1$. If $A = MN$ is a full-rank decomposition, then*

$$A^{\textcircled{\#}} = M(NM)^{-1}(M^*M)^{-1}M^*.$$

Lemma 2.8. [16] *(Core-EP decomposition) Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. Then A can be written as the sum of matrices A_1 and A_2, i.e., $A = A_1 + A_2$, where*

(1) $\mathrm{rk}(A_1^2) = \mathrm{rk}(A_1)$.

(2) $A_2^k = 0$.

(3) $A_1^*A_2 = A_2A_1 = 0$.

3. The First Approach

In this section, we present limit representations of the core-EP and dual core-EP inverse, which depend on the full-rank decomposition of a given matrix. In particular, limit representations of the core and dual core inverse are also given.

Theorem 3.1. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$, and the full-rank decomposition of A be as in Lemma 2.6. Then*

$$\begin{aligned} A^{\textcircled{\dagger}} &= \lim_{\lambda\to 0} B(BG_kB_k)^*(BG_kB_k(BG_kB_k)^* + \lambda I)^{-1} \\ &= \lim_{\lambda\to 0} B((BG_kB_k)^*BG_kB_k + \lambda I)^{-1}(BG_kB_k)^* \end{aligned}$$

and

$$\begin{aligned} A_{\textcircled{\dagger}} &= \lim_{\lambda\to 0}((G_kB_kG)^*G_kB_kG + \lambda I)^{-1}(G_kB_kG)^*G \\ &= \lim_{\lambda\to 0}(G_kB_kG)^*(G_kB_kG(G_kB_kG)^* + \lambda I)^{-1}G, \end{aligned}$$

where $B = B_1B_2\cdots B_k$ and $G = G_k\cdots G_2G_1$.

Proof. From [15], we know that

$$A^\dagger = \lim_{\lambda\to 0} A^*(AA^* + \lambda I)^{-1} = \lim_{\lambda\to 0}(A^*A + \lambda I)^{-1}A^*. \tag{3}$$

So it is sufficient to verify $X = B(BG_kB_k)^\dagger = A^{\textcircled{\dagger}}$. Since

$$AB = B_1G_1B_1\cdots B_k = B_1B_2G_2B_2\cdots B_k = \cdots = BG_kB_k, \tag{4}$$

$$XAX = B(BG_kB_k)^\dagger AB(BG_kB_k)^\dagger \overset{(4)}{=} B(BG_kB_k)^\dagger = X.$$

According to the full-rank decomposition of A be as in Lemma 2.6, we know that B is full column rank and G is full row rank. So it is easy to verify that B^*B and GG^* are invertible. From Lemma 2.6, we obtain

$$B(BG_kB_k)^\dagger = B(G_kB_k)^{-1}(B^*B)^{-1}B^*.$$

Therefore,

$$X^* = B(B^*B)^{-1}((G_kB_k)^{-1})^*B^*.$$

Hence

$$\begin{gathered} \mathrm{rk}(B) = \mathrm{rk}(B(G_kB_k)^*B^*B) \le \mathrm{rk}(B(BG_kB_k)^*) \le \mathrm{rk}(B), \\ \mathrm{rk}(B) = \mathrm{rk}(B(B^*B)^{-1}((G_kB_k)^{-1})^*B^*B) \le \mathrm{rk}(X^*) \le \mathrm{rk}(B), \\ \mathrm{rk}(B) = \mathrm{rk}(B(G_kB_k)^{-k-1}GG^*) \le \mathrm{rk}(B(G_kB_k)^{-k-1}G) \le \mathrm{rk}(B). \end{gathered}$$

Thus, we have the following equalities:

$$\begin{gathered} R(X) = R(B(BG_kB_k)^\dagger) = R(B(BG_kB_k)^*) = R(B), \\ R(X^*) = R(B(B^*B)^{-1}((G_kB_k)^{-1})^*B^*) = R(B), \\ R(A^D) = R(B(G_kB_k)^{-k-1}G) = R(B). \end{gathered}$$

Namely,

$$R(X) = R(X^*) = R(A^D).$$

Similarly, we can verify $A_{\textcircled{\dagger}} = (G_kB_kG)^\dagger G$. This completes the proof. □

Let $k = 1$ in Theorem 3.1. Then we obtain the following corollary.

Corollary 3.2. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = 1$. If $A = BG$ is a full-rank decomposition, then*

$$\begin{aligned} A^{\textcircled{\#}} &= \lim_{\lambda\to 0} B(AB)^*(AB(AB)^* + \lambda I)^{-1} \\ &= \lim_{\lambda\to 0} B((AB)^*AB + \lambda I)^{-1}(AB)^* \end{aligned}$$

and

$$\begin{aligned} A_{\textcircled{\#}} &= \lim_{\lambda\to 0}((GA)^*GA + \lambda I)^{-1}(GA)^*G \\ &= \lim_{\lambda\to 0}(GA)^*(GA(GA)^* + \lambda I)^{-1}G. \end{aligned}$$

4. The Second and Third Approaches

In this section, we present two types of limit representations of the core-EP and dual core-EP inverse, which depend on their own explicit representation. In particular, limit representations of the core and dual core inverse are also given.

Based on Definition 2.4 and Lemma 2.5, we present the second approach by using the following equation firstly:

$$A^{\textcircled{\dagger}}A^{k+1} = A^D A^k (A^k)^{\dagger} A^{k+1} = A^k,$$

where $k = \mathrm{ind}(A)$.

Theorem 4.1. *Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. Then*

(1) $A^{\textcircled{\dagger}} = \lim\limits_{\lambda\to 0} A^k(A^k)^*(A^{k+1}(A^k)^* + \lambda I)^{-1}$,

(2) $A_{\textcircled{\dagger}} = \lim\limits_{\lambda\to 0}((A^k)^*A^{k+1} + \lambda I)^{-1}(A^k)^*A^k$.

Proof. Suppose that $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$. Let the core-EP decomposition of A be as in Lemma 2.8. From [16], we know that there exists a unitary matrix U such that $A = U\begin{bmatrix} T & S \\ 0 & N\end{bmatrix}U^*$, where T is a nonsingular matrix, $\mathrm{rk}(T) = \mathrm{rk}(A^k)$ and N is a nilpotent matrix of index k. The core-EP inverse of A is $A^{\textcircled{\dagger}} = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0\end{bmatrix}U^*$. By a direct computation, we obtain

$$A^k = U\begin{bmatrix} T^k & \hat{T} \\ 0 & 0\end{bmatrix}U^*, A^{k+1} = U\begin{bmatrix} T^{k+1} & T\hat{T} \\ 0 & 0\end{bmatrix}U^*,$$

$$(A^k)^* = U\begin{bmatrix} (T^k)^* & 0 \\ \hat{T}^* & 0\end{bmatrix}U^*, \text{ where } \hat{T} = \sum_{i=0}^{k-1} T^i S N^{k-1-i}, \tag{5}$$

$$A^k(A^k)^* = U\begin{bmatrix} T^k(T^k)^* + \hat{T}(\hat{T})^* & 0 \\ 0 & 0\end{bmatrix}U^*, \tag{6}$$

$$A^{k+1}(A^k)^* + \lambda I_n = U\begin{bmatrix} T^{k+1}(T^k)^* + T\hat{T}(\hat{T})^* + \lambda I_{\mathrm{rk}(T)} & 0 \\ 0 & \lambda I_{n-\mathrm{rk}(T)}\end{bmatrix}U^*. \tag{7}$$

Since T is nonsingular, $T^k(T^k)^* + \hat{T}(\hat{T})^*$ is positive definite matrix and the matrix $T^{k+1}(T^k)^* + T\hat{T}(\hat{T})^*$ is nonsingular. Here $\lambda \to 0$ means that $\lambda \to 0$ through neighborhood of 0 in $\mathbb{C}$ which excludes the negative nonzero eigenvalues of $T^{k+1}(T^k)^* + \hat{T}(\hat{T})^*$. When $\lambda \neq 0$, the matrix $T^{k+1}(T^k)^* + T\hat{T}(\hat{T})^* + \lambda I_{\mathrm{rk}(T)}$ is invertible. Combining with (6) and (7), we have

$$\begin{aligned}
&\lim_{\lambda\to 0} A^k(A^k)^*(A^{k+1}(A^k)^* + \lambda I)^{-1} = \lim_{\lambda\to 0} U\begin{bmatrix} T^k(T^k)^* + \hat{T}(\hat{T})^* & 0 \\ 0 & 0\end{bmatrix} \\
&\times U^*U\begin{bmatrix} T^{k+1}(T^k)^* + T\hat{T}(\hat{T})^* + \lambda I_{\mathrm{rk}(T)} & 0 \\ 0 & \lambda I_{n-\mathrm{rk}(T)}\end{bmatrix}^{-1} U^* = \\
&\lim_{\lambda\to 0} U\begin{bmatrix} (T^k(T^k)^* + \hat{T}(\hat{T})^*)(T^{k+1}(T^k)^* + T\hat{T}(\hat{T})^* + \lambda I_{\mathrm{rk}(T)})^{-1} & 0 \\ 0 & 0\end{bmatrix}^{-1} U^* = \\
&U\begin{bmatrix} (T^k(T^k)^* + \hat{T}(\hat{T})^*)(T^{k+1}(T^k)^* + T\hat{T}(\hat{T})^*)^{-1} & 0 \\ 0 & 0\end{bmatrix}U^* = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0\end{bmatrix}U^*.
\end{aligned}$$

(2) It is analogous. □

Remark 4.2. *Here $\lambda \to 0$ means that $\lambda \to 0$ through any neighborhood of 0 in $\mathbb{C}$, which excludes the negative nonzero eigenvalues of $A^{k+1}(A^k)^*$ and $(A^k)^*A^{k+1}$ in Theorem 4.1 (1) and (2), respectively. As for the following corollary 4.3, λ is not equal to the negative nonzero eigenvalues of A^2A^* (or A^*A^2) in corollary 4.3 (1)(or (2)) when $\lambda \neq 0$.*

Let $k = 1$, we have the following corollary.

Corollary 4.3. *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = 1$. *Then*

(1) $A^{\textcircled{\#}} = \lim\limits_{\lambda\to 0} AA^*(A^2A^* + \lambda I)^{-1}$,

(2) $A_{\textcircled{\#}} = \lim\limits_{\lambda\to 0}(A^*A^2 + \lambda I)^{-1}A^*A$.

Next, we present the third approach of limit representations of the core-EP inverse. From Definition 2.4 and equation (2), it is easy to know that

$$A^{\textcircled{\dagger}} = A^k(A^{k+1})^\dagger \text{ and } A_{\textcircled{\dagger}} = (A^{k+1})^\dagger A^k,$$

where $k = \operatorname{ind}(A)$. Combining with limit representation of the Moore-Penrose inverse of A, we have the following theorem.

Theorem 4.4. *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = k$. *Then*

$$\begin{aligned} A^{\textcircled{\dagger}} &= \lim_{\lambda\to 0} A^k(A^{k+1})^*(A^{k+1}(A^{k+1})^* + \lambda I)^{-1} \\ &= \lim_{\lambda\to 0} A^k((A^{k+1})^*A^{k+1} + \lambda I)^{-1}(A^{k+1})^* \end{aligned}$$

and

$$\begin{aligned} A_{\textcircled{\dagger}} &= \lim_{\lambda\to 0}((A^{k+1})^*A^{k+1} + \lambda I)^{-1}(A^{k+1})^*A^k \\ &= \lim_{\lambda\to 0}(A^{k+1})^*(A^{k+1}(A^{k+1})^* + \lambda I)^{-1}A^k \end{aligned}$$

Let $k = 1$, we have the following corollary.

Corollary 4.5. *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = 1$. *Then*

$$\begin{aligned} A^{\textcircled{\#}} &= \lim_{\lambda\to 0} A(A^2)^*(A^2(A^2)^* + \lambda I)^{-1} \\ &= \lim_{\lambda\to 0} A((A^2)^*A^2 + \lambda I)^{-1}(A^2)^* \end{aligned}$$

and

$$\begin{aligned} A_{\textcircled{\#}} &= \lim_{\lambda\to 0}((A^2)^*A^2 + \lambda I)^{-1}(A^2)^*A \\ &= \lim_{\lambda\to 0}(A^2)^*(A^2(A^2)^* + \lambda I)^{-1}A. \end{aligned}$$

Let $k = 1$. From [1], we know that $A^{\textcircled{\#}} = (A^2A^\dagger)^\dagger$. According to the above corollary, it is known that $A^{\textcircled{\#}} = A(A^2)^\dagger$. Since the core inverse is unique, we obtain the following corollary.

Corollary 4.6. *Let $A \in \mathbb{C}^{n\times n}$ with* $\operatorname{ind}(A) = 1$. *Then* $(A^2A^\dagger)^\dagger = A(A^2)^\dagger$.

5. Examples

In this section, we present two examples to illustrate the efficacy of the established limit representations in this paper.

Example 5.1. *Let* $A = \begin{bmatrix} 1 & 1 & 2 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 3 & 1 \\ 1 & 0 & 1 & 4 \end{bmatrix}$*, then* $\mathrm{ind}(A) = 2$ *and* $\mathrm{rk}(A) = 3$*. Let* $B_1 = \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 1 \\ 0 & 3 & 1 \\ 1 & 0 & 4 \end{bmatrix}$, $G_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,
$B_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, $G_2 = \begin{bmatrix} 0 & 4 & 2 \\ 1 & 0 & 4 \end{bmatrix}$.
The exact core-EP inverse of A *is equal to*
$A^{\textcircled{\dagger}} = \frac{1}{756}\begin{bmatrix} 80 & 4 & -24 & 76 \\ 8 & 13 & 48 & -5 \\ -8 & 50 & 204 & -58 \\ 72 & -9 & -72 & 81 \end{bmatrix}$*. Set* $B = B_1B_2$ *and* $E = G_2B_2$*.*
Using matlab, we have

$$\lim_{\lambda\to 0} B(BE)^*(BE(BE)^* + \lambda I)^{-1} = \begin{bmatrix} \frac{20}{189} & \frac{1}{189} & -\frac{2}{63} & \frac{19}{189} \\ \frac{2}{189} & \frac{13}{756} & \frac{4}{63} & -\frac{5}{756} \\ -\frac{2}{189} & \frac{25}{378} & \frac{17}{63} & -\frac{29}{378} \\ \frac{2}{21} & -\frac{1}{84} & -\frac{2}{21} & \frac{3}{28} \end{bmatrix},$$

$$\lim_{\lambda\to 0} A^2(A^2)^*(A^3(A^2)^* + \lambda I)^{-1} = \begin{bmatrix} \frac{20}{189} & \frac{1}{189} & -\frac{2}{63} & \frac{19}{189} \\ \frac{2}{189} & \frac{13}{756} & \frac{4}{63} & -\frac{5}{756} \\ -\frac{2}{189} & \frac{25}{378} & \frac{17}{63} & -\frac{29}{378} \\ \frac{2}{21} & -\frac{1}{84} & -\frac{2}{21} & \frac{3}{28} \end{bmatrix},$$

$$\lim_{\lambda\to 0} A^2(A^3)^*(A^3(A^3)^* + \lambda I)^{-1} = \begin{bmatrix} \frac{20}{189} & \frac{1}{189} & -\frac{2}{63} & \frac{19}{189} \\ \frac{2}{189} & \frac{13}{756} & \frac{4}{63} & -\frac{5}{756} \\ -\frac{2}{189} & \frac{25}{378} & \frac{17}{63} & -\frac{29}{378} \\ \frac{2}{21} & -\frac{1}{84} & -\frac{2}{21} & \frac{3}{28} \end{bmatrix}.$$

Example 5.2. *Let* $A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & 0 & 2 \\ 2 & 0 & 1 \end{bmatrix}$*, then* $\mathrm{rk}(A) = 2$ *and* $\mathrm{ind}(A) = 1$*. Let*
$B = \begin{bmatrix} 1 & 3 \\ 4 & 2 \\ 2 & 1 \end{bmatrix}$, $G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$*. According to Lemma 2.7, the exact core inverse of* A *is equal to*

$$A^{\textcircled{\#}} = B(B^*BGB)^{-1}B^* = \begin{bmatrix} -0.2000 & 0.2400 & 0.1200 \\ 0.8000 & -0.1600 & -0.0800 \\ 0.4000 & -0.0800 & -0.0400 \end{bmatrix}.$$

Using matlab, we obtain

$$\lim_{\lambda\to 0} B(AB)^*(AB(AB)^* + \lambda I)^{-1} = \begin{bmatrix} -\frac{1}{5} & \frac{6}{25} & \frac{3}{25} \\ \frac{4}{5} & -\frac{4}{25} & -\frac{2}{25} \\ \frac{2}{5} & -\frac{2}{25} & -\frac{1}{25} \end{bmatrix},$$

$$\lim_{\lambda\to 0} AA^*(A^2A^* + \lambda I)^{-1} = \begin{bmatrix} -\frac{1}{5} & \frac{6}{25} & \frac{3}{25} \\ \frac{4}{5} & -\frac{4}{25} & -\frac{2}{25} \\ \frac{2}{5} & -\frac{2}{25} & -\frac{1}{25} \end{bmatrix},$$

$$\lim_{\lambda\to 0} A(A^2)^*(A^2(A^2)^* + \lambda I)^{-1} = \begin{bmatrix} -\frac{1}{5} & \frac{6}{25} & \frac{3}{25} \\ \frac{4}{5} & -\frac{4}{25} & -\frac{2}{25} \\ \frac{2}{5} & -\frac{2}{25} & -\frac{1}{25} \end{bmatrix}.$$

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