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A New Method for Finding the Shape Operator of a Hypersurface in Euclidean 4-Space

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Abstract. In this paper, by taking a Frenet curve lying on a parametric or implicit hypersurface and using the extended Darboux frame field of this curve, we give a new method for calculating the shape operator's matrix of the hypersurface depending on the extended Darboux frame curvatures. This new method enables us to obtain the Gaussian and mean curvatures of the hypersurface depending on the geodesic torsions of the curve and the normal curvatures of the hypersurface.

1. Introduction

In differential geometry, the shape of a geometric object (curve, surface, etc.) has always been a point of interest. In Euclidean space, along a space curve we have only one direction to move. This direction is determined by the tangent vector of the curve. The curvature of the curve measures the rate of change of its tangent vector, and together with the torsion of the curve they give us some information to its shape. However, along a surface we have different directions to move. These directions constitute a plane at a point, called the tangent plane of the surface at that point. When we move along a given direction, the tangent plane and thereby the normal vector of this plane, that is, the unit normal vector of the surface will change its direction except for some special cases. It is well-known that the rate of change of the unit normal vector of a surface has been measured by the shape operator. That's why computing the shape operator of a surface is important in surface theory. The calculation of the shape operator of a surface in Euclidean 3-space is well-known [3, 4]. The shape operator's matrix of a surface can be expressed by depending on the first and second fundamental form coefficients of the surface by using the Weingarten equations [3].

In this paper, we give a method which enables us to compute the shape operator of a hypersurface in Euclidean 4-space. We consider a Frenet curve lying on a hypersurface, and use the extended Darboux frame of the Frenet curve in Euclidean 4-space while constructing this new method. Our method includes two cases in which one can obtain directly the shape operator's matrix of a hypersurface along a Frenet curve. The elements of obtained new matrix are composed of the geodesic torsions of first and second order of the curve and the normal curvatures of the hypersurface.

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2. Preliminaries

Definition 2.1. *Let $\mathcal{M}$ be an orientable hypersurface with the unit normal vector field* N. *The shape operator* $\mathcal{S} : \chi(\mathcal{M}) \to \chi(\mathcal{M})$ *of $\mathcal{M}$ is defined by* $\mathcal{S}(\mathsf{X}) = -\mathcal{D}_{\mathsf{X}}^{\mathsf{N}}$ *for* $\mathsf{X} \in \chi(\mathcal{M})$, *where $\mathcal{D}$ is the Riemannian connection of* $\mathbb{E}^4$, *[5].*

Definition 2.2. *Let $\mathcal{M}$ be an orientable hypersurface in $\mathbb{E}^4$ and P be a point of $\mathcal{M}$. Then* $\mathsf{K}(P) = \det \mathcal{S}_P$ *and* $\mathsf{H}(P) = \frac{1}{3}\mathit{trace}(\mathcal{S}_P)$ *are called the Gaussian and the mean curvatures of $\mathcal{M}$ at P, respectively, [5].*

Definition 2.3. *Let $\mathcal{M}$ be an orientable hypersurface in $\mathbb{E}^4$, P be a point of $\mathcal{M}$, and* $\mathsf{X}_P \in T_P(\mathcal{M})$. *The normal curvature at P of $\mathcal{M}$ in the direction of X_P is defined by* $\kappa_n(\mathsf{X}_P) = \dfrac{\langle \mathcal{S}(\mathsf{X}_P), \mathsf{X}_P\rangle}{\langle \mathsf{X}_P, \mathsf{X}_P\rangle}$*,[3].*

Definition 2.4. *Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ be the standard basis of $\mathbb{R}^4$. The ternary product of the vectors* $\mathbf{u} = \sum_{i=1}^{4} u_i \mathbf{e_i}$, $\mathbf{v} = \sum_{i=1}^{4} v_i \mathbf{e_i}$, *and* $\mathbf{w} = \sum_{i=1}^{4} w_i \mathbf{e_i}$ *is defined by the vector [7]*

$$\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix},$$

and satisfies the following properties:

$$\mathbf{x} \otimes \mathbf{y} \otimes (\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}) = \begin{vmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \\ \langle \mathbf{u}, \mathbf{y}\rangle & \langle \mathbf{v}, \mathbf{y}\rangle & \langle \mathbf{w}, \mathbf{y}\rangle \\ \langle \mathbf{u}, \mathbf{x}\rangle & \langle \mathbf{v}, \mathbf{x}\rangle & \langle \mathbf{w}, \mathbf{x}\rangle \end{vmatrix},$$

$$\Big\langle \mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}, \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} \Big\rangle = \begin{vmatrix} \langle \mathbf{x}, \mathbf{u}\rangle & \langle \mathbf{x}, \mathbf{v}\rangle & \langle \mathbf{x}, \mathbf{w}\rangle \\ \langle \mathbf{y}, \mathbf{u}\rangle & \langle \mathbf{y}, \mathbf{v}\rangle & \langle \mathbf{y}, \mathbf{w}\rangle \\ \langle \mathbf{z}, \mathbf{u}\rangle & \langle \mathbf{z}, \mathbf{v}\rangle & \langle \mathbf{z}, \mathbf{w}\rangle \end{vmatrix}.$$

2.1. Curves on a parametric hypersurface in $\mathbb{E}^4$

Let $\mathcal{M}$ be a regular hypersurface given by its parametric equation $\mathbf{R} = \mathbf{R}(u_1, u_2, u_3)$ and $\beta : I \subset \mathbb{R} \to \mathcal{M}$ be a Frenet curve with arc-length parametrization. Since $\mathcal{M}$ is regular, $\mathbf{R}_1 \otimes \mathbf{R}_2 \otimes \mathbf{R}_3 \neq \mathbf{0}$ at every point of $\mathcal{M}$, where $\mathbf{R}_i = \frac{\partial \mathbf{R}}{\partial u_i}$. Then, the unit normal vector field of $\mathcal{M}$ is given by

$$\mathbf{N} = \frac{\mathbf{R}_1 \otimes \mathbf{R}_2 \otimes \mathbf{R}_3}{\|\mathbf{R}_1 \otimes \mathbf{R}_2 \otimes \mathbf{R}_3\|}. \tag{1}$$

Moreover, since the curve $\beta(s)$ lies on $\mathcal{M}$, we may write $\beta(s) = \mathbf{R}\Big(u_1(s), u_2(s), u_3(s)\Big)$. Then we have

$$\beta'(s) = \sum_{i=1}^{3} \mathbf{R}_i u_i', \tag{2}$$

$$\beta''(s) = \sum_{i=1}^{3} \mathbf{R}_i u_i'' + \sum_{i,j=1}^{3} \mathbf{R}_{ij} u_i' u_j', \tag{3}$$

$$\beta'''(s) = \sum_{i=1}^{3} \mathbf{R}_i u_i''' + 3\sum_{i,j=1}^{3} \mathbf{R}_{ij} u_i'' u_j' + \sum_{i,j,k=1}^{3} \mathbf{R}_{ijk} u_i' u_j' u_k',$$

where $\mathbf{R}_{ij} = \frac{\partial^2 \mathbf{R}}{\partial u_j \partial u_i}$, and $\mathbf{R}_{ijk} = \frac{\partial^3 \mathbf{R}}{\partial u_k \partial u_j \partial u_i}$, $1 \leq i, j, k \leq 3$.

2.2. Curves on an implicit hypersurface in $\mathbb{E}^4$

Let $\mathcal{M}$ be a regular hypersurface given by $f(x,y,z,w)=0$, and $\beta(s)=(x(s),y(s),z(s),w(s))$ be a Frenet curve on $\mathcal{M}$. Since $\mathcal{M}$ is regular, at every point of $\mathcal{M}$ we have $\mathsf{N}(s)=\frac{\nabla f}{\|\nabla f\|}(s)$ for the unit normal vector field of $\mathcal{M}$ along β. Besides, we have [1]

$$\begin{aligned}
&\langle \nabla f, \beta' \rangle = 0,\\
&\langle \nabla f, \beta'' \rangle = -\beta' H_f\,(\beta')^t,\\
&\langle \nabla f, \beta''' \rangle = -3\beta' H_f\,(\beta'')^t - \beta' \frac{d(H_f)}{ds}\,(\beta')^t,
\end{aligned} \tag{4}$$

where $\beta'=[x'\ y'\ z'\ w']$, $\beta''=[x''\ y''\ z''\ w'']$, $\beta'''=[x'''\ y'''\ z'''\ w''']$, $\nabla f=[f_x\ f_y\ f_z\ f_w]$, and

$$H_f=\begin{bmatrix} f_{xx} & f_{xy} & f_{xz} & f_{xw}\\ f_{yx} & f_{yy} & f_{yz} & f_{yw}\\ f_{zx} & f_{zy} & f_{zz} & f_{zw}\\ f_{wx} & f_{wy} & f_{wz} & f_{ww}\end{bmatrix}, \qquad \frac{d\left(H_f\right)}{ds}=\begin{bmatrix} \frac{\partial H_f}{\partial x}(\beta')^t & \cdots & \frac{\partial H_f}{\partial w}(\beta')^t \end{bmatrix},$$

$$\frac{\partial H_f}{\partial x}=\begin{bmatrix} f_{xxx} & f_{xyx} & f_{xzx} & f_{xwx}\\ f_{yxx} & f_{yyx} & f_{yzx} & f_{ywx}\\ f_{zxx} & f_{zyx} & f_{zzx} & f_{zwx}\\ f_{wxx} & f_{wyx} & f_{wzx} & f_{wwx}\end{bmatrix},\ldots,\ \frac{\partial H_f}{\partial w}=\begin{bmatrix} f_{xxw} & f_{xyw} & f_{xzw} & f_{xww}\\ f_{yxw} & f_{yyw} & f_{yzw} & f_{yww}\\ f_{zxw} & f_{zyw} & f_{zzw} & f_{zww}\\ f_{wxw} & f_{wyw} & f_{wzw} & f_{www}\end{bmatrix}.$$

*2.3. The extended Darboux frame(*ED*-frame) field in* $\mathbb{E}^4$

Let $\mathcal{M}\subset\mathbb{E}^4$ be an orientable hypersurface and $\beta: I\subset\mathbb{R}\to\mathcal{M}$ be a Frenet curve in Euclidean 4-space $\mathbb{E}^4$. Let T and N denote the unit tangent vector field of the curve β and the unit normal vector field of $\mathcal{M}$ restricted to the curve β, respectively. Then the extended Darboux frame (ED-frame) field along β is denoted by $\{\mathsf{T},\mathsf{E},\mathsf{D},\mathsf{N}\}$ [2], where

$$\mathsf{E}=\frac{\beta''-\langle\beta'',\mathsf{N}\rangle\mathsf{N}}{\|\beta''-\langle\beta'',\mathsf{N}\rangle\mathsf{N}\|}, \qquad \text{if } \{\mathsf{N},\mathsf{T},\beta''\} \text{ is linearly independent (Case 1)}, \tag{5}$$

$$\mathsf{E}=\frac{\beta'''-\langle\beta''',\mathsf{N}\rangle\mathsf{N}-\langle\beta''',\mathsf{T}\rangle\mathsf{T}}{\|\beta'''-\langle\beta''',\mathsf{N}\rangle\mathsf{N}-\langle\beta''',\mathsf{T}\rangle\mathsf{T}\|}, \qquad \text{if } \{\mathsf{N},\mathsf{T},\beta''\} \text{ is linearly dependent (Case 2)}$$

and

$$\mathsf{D}=\mathsf{N}\otimes\mathsf{T}\otimes\mathsf{E}, \qquad \text{in both cases.}$$

The differential equations for ED-frame field have the form

$$\begin{bmatrix}\mathsf{T}'\\ \mathsf{E}'\\ \mathsf{D}'\\ \mathsf{N}'\end{bmatrix}=\begin{bmatrix} 0 & \kappa_g^1 & 0 & \kappa_n\\ -\kappa_g^1 & 0 & \kappa_g^2 & \tau_g^1\\ 0 & -\kappa_g^2 & 0 & \tau_g^2\\ -\kappa_n & -\tau_g^1 & -\tau_g^2 & 0\end{bmatrix}\begin{bmatrix}\mathsf{T}\\ \mathsf{E}\\ \mathsf{D}\\ \mathsf{N}\end{bmatrix} \qquad \text{(Case 1)},$$

$$\begin{bmatrix}\mathsf{T}'\\ \mathsf{E}'\\ \mathsf{D}'\\ \mathsf{N}'\end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 & \kappa_n\\ 0 & 0 & \kappa_g^2 & \tau_g^1\\ 0 & -\kappa_g^2 & 0 & 0\\ -\kappa_n & -\tau_g^1 & 0 & 0\end{bmatrix}\begin{bmatrix}\mathsf{T}\\ \mathsf{E}\\ \mathsf{D}\\ \mathsf{N}\end{bmatrix} \qquad \text{(Case 2)},$$

where κ_g^i and τ_g^i are the geodesic curvature and the geodesic torsion of order i, $(i=1,2)$, respectively [2].

3. Shape Operator's Matrix of a Hypersurface

Let us consider an orientable hypersurface $\mathcal{M}$ in 4-dimensional Euclidean space and take a Frenet curve $\beta(s)$ lying on this hypersurface. Let us denote the extended Darboux frame field along β with $\{\mathsf{T}, \mathsf{E}, \mathsf{D}, \mathsf{N}\}$. It is obvious from the definition that $\{\mathsf{T}, \mathsf{E}, \mathsf{D}\}$ constitutes a basis for $\chi(\mathcal{M})$ along β. Our aim is now to obtain the matrix of the shape operator along the curve and to give the Gaussian and the mean curvatures of the hypersurface depending on the geodesic torsions of first and second order of the curve and the normal curvatures in the directions of E, D and $\mathsf{E} + \mathsf{D}$ of the hypersurface for each case.

3.1. *Case 1*

Proposition 3.1. *Let $\mathcal{M}$ be an orientable hypersurface, β be a Frenet curve lying on $\mathcal{M}$, and $\{\mathsf{T}, \mathsf{E}, \mathsf{D}, \mathsf{N}\}$ denotes the* ED*-frame field of β. Then, the shape operator's matrix of $\mathcal{M}$ along β with respect to the basis $\{\mathsf{T}, \mathsf{E}, \mathsf{D}\}$ is obtained by*

$$\mathcal{S} = \begin{bmatrix} \kappa_n(\mathsf{T}) & \tau_g^1 & \tau_g^2 \\ \tau_g^1 & \kappa_n(\mathsf{E}) & \Psi \\ \tau_g^2 & \Psi & \kappa_n(\mathsf{D}) \end{bmatrix}, \tag{6}$$

where

$$\Psi = \kappa_n(\mathsf{E} + \mathsf{D}) - \frac{1}{2}\Big(\kappa_n(\mathsf{E}) + \kappa_n(\mathsf{D})\Big). \tag{7}$$

Proof. Let

$$\begin{cases} \mathcal{S}(\mathsf{T}) = a_{11}\mathsf{T} + a_{12}\mathsf{E} + a_{13}\mathsf{D} \\ \mathcal{S}(\mathsf{E}) = a_{21}\mathsf{T} + a_{22}\mathsf{E} + a_{23}\mathsf{D} \\ \mathcal{S}(\mathsf{D}) = a_{31}\mathsf{T} + a_{32}\mathsf{E} + a_{33}\mathsf{D} \end{cases}. \tag{8}$$

Then, we obtain

$$a_{11} = \langle \mathcal{S}(\mathsf{T}), \mathsf{T} \rangle = \kappa_n(\mathsf{T}), \quad a_{22} = \langle \mathcal{S}(\mathsf{E}), \mathsf{E} \rangle = \kappa_n(\mathsf{E}), \quad a_{33} = \langle \mathcal{S}(\mathsf{D}), \mathsf{D} \rangle = \kappa_n(\mathsf{D}),$$

$$a_{12} = a_{21} = \langle \mathcal{S}(\mathsf{T}), \mathsf{E} \rangle = \langle \mathsf{E}', \mathsf{N} \rangle = \tau_g^1, \qquad a_{13} = a_{31} = \langle \mathcal{S}(\mathsf{T}), \mathsf{D} \rangle = \langle \mathsf{D}', \mathsf{N} \rangle = \tau_g^2.$$

On the other hand, we may write

$$\langle \mathcal{S}(\mathsf{E} + \mathsf{D}), \mathsf{E} + \mathsf{D} \rangle = \langle \mathcal{S}(\mathsf{E}), \mathsf{E} \rangle + \langle \mathcal{S}(\mathsf{D}), \mathsf{D} \rangle + 2\langle \mathcal{S}(\mathsf{E}), \mathsf{D} \rangle$$

i.e.

$$2\kappa_n(\mathsf{E} + \mathsf{D}) = \kappa_n(\mathsf{E}) + \kappa_n(\mathsf{D}) + 2\langle \mathcal{S}(\mathsf{E}), \mathsf{D} \rangle.$$

Thus, we have

$$a_{23} = a_{32} = \langle \mathcal{S}(\mathsf{E}), \mathsf{D} \rangle = \kappa_n(\mathsf{E} + \mathsf{D}) - \frac{1}{2}\Big(\kappa_n(\mathsf{E}) + \kappa_n(\mathsf{D})\Big) =: \Psi.$$

Substituting the obtained results into (8), we find the shape operator's matrix $\mathcal{S}$ as desired. □

The computations of $\kappa_n(\mathsf{T})$, τ_g^1 and τ_g^2 are known for not only parametric hypersurfaces [6] but also for implicit hypersurfaces [2] in Euclidean 4-space. To make the matrix of the shape operator in the Proposition 3.1 understandable, we now show how the normal curvatures in the directions of E, D and $\mathsf{E} + \mathsf{D}$ can be obtained for both parametric and implicit hypersurfaces.

Lemma 3.2. *Let $\mathcal{M}$ be an orientable hypersurface given by $R(u_1, u_2, u_3)$ and $\beta(s) = R(u_1(s), u_2(s), u_3(s))$ be a Frenet curve lying on $\mathcal{M}$. Denoting the ED-frame field of β by $\{\mathsf{T}, \mathsf{E}, \mathsf{D}, \mathsf{N}\}$, the normal curvature of $\mathcal{M}$ in the direction of T is (also in Case 2) given by [6]*

$$\kappa_n(\mathsf{T}) = \sum_{i,j=1}^{3} h_{ij} u_i' u_j',$$

where $h_{ij} = \langle R_{ij}, \mathsf{N}\rangle$, $(1 \leq i, j \leq 3)$, denotes the second fundamental form coefficients of $\mathcal{M}$.

Lemma 3.3. *Let $\mathcal{M}$ be an orientable hypersurface given by $R(u_1, u_2, u_3)$, and $\beta(s) = R(u_1(s), u_2(s), u_3(s))$ be a Frenet curve with its ED-frame field $\{\mathsf{T}, \mathsf{E}, \mathsf{D}, \mathsf{N}\}$. Then, the normal curvature of $\mathcal{M}$ in the direction of E is found as*

$$\kappa_n(\mathsf{E}) = \sum_{i,j=1}^{3} h_{ij} a_i a_j, \tag{9}$$

where

$$a_i = \frac{1}{\Delta^2}\Big\{(g_{jj}g_{kk} - g_{jk}^2)\langle R_i, \mathsf{E}\rangle + (g_{ik}g_{jk} - g_{ij}g_{kk})\langle R_j, \mathsf{E}\rangle + (g_{ij}g_{jk} - g_{ik}g_{jj})\langle R_k, \mathsf{E}\rangle\Big\}, \tag{10}$$

i,j,k=1,2,3 (cyclic),

$$\Delta^2 = \|R_1 \otimes R_2 \otimes R_3\|^2 = g_{11}g_{22}g_{33} + 2g_{12}g_{13}g_{23} - g_{11}g_{23}^2 - g_{22}g_{13}^2 - g_{33}g_{12}^2, \tag{11}$$

$$\langle R_m, \mathsf{E}\rangle = \frac{1}{\mu}\Big\{\sum_{i=1}^{3} g_{im}u_i'' + \sum_{i,j=1}^{3}\langle R_{ij}, R_m\rangle u_i'u_j'\Big\}, \quad m = 1, 2, 3,$$

$$\mu = \Big\{\sum_{i,j=1}^{3} g_{ij}u_i''u_j'' + 2\sum_{i,j,k=1}^{3}\langle R_{ij}, R_k\rangle u_i'u_j'u_k'' + \sum_{i,j,k,\ell=1}^{3}\langle R_{ij}, R_{k\ell}\rangle u_i'u_j'u_k'u_\ell' - \Big(\sum_{i,j=1}^{3} h_{ij}u_i'u_j'\Big)^2\Big\}^{\frac{1}{2}},$$

and $g_{ij} = \langle R_i, R_j\rangle$, $(1 \leq i, j \leq 3)$, denotes the first fundamental form coefficients of $\mathcal{M}$.

Proof. We may write

$$\mathsf{E} = \sum_{i=1}^{3} R_i a_i. \tag{12}$$

Taking the scalar product both hand sides of (12) with R_j, $j = 1, 2, 3$, we obtain the linear equations $\langle R_j, \mathsf{E}\rangle = \sum_{i=1}^{3} g_{ij}a_i$. These equations constitute a linear system which has nonzero coefficients determinant $\Delta^2 = \|R_1 \otimes R_2 \otimes R_3\|^2$. By solving this system, we find the scalars a_i as

$$a_i = \frac{1}{\Delta^2}\Big\{(g_{jj}g_{kk} - g_{jk}^2)\langle R_i, \mathsf{E}\rangle + (g_{ik}g_{jk} - g_{ij}g_{kk})\langle R_j, \mathsf{E}\rangle + (g_{ij}g_{jk} - g_{ik}g_{jj})\langle R_k, \mathsf{E}\rangle\Big\},$$

i,j,k=1,2,3 (cyclic). If we use (5) and (3), we get

$$\langle R_m, \mathsf{E}\rangle = \frac{1}{\mu}\Big\{\sum_{i=1}^{3} g_{im}u_i'' + \sum_{i,j=1}^{3}\langle R_{ij}, R_m\rangle u_i'u_j'\Big\}, \quad 1 \leq m \leq 3,$$

where

$$\begin{aligned}\mu &= \|\beta'' - \langle \beta'', \mathsf{N}\rangle \mathsf{N}\| = \{\langle \beta'', \beta''\rangle - \langle \beta'', \mathsf{N}\rangle^2\}^{\frac{1}{2}} \\ &= \Big\{\sum_{i,j=1}^{3} g_{ij}u_i''u_j'' + 2\sum_{i,j,k=1}^{3}\langle R_{ij}, R_k\rangle u_i'u_j'u_k'' + \sum_{i,j,k,\ell=1}^{3}\langle R_{ij}, R_{k\ell}\rangle u_i'u_j'u_k'u_\ell' - \Big(\sum_{i,j=1}^{3} h_{ij}u_i'u_j'\Big)^2\Big\}^{\frac{1}{2}}.\end{aligned}$$

Since $\kappa_n(\mathsf{E}) = \langle \mathcal{S}(\mathsf{E}), \mathsf{E}\rangle$, using Lemma 3.2, the normal curvature of $\mathcal{M}$ in the direction of E is obtained as desired. □

Lemma 3.4. *Let $\mathcal{M}$ be an orientable hypersurface given by $R(u_1, u_2, u_3)$ and $\beta(s) = R(u_1(s), u_2(s), u_3(s))$ be a Frenet curve lying on $\mathcal{M}$. Denoting the* ED*-frame field of β by $\{\mathsf{T}, \mathsf{E}, \mathsf{D}, \mathsf{N}\}$, the normal curvature of $\mathcal{M}$ in the direction of* D *is obtained by*

$$\kappa_n(\mathsf{D}) = \sum_{i,j=1}^{3} h_{ij} A_i A_j, \tag{13}$$

where

$$A_i = \frac{1}{\Delta}\left\{\langle R_j, \mathsf{E}\rangle \sum_{m=1}^{3} g_{km} u'_m - \langle R_k, \mathsf{E}\rangle \sum_{m=1}^{3} g_{jm} u'_m\right\}, \quad i, j, k = 1, 2, 3(cyclic). \tag{14}$$

Proof. Let $\mathsf{D} = \sum_{i=1}^{3} R_i A_i$. Since $\mathsf{D} = \mathsf{N} \otimes \mathsf{T} \otimes \mathsf{E}$, using (1) we get

$$\mathsf{D} = \frac{1}{\Delta} \sum_{i=1}^{3} \Big\{\langle R_j, \mathsf{E}\rangle\langle R_k, \mathsf{T}\rangle - \langle R_j, \mathsf{T}\rangle\langle R_k, \mathsf{E}\rangle\Big\} R_i, \quad i, j, k = 1, 2, 3(cyclic).$$

Then, using (2), we find A_i as given in (14). So, Lemma 3.2 yields the desired normal curvature. □

Lemma 3.5. *Let $\mathcal{M}$ be an orientable hypersurface given by $R(u_1, u_2, u_3)$, and $\beta(s) = R(u_1(s), u_2(s), u_3(s))$ be a Frenet curve with the* ED*-frame field $\{\mathsf{T}, \mathsf{E}, \mathsf{D}, \mathsf{N}\}$ lying on $\mathcal{M}$. Then, the normal curvature of $\mathcal{M}$ in the direction of $\mathsf{E} + \mathsf{D}$ is given by*

$$\kappa_n(\mathsf{E} + \mathsf{D}) = \frac{1}{2} \sum_{i,j=1}^{3} h_{ij} B_i B_j, \tag{15}$$

where $B_i = a_i + A_i$.

Proof. We may write $\mathsf{E} + \mathsf{D} = \sum_{i=1}^{3} R_i(a_i + A_i) = \sum_{i=1}^{3} R_i B_i$. Then it follows from

$$\kappa_n(\mathsf{E} + \mathsf{D}) = \frac{1}{2}\langle \mathcal{S}(\mathsf{E} + \mathsf{D}), \mathsf{E} + \mathsf{D}\rangle.$$

□

Lemma 3.6. *Let $\mathcal{M} : f(x, y, z, w) = 0$ be an orientable hypersurface and $\beta(s)$ be a Frenet curve lying on $\mathcal{M}$. Denoting the* ED*-frame field of β by $\{\mathsf{T}, \mathsf{E}, \mathsf{D}, \mathsf{N}\}$, the normal curvature of $\mathcal{M}$ in the direction of* T *is given by [2]*

$$\kappa_n(\mathsf{T}) = \frac{-1}{\|\nabla f\|} \beta' H_f (\beta')^t,$$

where H_f denotes the Hessian of f.

Lemma 3.7. *Let $\mathcal{M} : f(x, y, z, w) = 0$ be an orientable hypersurface, $\beta(s)$ be a Frenet curve lying on $\mathcal{M}$ and $\{\mathsf{T}, \mathsf{E}, \mathsf{D}, \mathsf{N}\}$ denotes the* ED*-frame field of β. Then the normal curvature of $\mathcal{M}$ in the direction of* E *is obtained by*

$$\kappa_n(\mathsf{E}) = \frac{-1}{\delta^2 \|\nabla f\|}\Big\{\beta'' H_f (\beta'')^t + \lambda \beta'' H_f (\nabla f)^t + \lambda \nabla f H_f (\beta'')^t + \lambda^2 \nabla f H_f (\nabla f)^t\Big\},$$

where

$$\lambda = \frac{1}{\|\nabla f\|^2} \beta' H_f (\beta')^t, \quad \delta^2 = \beta''(\beta'')^t - \lambda^2 \|\nabla f\|^2.$$

Proof. If we substitute $\mathsf{N} = \frac{\nabla f}{\|\nabla f\|}$ into (5), and use (4), we find

$$\mathsf{E} = \frac{1}{\delta}(\beta'' + \lambda \nabla f), \tag{16}$$

where

$$\lambda = -\frac{1}{\|\nabla f\|^2}\langle \beta'', \nabla f\rangle = \frac{1}{\|\nabla f\|^2}\beta' H_f (\beta')^t,$$

$$\delta^2 = \langle \beta'', \beta''\rangle - \langle \beta'', \mathsf{N}\rangle^2 = \beta''(\beta'')^t - \lambda^2 \|\nabla f\|^2.$$

Then, the desired result is obtained from Lemma 3.6. □

Lemma 3.8. *Let $\mathcal{M} : f(x, y, z, w) = 0$ be an orientable hypersurface and $\{\mathsf{T}, \mathsf{E}, \mathsf{D}, \mathsf{N}\}$ denotes the ED-frame field of the Frenet curve $\beta(s) = (\beta_1(s), \beta_2(s), \beta_3(s), \beta_4(s))$ lying on $\mathcal{M}$. Then, the normal curvature of $\mathcal{M}$ in the direction of D is found as*

$$\kappa_n(\mathsf{D}) = \frac{-1}{\delta^2 \|\nabla f\|^3}\Phi H_f \Phi^t,$$

where $\Phi = [\phi_1 \quad \phi_2 \quad \phi_3 \quad \phi_4]$, $\nabla f = (f_1, f_2, f_3, f_4)$, and

$$\phi_i = (-1)^j f_j\Big(\beta'_k\beta''_\ell - \beta''_k\beta'_\ell\Big) + (-1)^k f_k\Big(\beta'_j\beta''_\ell - \beta''_j\beta'_\ell\Big) + (-1)^\ell f_\ell\Big(\beta'_j\beta''_k - \beta''_j\beta'_k\Big),$$

$i, j, k, \ell = 1, 2, 3, 4$ (cyclic).

Proof. Substituting $\mathsf{N} = \dfrac{\nabla f}{\|\nabla f\|}$ and $\mathsf{E} = \dfrac{1}{\delta}(\beta'' + \lambda\nabla f)$ into $\mathsf{D} = \mathsf{N} \otimes \mathsf{T} \otimes \mathsf{E}$ gives

$$\mathsf{D} = \frac{1}{\delta\|\nabla f\|}\nabla f \otimes \mathsf{T} \otimes \beta'' = \frac{1}{\delta\|\nabla f\|}\begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ f_1 & f_2 & f_3 & f_4 \\ \beta'_1 & \beta'_2 & \beta'_3 & \beta'_4 \\ \beta''_1 & \beta''_2 & \beta''_3 & \beta''_4 \end{vmatrix} = \frac{1}{\delta\|\nabla f\|}\Phi, \tag{17}$$

where $\Phi = [\phi_1 \quad \phi_2 \quad \phi_3 \quad \phi_4]$. Then, the normal curvature of $\mathcal{M}$ in the direction of D follows from Lemma 3.6. □

Lemma 3.9. *Let $\mathcal{M} : f(x, y, z, w) = 0$ be an orientable hypersurface, $\beta(s)$ be a Frenet curve lying on $\mathcal{M}$ and $\{\mathsf{T}, \mathsf{E}, \mathsf{D}, \mathsf{N}\}$ denotes the ED-frame field of β. Then the normal curvature of $\mathcal{M}$ in the direction of $\mathsf{E} + \mathsf{D}$ is obtained by*

$$\kappa_n(\mathsf{E} + \mathsf{D}) = \frac{-1}{2\delta^2\|\nabla f\|}\Omega H_f \Omega^t,$$

where

$$\Omega = \beta'' + \lambda\nabla f + \frac{1}{\|\nabla f\|}\Phi.$$

Proof. The result can be seen by using (16) and (17). □

3.2. Case 2

The proofs of the following results can be done similar to the proofs given in Case 1.

Proposition 3.10. *Let $\mathcal{M}$ be an orientable hypersurface, β be a Frenet curve lying on $\mathcal{M}$, and $\{\mathsf{T},\mathsf{E},\mathsf{D},\mathsf{N}\}$ denotes the ED-frame field of β. Then, the shape operator's matrix of $\mathcal{M}$ along β with respect to the basis $\{\mathsf{T},\mathsf{E},\mathsf{D}\}$ is found as*

$$\mathcal{S}=\begin{bmatrix}\kappa_n(\mathsf{T}) & \tau_g^1 & 0\\ \tau_g^1 & \kappa_n(\mathsf{E}) & \Psi\\ 0 & \Psi & \kappa_n(\mathsf{D})\end{bmatrix}, \tag{18}$$

where $\Psi=\kappa_n(\mathsf{E}+\mathsf{D})-\frac{1}{2}\Big(\kappa_n(\mathsf{E})+\kappa_n(\mathsf{D})\Big)$.

Lemma 3.11. *Let $\mathcal{M}$ be an orientable hypersurface given by $R(u_1,u_2,u_3)$, and $\beta(s)=R(u_1(s),u_2(s),u_3(s))$ be a Frenet curve with its ED-frame field $\{\mathsf{T},\mathsf{E},\mathsf{D},\mathsf{N}\}$. Then, the normal curvature of $\mathcal{M}$ in the direction of E is given by*

$$\kappa_n(\mathsf{E})=\sum_{i,j=1}^{3}h_{ij}a_ia_j,$$

where a_i and Δ^2 are as given in (10) and (11),

$$\begin{aligned}\langle R_n,\mathsf{E}\rangle \;&= \tfrac{1}{\nu}\Big\{\sum_{i=1}^{3}g_{in}u_i''' + 3\sum_{i,j=1}^{3}\langle R_{ij},R_n\rangle u_i''u_j' + \sum_{i,j,k=1}^{3}\langle R_{ijk},R_n\rangle u_i'u_j'u_k'\\ &\quad-\sum_{i=1}^{3}g_{in}u_i'\Big(\sum_{i,j=1}^{3}g_{ij}u_i'''u_j' + 3\sum_{i,j,k=1}^{3}\langle R_{ij},R_k\rangle u_i''u_j'u_k' + \sum_{i,j,k,\ell=1}^{3}\langle R_{ijk},R_\ell\rangle u_i'u_j'u_k'u_\ell'\Big)\Big\},\end{aligned} \tag{19}$$

$n=1,2,3$, and

$$\begin{aligned}\nu \;&= \|\beta'''-\langle\beta''',\mathsf{N}\rangle\mathsf{N}-\langle\beta''',\mathsf{T}\rangle\mathsf{T}\| = \{\langle\beta''',\beta'''\rangle-\langle\beta''',\mathsf{N}\rangle^2-\langle\beta''',\mathsf{T}\rangle^2\}^{\frac{1}{2}}\\ &= \Big\{\sum_{i,j=1}^{3}g_{ij}u_i'''u_j''' + 6\sum_{i,j,k=1}^{3}\langle R_{ij},R_k\rangle u_i'''u_j'u_k' + 9\sum_{i,j,k,\ell=1}^{3}\langle R_{ij},R_{k\ell}\rangle u_i''u_j''u_k'u_\ell'\\ &\quad+2\sum_{i,j,k,\ell=1}^{3}\langle R_{ijk},R_\ell\rangle u_i'''u_j'u_k'u_\ell' + 6\sum_{i,j,k,\ell,m=1}^{3}\langle R_{ijk},R_{\ell m}\rangle u_i''u_j'u_k'u_\ell'u_m'\\ &\quad+\sum_{i,j,k,\ell,m,n=1}^{3}\langle R_{ijk},R_{\ell mn}\rangle u_i'u_j'u_k'u_\ell'u_m'u_n' - \Big(3\sum_{i,j=1}^{3}h_{ij}u_i''u_j' + \sum_{i,j,k=1}^{3}\langle R_{ijk},\mathsf{N}\rangle u_i'u_j'u_k'\Big)^2\\ &\quad-\Big(\sum_{i,j=1}^{3}g_{ij}u_i'''u_j' + 3\sum_{i,j,k=1}^{3}\langle R_{ij},R_k\rangle u_i''u_j'u_k' + \sum_{i,j,k,\ell=1}^{3}\langle R_{ijk},R_\ell\rangle u_i'u_j'u_k'u_\ell'\Big)^2\Big\}^{\frac{1}{2}}.\end{aligned}$$

Lemma 3.12. *Let $\mathcal{M}$ be an orientable hypersurface given by $R(u_1,u_2,u_3)$ and $\beta(s)=R(u_1(s),u_2(s),u_3(s))$ be a Frenet curve lying on $\mathcal{M}$. $\{\mathsf{T},\mathsf{E},\mathsf{D},\mathsf{N}\}$ denotes ED-frame field of β, then the normal curvature of $\mathcal{M}$ in the direction of D is obtained by*

$$\kappa_n(\mathsf{D})=\sum_{i,j=1}^{3}h_{ij}A_iA_j,$$

where A_i is as given in (14), and $\langle R_j,\mathsf{E}\rangle$ is as given in (19).

Lemma 3.13. *Let $\mathcal{M}$ be an orientable hypersurface given by $R(u_1, u_2, u_3)$, and $\beta(s) = R(u_1(s), u_2(s), u_3(s))$ be a Frenet curve lying on $\mathcal{M}$. Let $\{\mathsf{T}, \mathsf{E}, \mathsf{D}, \mathsf{N}\}$ denotes the ED-frame field of β. Then, the normal curvature of $\mathcal{M}$ in the direction of $\mathsf{E} + \mathsf{D}$ is found as*

$$\kappa_n(\mathsf{E} + \mathsf{D}) = \frac{1}{2} \sum_{i,j=1}^{3} h_{ij} B_i B_j,$$

where $B_i = a_i + A_i$.

Lemma 3.14. *Let $\mathcal{M} : f(x, y, z, w) = 0$ be an orientable hypersurface, $\beta(s)$ be a Frenet curve lying on $\mathcal{M}$ and $\{\mathsf{T}, \mathsf{E}, \mathsf{D}, \mathsf{N}\}$ denotes the ED-frame field of β. Then the normal curvature of $\mathcal{M}$ in the direction of E is given by*

$$\kappa_n(\mathsf{E}) = \frac{-1}{\epsilon^2 \|\nabla f\|} \Big\{ \beta''' H_f(\beta''')^t + \zeta_1 \beta''' H_f(\nabla f)^t + \zeta_2 \beta''' H_f(\beta')^t + \zeta_1 \nabla f H_f(\beta''')^t$$

$$+ \zeta_1^2 \nabla f H_f(\nabla f)^t + \zeta_1 \zeta_2 \nabla f H_f(\beta')^t + \zeta_2 \beta' H_f(\beta''')^t + \zeta_1 \zeta_2 \beta' H_f(\nabla f)^t + \zeta_2^2 \beta' H_f(\beta')^t \Big\},$$

where

$$\zeta_1 = \frac{1}{\|\nabla f\|^2} \Big(3\beta' H_f(\beta'')^t + \beta' \frac{d(H_f)}{ds} (\beta')^t \Big),$$

$$\zeta_2 = -\beta'(\beta''')^t,$$

$$\epsilon^2 = \beta'''(\beta''')^t - \zeta_1^2 \|\nabla f\|^2 - \zeta_2^2.$$

Lemma 3.15. *Let $\mathcal{M} : f(x, y, z, w) = 0$ be an orientable hypersurface and $\{\mathsf{T}, \mathsf{E}, \mathsf{D}, \mathsf{N}\}$ denotes the ED-frame field of the Frenet curve $\beta(s) = (\beta_1(s), \beta_2(s), \beta_3(s), \beta_4(s))$ lying on $\mathcal{M}$. Then the normal curvature of $\mathcal{M}$ in the direction of D is found as*

$$\kappa_n(\mathsf{D}) = \frac{-1}{\epsilon^2 \|\nabla f\|^3} \Gamma H_f \Gamma^t,$$

where $\Gamma = [\gamma_1 \quad \gamma_2 \quad \gamma_3 \quad \gamma_4]$ and $\gamma_i = (-1)^j f_j \big(\beta'_k \beta'''_\ell - \beta'''_k \beta'_\ell \big) + (-1)^k f_k \big(\beta'_j \beta'''_\ell - \beta'''_j \beta'_\ell \big) + (-1)^\ell f_\ell \big(\beta'_j \beta'''_k - \beta'''_j \beta'_k \big)$, $i, j, k, \ell = 1, 2, 3, 4$(cyclic).

Lemma 3.16. *Let $\mathcal{M} : f(x, y, z, w) = 0$ be an orientable hypersurface, $\beta(s)$ be a Frenet curve lying on $\mathcal{M}$ and $\{\mathsf{T}, \mathsf{E}, \mathsf{D}, \mathsf{N}\}$ denotes the ED-frame field of β. Then the normal curvature of $\mathcal{M}$ in the direction of $\mathsf{E} + \mathsf{D}$ is obtained by*

$$\kappa_n(\mathsf{E} + \mathsf{D}) = \frac{-1}{2\epsilon^2 \|\nabla f\|} \Lambda H_f \Lambda^t,$$

where $\Lambda = \beta''' + \zeta_1 \nabla f + \zeta_2 \beta' + \frac{1}{\|\nabla f\|} \Gamma$.

If we use the matrices for the shape operator in each case, we may give the following corollaries:

Corollary 3.17. *Let $\mathcal{M}$ be an orientable hypersurface, β be a Frenet curve lying on $\mathcal{M}$, and $\{\mathsf{T}, \mathsf{E}, \mathsf{D}, \mathsf{N}\}$ denotes the ED-frame field of β. Then, the Gaussian curvature and the mean curvature of the hypersurface along β can be given by*

Case 1: $\mathsf{K}_{\beta(s)} = \kappa_n(\mathsf{T})\kappa_n(\mathsf{E})\kappa_n(\mathsf{D}) - \Psi^2 \kappa_n(\mathsf{T}) - (\tau_g^2)^2 \kappa_n(\mathsf{E}) - (\tau_g^1)^2 \kappa_n(\mathsf{D}) + 2\tau_g^1 \tau_g^2 \Psi,$

Case 2: $\mathsf{K}_{\beta(s)} = \kappa_n(\mathsf{T})\kappa_n(\mathsf{E})\kappa_n(\mathsf{D}) - \Psi^2 \kappa_n(\mathsf{T}) - (\tau_g^1)^2 \kappa_n(\mathsf{D}),$

and in both cases

$$\mathsf{H}_{\beta(s)} = \frac{1}{3} \Big(\kappa_n(\mathsf{T}) + \kappa_n(\mathsf{E}) + \kappa_n(\mathsf{D}) \Big).$$

Corollary 3.18. *Let $\mathcal{M}$ be an orientable hypersurface, β be a line of curvature on $\mathcal{M}$. Then* T *corresponds to the principal direction, i.e.* $\mathcal{S}(\mathsf{T}) = k_1\mathsf{T}$. *If the principal curvature* $k_1 \neq 0$ *and* E, D *correspond to principal directions, then we have* $\Psi = 0$ *in each case. In this case,* $\kappa_n(\mathsf{E}+\mathsf{D})$ *corresponds to the arithmetic mean of the normal curvatures* $\kappa_n(\mathsf{E})$ *and* $\kappa_n(\mathsf{D})$.

4. Examples

In this part, we calculate the Gaussian curvatures and the mean curvatures of the hypersurfaces depending on the ED-frame curvatures by finding the matrices of the shape operators of hypersurfaces given by parametric or implicit equations.

Example 4.1. Let us consider the hypercylinder $\mathcal{M}$ given by its parametric equation $\mathbf{R}(u_1,u_2,u_3) = (\cos u_1 \cos u_2, \sin u_1 \cos u_2, \sin u_2, u_3)$ and the curve $\beta(s) = \mathbf{R}\left(\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}, \cos\frac{s}{\sqrt{2}}\right)$ lying on $\mathcal{M}$. Since Case 1 is valid along β, at the point $\beta(0) = (1,0,0,1)$ we obtain $\kappa_n(\mathsf{T}) = -1$, $\tau_g^1 = \tau_g^2 = 0$ [6]. We also find $\Delta = 1$, $\langle R_1, \mathsf{E}\rangle = \langle R_2, \mathsf{E}\rangle = 0$, $\langle R_3, \mathsf{E}\rangle = -1$. Substituting these values into (10) gives $a_1 = a_2 = 0, a_3 = -1$. So, from (9) we obtain $\kappa_n(\mathsf{E}) = 0$. If we use (14), we get $A_1 = -A_2 = \frac{1}{\sqrt{2}}$, $A_3 = 0$. Then, from (13) we have $\kappa_n(\mathsf{D}) = -1$. Besides, we find $B_1 = -B_2 = \frac{1}{\sqrt{2}}$, $B_3 = -1$ and from (15), we get $\kappa_n(\mathsf{E}+\mathsf{D}) = -\frac{1}{2}$. So, from (7) we obtain $\Psi = 0$.

Finally, the shape operator's matrix of $\mathcal{M}$ at $\beta(0) = (1,0,0,1)$ is obtained from (6) as

$$\mathcal{S}_{\beta(0)} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

which yields the Gaussian curvature and mean curvature of $\mathcal{M}$ as $\mathsf{K}_{(\beta(0))} = 0$ and $\mathsf{H}_{(\beta(0))} = -\frac{2}{3}$, respectively.

Example 4.2. Let us consider the unit speed curve $\alpha(s) = \left(\frac{\sqrt{3}}{2}\cos\left(\frac{s}{2}\right), \frac{1}{2}\cos\left(\frac{s}{2}\right), \sin\left(\frac{s}{2}\right), \frac{\sqrt{3}}{2}s\right)$ lying on the hypercylinder $\mathcal{M}$ given by its implicit equation $x^2+y^2+z^2 = 1$. Since Case 2 is valid along α, at the point $\alpha(0) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}, 0, 0\right)$ we have

$$\tau_g^1 = -\frac{\sqrt{3}}{4}, \quad \kappa_n(\mathsf{T}) = -\frac{1}{4}, \quad \kappa_n(\mathsf{E}) = -\frac{3}{4}, \quad \kappa_n(\mathsf{D}) = -1, \quad \kappa_n(\mathsf{E}+\mathsf{D}) = -\frac{7}{8}, \quad \Psi = 0.$$

The shape operator's matrix is then follows from (18) as

$$\mathcal{S}_{\alpha(0)} = \begin{bmatrix} -\frac{1}{4} & -\frac{\sqrt{3}}{4} & 0 \\ -\frac{\sqrt{3}}{4} & -\frac{3}{4} & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

We again have

$$\mathsf{K}_{(\alpha(0))} = 0, \quad \mathsf{H}_{(\alpha(0))} = -\frac{2}{3}.$$

References

[1] O. Aléssio, Differential geometry of intersection curves in R^4 of three implicit surfaces, Comput Aided Geom Design 26 (2009) 455-471.

[2] M. Düldül, B. Uyar Düldül, N. Kuruoğlu, E. Özdamar, Extension of the Darboux frame into Euclidean 4-space and its invariants, Turkish Journal of Mathematics 41(6) (2017) 1628-1639.

[3] A. Gray, E. Abbena, S. Salamon, Modern differential geometry of curves and surfaces with Mathematica, (3rd Edition), Chapman & Hall CRC, 2006.

[4] B. O'Neill, Elementary Differential Geometry, Academic Press, 1966.

[5] L.W. Tu, Differential Geometry: Connections, Curvature, and Characteristic Classes, (1st edition), Springer, 2017.

[6] B. Uyar Düldül, M. Düldül, Extended Darboux frame curvatures of Frenet curves lying on parametric 3-surfaces, Kuwait Journal of Science, 46(1) (2019) 32-40.

[7] M.Z. Williams, F.M. Stein, A triple product of vectors in four-space, Math. Mag. 37 (1964) 230-235.