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The Fejér Inequality and its Generalizations

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Abstract. The article provides generalizations of the Fejér inequality. These include extensions, and refinements by inserting discrete and integral terms. The fundamental issue underlying this research is the barycenter of a nonnegative integrable function. Appropriate properties of the convexity related to the function barycenter are used in the construction of main results.

1. Introduction

Throughout the paper, we consider certain functions defined on a bounded closed interval $[a, b] \subset \mathbb{R}$ with endpoints $a < b$.

Each point $x \in \mathbb{R}$ can be represented by the combination of the points a and b in the form

$$x = \alpha(x)a + \beta(x)b, \tag{1}$$

where the coefficients

$$\alpha(x) = \frac{b-x}{b-a},\ \beta(x) = \frac{x-a}{b-a}. \tag{2}$$

The sum $\alpha(x) + \beta(x)$ is equal to 1, which guarantees the uniqueness of $\alpha(x)$ and $\beta(x)$, and the combination in formula (1) is affine. If $x \in [a, b]$, then the coefficients $\alpha(x)$ and $\beta(x)$ are nonnegative, and the combination in formula (1) is convex.

A function $h : \mathbb{R} \to \mathbb{R}$ is affine if the equality $h(\alpha x + \beta y) = \alpha h(x) + \beta h(y)$ holds for every affine combination $\alpha x + \beta y$ with $x, y \in \mathbb{R}$ (two different fixed points x and y can also be used). The affine function h can be expressed by the explicit equation $h(x) = kx + l$, where $k = (h(b) - h(a))/(b - a)$ with $a \neq b$, and $l = h(0)$.

A function $f : [a, b] \to \mathbb{R}$ is convex if the inequality $f(\alpha x + \beta y) \le \alpha f(x) + \beta f(y)$ holds for every convex combination $\alpha x + \beta y$ with $x, y \in [a, b]$. The convex function f is bounded by two affine functions, a support line h_1 at an interior point $c \in (a, b)$ with a slope coefficient $k \in [f'(c-), f'(c+)]$, and the secant line h_2 at the endpoints a and b. The double inequality

$$k(x-c) + f(c) = h_1(x) \le f(x) \le h_2(x) = \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b) \tag{3}$$

holds for every $x \in [a, b]$. Using the equality $f(c) = h_1(c)$ with $c = \sum_{i=1}^{n} \alpha_i x_i$, applying the affinity of h_1, and employing the inequalities $h_1(x_i) \le f(x_i)$, we can derive the discrete form of the Jensen inequality (see [5]). Using the inequality in formula (3) with $c = (a + b)/2$, and integrating over $[a, b]$, we can obtain the Hermite-Hadamard inequality (see [3] and [2]).

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2. Function Barycenter and Fejér's Inequality

Regarding the concept of the function barycenter, we present Fejér's inequality (see [1]) and its extension. Lemma A and Theorem A explain the significance of Fejér's inequality, and moreover, Lemma B and Theorem B offer its extension.

Fejér's inequality uses a nonnegative integrable function $g : [a,b] \to \mathbb{R}$ such that $\int_a^b g(x)dx > 0$. It also requires that g is symmetric with respect to the midpoint $(a+b)/2$, indicating that g satisfies the equation $g(a+b-x) = g(x)$ for every $x \in [a,b]$. Some information relating to Fejér's inequality can be found in [8].

The barycenter of the function g can be understood as the barycenter $\overline{x}$ of a piece of wire on the segment $[a,b]$ with the line density g. The point $\overline{x}$ is determined by the equation $\int_a^b (x-\overline{x})g(x)dx = 0$, and so

$$\overline{x} = \frac{\int_a^b xg(x)dx}{\int_a^b g(x)dx}. \tag{4}$$

According to formulae (1) and (2), the barycenter $\overline{x}$ can be represented by the convex combination $\overline{x} = \alpha(\overline{x})a + \beta(\overline{x})b$ with respect to the coefficients

$$\alpha(\overline{x}) = \frac{\int_a^b (b-x)g(x)dx}{\int_a^b (b-a)g(x)dx},\ \ \beta(\overline{x}) = \frac{\int_a^b (x-a)g(x)dx}{\int_a^b (b-a)g(x)dx}. \tag{5}$$

The coefficients $\alpha(\overline{x})$ and $\beta(\overline{x})$ are positive, which implies that $a < \overline{x} < b$. A key point in the consideration of Fejér's inequality is the barycenter $\overline{x}$.

If $p(x) = g(x)/\int_a^b g(x)dx$, then p and g have the same barycenter, and meet the same equation of symmetry. Besides, p is normalized by means of $\int_a^b p(x)dx = 1$.

Lemma A. *Let $p : [a,b] \to \mathbb{R}$ be a nonnegative integrable normalized function that satisfies $p(a+b-x) = p(x)$ for every $x \in [a,b]$.*

Then the barycenter of the function p is the midpoint

$$\overline{x} = \frac{a+b}{2}. \tag{6}$$

Proof. Combing the substitution $x \mapsto a+b-x$ and the equation $p(a+b-x) = p(x)$, we obtain the equality

$$\int_a^b (b-x)p(x)dx = \int_a^b (x-a)p(x)dx,$$

and thus $\alpha(\overline{x}) = \beta(\overline{x})$ by formula (5). Since $\alpha(\overline{x}) + \beta(\overline{x}) = 1$, it follows that $\alpha(\overline{x}) = \beta(\overline{x}) = 1/2$, and so $\overline{x} = (a+b)/2$. □

Now we have a clearer insight into the following presentation of Fejér's inequality.

Theorem A. *Let $p : [a,b] \to \mathbb{R}$ be a nonnegative integrable normalized function that satisfies $p(a+b-x) = p(x)$ for every $x \in [a,b]$.*

Then each convex function $f : [a,b] \to \mathbb{R}$ satisfies the double inequality

$$f\left(\frac{a+b}{2}\right) \le \int_a^b f(x)p(x)dx \le \frac{f(a)+f(b)}{2}. \tag{7}$$

If $f(x) = x$, then formula (7) is reduced to formula (6). If $p(x) = 1/(b-a)$, then formula (7) presents the Hermite-Hadamard inequality.

Lemma B. *Let $p : [a,b] \to \mathbb{R}$ be a nonnegative integrable normalized function, and let $\overline{x}$ be its barycenter.*

Then each affine function $h : \mathbb{R} \to \mathbb{R}$ satisfies the equality

$$\int_a^b h(x)p(x)dx = h(\overline{x}). \tag{8}$$

Proof. The affine equation $h(x) = kx + l$ should be used. $\square$

The consequence of Lemma B is the following extension of Fejér's inequality.

Theorem B. *Let $p : [a, b] \to \mathbb{R}$ be a nonnegative integrable normalized function, let $\overline{x}$ be its barycenter, and let $\overline{x} = \alpha a + \beta b$ be the convex combination.*

Then each convex function $f : [a, b] \to \mathbb{R}$ satisfies the double inequality

$$f(\alpha a + \beta b) \leq \int_a^b f(x)p(x)dx \leq \alpha f(a) + \beta f(b). \tag{9}$$

Proof. Since $\overline{x} \in (a, b)$, the convexity of f admits a support line h_1 at $\overline{x}$. Let h_2 be the secant line of f. Then we have the coupled values $h_1(\overline{x}) = f(\overline{x})$, $h_2(a) = f(a)$ and $h_2(b) = f(b)$.

Multiplying formula (3) by $p(x)$, and integrating over $[a, b]$, we obtain

$$\int_a^b h_1(x)p(x)dx \leq \int_a^b f(x)p(x)dx \leq \int_a^b h_2(x)p(x)dx. \tag{10}$$

Applying formula (8) to h_1 and h_2, and using the coupled values, we find

$$\int_a^b h_1(x)p(x)dx = h_1(\overline{x}) = f(\overline{x}) = f(\alpha a + \beta b)$$

and

$$\int_a^b h_2(x)p(x)dx = h_2(\overline{x}) = \alpha h_2(a) + \beta h_2(b) = \alpha f(a) + \beta f(b),$$

whereby formula (10) becomes formula (9). $\square$

If f is affine, then formula (9) is reduced to formula (8). If $\overline{x} = (a + b)/2$, including the case that p is symmetric with respect to $(a + b)/2$, then formula (9) is reduced to formula (7). Theorem B shows that the matter of fact in Fejér's inequality is not the symmetry of p, but the barycenter of p.

As regards Fejér's inequality for multivariate functions, see [9] and [10].

3. Main Results

We expose expansions and refinements of the extended version of Fejér's inequality in formula (9) by using the function barycenter concept.

3.1. Expansions to Convex Combinations

To expand the double inequality in formula (9) to a convex combination of the barycenters of positive integrable functions, we can rely on the discrete form of Jensen's inequality.

Theorem 3.1. *Let $p_i : [a, b] \to \mathbb{R}$ be nonnegative integrable normalized functions, let $\overline{x}_i$ be their barycenters, let $\overline{x} = \sum_{i=1}^n \lambda_i \overline{x}_i$ be a convex combination, and let $\overline{x} = \alpha a + \beta b$ be the convex combination.*

Then each convex function $f : [a, b] \to \mathbb{R}$ satisfies the double inequality

$$f(\alpha a + \beta b) \leq \sum_{i=1}^n \lambda_i \int_a^b f(x)p_i(x)dx \leq \alpha f(a) + \beta f(b). \tag{11}$$

Proof. Taking the convex combinations $\bar{x}_i = \alpha_i a + \beta_i b$, and using the equality

$$\alpha a + \beta b = \sum_{i=1}^{n} \lambda_i \bar{x}_i = \Big(\sum_{i=1}^{n} \lambda_i \alpha_i\Big) a + \Big(\sum_{i=1}^{n} \lambda_i \beta_i\Big) b,$$

we have the coefficient connections

$$\alpha = \sum_{i=1}^{n} \lambda_i \alpha_i, \ \ \beta = \sum_{i=1}^{n} \lambda_i \beta_i. \tag{12}$$

Applying formula (9) to the barycenter $\alpha_i a + \beta_i b$ of the function p_i, we get

$$f(\alpha_i a + \beta_i b) \le \int_a^b f(x) p_i(x) dx \le \alpha_i f(a) + \beta_i f(b),$$

then multiplying by λ_i and summing, we obtain

$$\sum_{i=1}^{n} \lambda_i f(\alpha_i a + \beta_i b) \le \sum_{i=1}^{n} \lambda_i \int_a^b f(x) p_i(x) dx \le \sum_{i=1}^{n} \lambda_i \Big(\alpha_i f(a) + \beta_i f(b)\Big).$$

Since

$$f(\alpha a + \beta b) = f\left(\sum_{i=1}^{n} \lambda_i \bar{x}_i\right) \le \sum_{i=1}^{n} \lambda_i f(\bar{x}_i) = \sum_{i=1}^{n} \lambda_i f(\alpha_i a + \beta_i b)$$

by Jensen's inequality, and

$$\sum_{i=1}^{n} \lambda_i \Big(\alpha_i f(a) + \beta_i f(b)\Big) = \alpha f(a) + \beta f(b)$$

by formula (12), combining the last three formulae, we achieve the series of inequalities containing the double inequality in formula (11). □

3.2. *Refinements With Discrete Terms*

We aspire to refine the double inequality in formula (9).

Let $p : [a, b] \to \mathbb{R}$ be a positive integrable normalized function, and let $\bar{x}$ be the barycenter of p. Let $c \in (a, b)$ be a point, let $\bar{x}_1$ be the barycenter of the restriction $p_1 = p/[a, c]$, and let $\bar{x}_2$ be the barycenter of the restriction $p_2 = p/[c, b]$. Formula (4) yields

$$\bar{x}_1 = \frac{\int_a^c x p(x) dx}{\int_a^c p(x) dx}, \ \ \bar{x}_2 = \frac{\int_c^b x p(x) dx}{\int_c^b p(x) dx}. \tag{13}$$

By including the coefficients

$$\lambda_1 = \int_a^c p(x) dx, \ \ \lambda_2 = \int_c^b p(x) dx, \tag{14}$$

the barycenter $\bar{x}$ can be represented as the convex combination

$$\bar{x} = \lambda_1 \bar{x}_1 + \lambda_2 \bar{x}_2. \tag{15}$$

Since $\bar{x}_1 < \bar{x}_2$ and $\lambda_1, \lambda_2 > 0$, it follows that $\bar{x} \in (\bar{x}_1, \bar{x}_2)$.

Lemma 3.2. *Let $p : [a, b] \to \mathbb{R}$ be a positive integrable normalized function, and let $\overline{x}$ be its barycenter. Let $c \in (a, b)$ be a point, let $\overline{x}_1$ and $\overline{x}_2$ be the barycenters as in formula* (13), *and let λ_1 and λ_2 be the coefficients as in formula* (14). *Let $\overline{x} = \alpha a + \beta b$, $\overline{x}_1 = \alpha_1 a + \gamma_1 c$ and $\overline{x}_2 = \gamma_2 c + \beta_2 b$ be the convex combinations.*

Then each convex function $f : [a, b] \to \mathbb{R}$ satisfies the double inequality

$$\begin{aligned} f(\alpha a + \beta b) &\leq \lambda_1 f(\alpha_1 a + \gamma_1 c) + \lambda_2 f(\gamma_2 c + \beta_2 b) \\ &\leq \int_a^b f(x)p(x)dx. \end{aligned} \tag{16}$$

Proof. Applying the convexity of f to the valuation $f(\overline{x}) = f(\lambda_1\overline{x}_1 + \lambda_2\overline{x}_2)$, we prove the left-hand side of the inequality in formula (16) by

$$\begin{aligned} f(\alpha a + \beta b) = f(\overline{x}) &\leq \lambda_1 f(\overline{x}_1) + \lambda_2 f(\overline{x}_2) \\ &= \lambda_1 f(\alpha_1 a + \gamma_1 c) + \lambda_2 f(\gamma_2 c + \beta_2 b). \end{aligned}$$

Applying the left-hand side of the inequality in formula (9) to the valuations $f(\overline{x}_1) = f(\alpha_1 a + \gamma_1 c)$ and $f(\overline{x}_2) = f(\gamma_2 c + \beta_2 b)$, we prove the right-hand side of the inequality in formula (16) by

$$\begin{aligned} \lambda_1 f(\overline{x}_1) + \lambda_2 f(\overline{x}_2) &\leq \lambda_1 \frac{\int_a^c f(x)p(x)dx}{\int_a^c p(x)dx} + \lambda_2 \frac{\int_c^b f(x)p(x)dx}{\int_c^b p(x)dx} \\ &= \int_a^b f(x)p(x)\,dx. \end{aligned}$$

So, the proof is over. □

Lemma 3.3. *Let the assumptions of Lemma 3.2 be fulfilled.*

Then each convex function $f : [a, b] \to \mathbb{R}$ satisfies the double inequality

$$\begin{aligned} \int_a^b f(x)p(x)dx &\leq \lambda_1\alpha_1 f(a) + \lambda_2\beta_2 f(b) + (\lambda_1\gamma_1 + \lambda_2\gamma_2)f(c) \\ &\leq \alpha f(a) + \beta f(b). \end{aligned} \tag{17}$$

Proof. Applying the right-hand side of the inequality in formula (9) to the restrictions $p_1 = p/[a, c]$ and $p_2 = p/[c, b]$, we prove the left-hand side of the inequality in formula (17) by

$$\begin{aligned} \int_a^b f(x)p(x)dx &= \lambda_1 \frac{\int_a^c f(x)p(x)dx}{\int_a^c p(x)dx} + \lambda_2 \frac{\int_c^b f(x)p(x)dx}{\int_c^b p(x)dx} \\ &\leq \lambda_1[\alpha_1 f(a) + \gamma_1 f(c)] + \lambda_2[\gamma_2 f(c) + \beta_2 f(b)] \\ &= \lambda_1\alpha_1 f(a) + \lambda_2\beta_2 f(b) + (\lambda_1\gamma_1 + \lambda_2\gamma_2)f(c). \end{aligned}$$

Taking $c = \alpha_3 a + \beta_3 b$ as the convex combination, we have

$$\begin{aligned} \alpha a + \beta b &= \lambda_1\overline{x}_1 + \lambda_2\overline{x}_2 \\ &= \lambda_1[\alpha_1 a + \gamma_1(\alpha_3 a + \beta_3 b)] + \lambda_2[\gamma_2(\alpha_3 a + \beta_3 b) + \beta_2 b] \\ &= (\lambda_1\alpha_1 + \lambda_1\gamma_1\alpha_3 + \lambda_2\gamma_2\alpha_3)a + (\lambda_2\beta_2 + \lambda_1\gamma_1\beta_3 + \lambda_2\gamma_2\beta_3)b, \end{aligned}$$

from which it follows that

$$\alpha = \lambda_1\alpha_1 + \lambda_1\gamma_1\alpha_3 + \lambda_2\gamma_2\alpha_3, \;\; \beta = \lambda_2\beta_2 + \lambda_1\gamma_1\beta_3 + \lambda_2\gamma_2\beta_3. \tag{18}$$

Taking $y = \lambda_1\alpha_1 f(a) + \lambda_2\beta_2 f(b) + (\lambda_1\gamma_1 + \lambda_2\gamma_2) f(c)$ as the abbreviation, using the inequality $f(c) \leq \alpha_3 f(a) + \beta_3 f(b)$ and formula (18), we prove the right-hand side of the inequality in formula (17) by

$$\begin{aligned} y &\leq \lambda_1\alpha_1 f(a) + \lambda_2\beta_2 f(b) + \Big(\lambda_1\gamma_1 + \lambda_2\gamma_2\Big)\Big(\alpha_3 f(a) + \beta_3 f(b)\Big) \\ &= (\lambda_1\alpha_1 + \lambda_1\gamma_1\alpha_3 + \lambda_2\gamma_2\alpha_3) f(a) + (\lambda_2\beta_2 + \lambda_1\gamma_1\beta_3 + \lambda_2\gamma_2\beta_3) f(b) \\ &= \alpha f(a) + \beta f(b). \end{aligned}$$

The proof is done. □

In the theorem which follows, we bring together Lemma 3.2 and Lemma 3.3 by taking points $c_i \in (a, b)$ for $i = 1, 2$.

Theorem 3.4. *Let $p : [a, b] \to \mathbb{R}$ be a positive integrable normalized function, and let $\overline{x}$ be its barycenter. Let $c_i \in (a, b)$ be points, and let $\overline{x}_{i1}$, $\overline{x}_{i2}$, λ_{i1} and λ_{i2} be the barycenters and coefficients as in formulae* (13)-(14) *with c_i instead of c. Let $\overline{x} = \alpha a + \beta b$, $\overline{x}_{i1} = \alpha_{i1} a + \gamma_{i1} c$ and $\overline{x}_{i2} = \gamma_{i2} c + \beta_{i2} b$ be the convex combinations.*

Then each convex function $f : [a, b] \to \mathbb{R}$ satisfies the series of inequalities

$$\begin{aligned} f(\alpha a + \beta b) &\leq \lambda_{11} f(\alpha_{11} a + \gamma_{11} c_1) + \lambda_{12} f(\gamma_{12} c_1 + \beta_{12} b) \\ &\leq \int_a^b f(x) p(x) dx \\ &\leq \lambda_{21}\alpha_{21} f(a) + \lambda_{22}\beta_{22} f(b) + (\lambda_{21}\gamma_{21} + \lambda_{22}\gamma_{22}) f(c_2) \\ &\leq \alpha f(a) + \beta f(b). \end{aligned} \tag{19}$$

Proof. We need to apply Lemma 3.2 to c_1, and Lemma 3.3 to c_2. □

3.3. Refinements With Integral Terms

Let J be a finite union of bounded intervals. We may observe the barycenters of functions defined on the union J.

In view of this, Theorem B also applies to a nonnegative integrable normalized function p defined on J, and each convex function f defined on a bounded closed interval $[a, b]$ containing J. Namely, the function $\widetilde{p} : [a, b] \to \mathbb{R}$ defined by $\widetilde{p}/J = p$ and $\widetilde{p}/[a, b] \setminus J = 0$ may be used in formula (9).

Lemma 3.5. *Let $p : I = [a, b] \to \mathbb{R}$ be a positive integrable normalized function, let $\overline{x}$ be its barycenter, and let $c \in (a, \overline{x})$ be a point such that the integrals $\int_J p(x)\,dx$ and $\int_{I\setminus J} p(x)\,dx$ are positive.*

Then there is a point $d \in (\overline{x}, b)$ so that the interval $J = [c, d]$ satisfies the double equality

$$\int_I xp(x)dx = \frac{\int_J xp(x)dx}{\int_J p(x)dx} = \frac{\int_{I\setminus J} xp(x)dx}{\int_{I\setminus J} p(x)dx}. \tag{20}$$

Proof. We utilize intervals $J(x) = [c, x]$ relating to points $x \in [\overline{x}, b]$. Let us take such an interval $J(x)$. Let $u_1(x)$ be the barycenter of the restriction $p/J(x)$, and let $u_2(x)$ be the barycenter of the restriction $p/I \setminus J(x)$. Thus we have

$$u_1(x) = \frac{\int_{J(x)} tp(t)dt}{\int_{J(x)} p(t)dt}, \quad u_2(x) = \frac{\int_{I\setminus J(x)} tp(t)dt}{\int_{I\setminus J(x)} p(t)dt},$$

and the barycenter $\overline{x}$ can be represented by the convex combination

$$\overline{x} = \lambda_1(x)u_1(x) + \lambda_2(x)u_2(x), \tag{21}$$

where the coefficients

$$\lambda_1(x) = \int_{J(x)} p(t)dt, \ \ \lambda_2(x) = \int_{I\setminus J(x)} p(t)dt.$$

The coefficients $\lambda_1(x)$ and $\lambda_2(x)$ are positive, and so we have $u_1(x) < \overline{x} < u_2(x)$ or $u_2(x) < \overline{x} < u_1(x)$. This applies to each interval $J(x)$.

Since $u_1(\overline{x}) < \overline{x}$, it follows that $u_2(\overline{x}) > \overline{x}$. Further, $u_2(b) < c < \overline{x}$ implies $u_1(b) > \overline{x}$. Observing u_1 as the continuous function on $[\overline{x}, b]$, we can find $d \in (\overline{x}, b)$ such that $u_1(d) = \overline{x}$. Using formula (21), we get $u_2(d) = \overline{x}$. Thus $\overline{x} = u_1(d) = u_2(d)$, which proves the double equality in formula (20) for $J = J(d) = [c, d]$. □

Remark 3.6. There is only one point d in Lemma 3.5. Since the function p is almost everywhere continuous, the function u_1 is almost everywhere differentiable. Particularly, u_1 has the one-sided derivatives on the whole interval $[\overline{x}, b]$. It can be demonstrated that the one-sided derivatives of u_1 are positive. Thus u_1 increases on $[\overline{x}, b]$, ensuring the uniqueness of the point d.

Lemma 3.5 provides an interval $J \subset (a, b)$ such that the functions p, p/J and $p/I \setminus J$ have the common barycenter. It is easy to prove that if one of the equalities in formula (20) is valid, then the other two are also valid. So, if two of the above functions have the common barycenter, then it applies to the third.

Theorem 3.7. *Let $p : I = [a, b] \to \mathbb{R}$ be a positive integrable normalized function, let $J \subset (a, b)$ be an interval such that p and p/J have the common barycenter, let $\overline{x}$ be that barycenter, and let $\overline{x} = \alpha a + \beta b$ be the convex combination.*

Then each convex function $f : I \to \mathbb{R}$ satisfies the series of inequalities

$$\begin{aligned} f(\alpha a + \beta b) &\leq \min\left\{ \frac{\int_J f(x)p(x)dx}{\int_J p(x)dx}, \frac{\int_{I\setminus J} f(x)p(x)dx}{\int_{I\setminus J} p(x)dx} \right\} \\ &\leq \int_I f(x)p(x)dx \\ &\leq \max\left\{ \frac{\int_J f(x)p(x)dx}{\int_J p(x)dx}, \frac{\int_{I\setminus J} f(x)p(x)dx}{\int_{I\setminus J} p(x)dx} \right\} \\ &\leq \alpha f(a) + \beta f(b). \end{aligned} \tag{22}$$

Proof. The functions p, p/J and $p/I \setminus J$ have the common barycenter $\overline{x}$. Applying the left-hand (resp. right-hand) side of the double inequality in formula (9) to the restrictions p/J and $p/I \setminus J$, we get the inequality of the first (resp. last) two members in formula (22).

The convex combination equality

$$\int_I f(x)p(x)dx = \int_J p(x)dx \cdot \frac{\int_J f(x)p(x)dx}{\int_J p(x)dx} + \int_{I\setminus J} p(x)dx \cdot \frac{\int_{I\setminus J} f(x)p(x)dx}{\int_{I\setminus J} p(x)dx}$$

says that the number

$$\int_I f(x)p(x)dx$$

is located between the numbers

$$\frac{\int_J f(x)p(x)dx}{\int_J p(x)dx}, \frac{\int_{I\setminus J} f(x)p(x)dx}{\int_{I\setminus J} p(x)dx}.$$

Thus, the above combination generates the double inequality including the second, third and fourth member in formula (22). □

Some refinements of Fejér's inequality by inserting integral terms were obtained in [7] and [4].

4. Applications to Means

The next version of the integral form of Jensen's inequality (see [6]) is often used. Let $p : [a,b] \to \mathbb{R}$ be a nonnegative integrable normalized function, let $g : [a,b] \to \mathbb{R}$ be an integrable function, and let f be a convex function whose domain contains the image of g. Using the equality $f(c) = h_1(c)$ with $c = \int_a^b g(x)p(x)dx$, applying the affinity of h_1 by way of $h_1\Big(\int_a^b g(x)p(x)dx\Big) = \int_a^b h_1(g(x))p(x)dx$, and utilizing the fact $h_1(g(x)) \le f(g(x))$, we obtain the inequality

$$f\left(\int_a^b g(x)p(x)dx\right) \le \int_a^b f(g(x))p(x)dx. \tag{23}$$

If g is strictly monotone, then the equality is valid in formula (23) if and only if f is affine on the image of g. For more information on the integral form of Jensen's inequality, see the books [12] and [11].

What we need now is a strictly monotone continuous function $\varphi : [a,b] \to \mathbb{R}$. The φ-integral quasi-arithmetic mean of numbers a and b with nonnegative coefficients α and β such that $\alpha + \beta = 1$ is defined by the number

$$M_\varphi(a,b;\alpha,\beta) = \varphi^{-1}\left(\alpha\varphi(a) + \beta\varphi(b)\right). \tag{24}$$

We include a function p into the concept of integral means. The φ-integral quasi-arithmetic mean of numbers a and b with a nonnegative integrable normalized function p on $[a,b]$ can be defined by the number

$$M_\varphi(a,b;p) = \varphi^{-1}\left(\int_a^b \varphi(x)p(x)dx\right). \tag{25}$$

To present the basic integral means, we have to assume that a and b are positive numbers. Applying the identity function $\varphi(x) = x$, we get the arithmetic mean

$$A(a,b;p) = \int_a^b xp(x)dx = \overline{x},$$

applying the logarithmic function $\varphi(x) = \ln(x)$, we get the identric mean

$$I(a,b;p) = \exp\left(\int_a^b \ln(x)p(x)dx\right),$$

and applying the hyperbolic function $\varphi(x) = 1/x$, we get the logarithmic mean

$$L(a,b;p) = \left(\int_a^b (1/x)p(x)dx\right)^{-1}.$$

If $p(x) = 1/(b-a)$, we get the standard quasi-arithmetic and basic integral means of numbers a and b. As in the standard case, we have

$$L(a,b;p) < I(a,b;p) < A(a,b;p),$$

which will be demonstrated in two steps as follows.

Using the function $g(x) = 1/x$, and the convex function $f(x) = -\ln(x)$ in the inequality in formula (23), we get

$$-\ln\left(\int_a^b (1/x)p(x)dx\right) < \int_a^b \ln(x)p(x)dx.$$

Moving -1 to the place of an exponent, and acting with the exponential function, we obtain $L(a,b;p) < I(a,b;p)$.

Using the function $g(x) = x$, and the concave function $f(x) = \ln(x)$ in the reverse inequality in formula (23), we get

$$\ln\left(\int_a^b xp(x)dx\right) > \int_a^b \ln(x)p(x)dx.$$

Acting with the exponential function, and preferring an increasing order, we obtain $I(a,b;p) < A(a,b;p)$.

In order to present the quasi-arithmetic version of Fejér's inequality, we need the generalization of Theorem B which includes the function φ.

Corollary 4.1. *Let $p : [a,b] \to \mathbb{R}$ be a nonnegative integrable normalized function, let $\varphi : [a,b] \to \mathbb{R}$ be a strictly monotone continuous function, and let $\alpha\varphi(a) + \beta\varphi(b)$ be the convex combination such that*

$$\int_a^b \varphi(x)p(x)dx = \alpha\varphi(a) + \beta\varphi(b). \tag{26}$$

Then each convex function f whose domain contains the image of φ satisfies the double inequality

$$f\Big(\alpha\varphi(a) + \beta\varphi(b)\Big) \le \int_a^b f(\varphi(x))p(x)dx \le \alpha f(\varphi(a)) + \beta f(\varphi(b)). \tag{27}$$

Proof. Let us denote the left member of equation (26) by $\overline{\varphi(x)}$. The point $\overline{\varphi(x)}$ belongs to the interior of the interval $\varphi([a,b])$, and we can apply the proof of Theorem B to the convex function $f : \varphi([a,b]) \to \mathbb{R}$. □

If $\varphi, \psi : [a,b] \to \mathbb{R}$ are strictly monotone continuous functions, then it is said that ψ is φ-convex if the composition function $\psi \circ \varphi^{-1}$ is convex. The same notation is used for concavity.

Now we can demonstrate the quasi-arithmetic version of Fejér's inequality.

Corollary 4.2. *Let $p : [a,b] \to \mathbb{R}$ be a nonnegative integrable normalized function, let $\varphi, \psi : [a,b] \to \mathbb{R}$ be strictly monotone continuous functions, and let $\alpha\varphi(a) + \beta\varphi(b)$ be the convex combination such that*

$$M_\varphi(a,b;p) = M_\varphi(a,b;\alpha,\beta). \tag{28}$$

If ψ is either φ-convex and increasing or φ-concave and decreasing, then we have the double inequality

$$M_\varphi(a,b;\alpha,\beta) \le M_\psi(a,b;p) \le M_\psi(a,b;\alpha,\beta). \tag{29}$$

If ψ is either φ-convex and decreasing or φ-concave and increasing, then we have the reverse double inequality in formula (29)*.*

Proof. Acting with the function φ to equation (28), we get equation (26). Referring to Corollary 4.1, we can use the double inequality in formula (27).

To prove the case that ψ is φ-convex and increasing, we use the convex function $f = \psi \circ \varphi^{-1}$ in formula (27),

$$\Big(\psi \circ \varphi^{-1}\Big)\Big(\alpha\varphi(a) + \beta\varphi(b)\Big) \le \int_a^b \psi(x)p(x)dx \le \alpha\psi(a) + \beta\psi(b),$$

then act with the increasing function ψ^{-1}, and the result is formula (29). We similarly proceed in other cases. □

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