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A New Approach for Soft Semi-Topological Groups Based on Soft Element

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Abstract. Molodtsov introduced a theory of soft sets which can be seen as an effective tool to deal with uncertainties in [16] and then, the development of the theory gradually increased. In this paper we define the notion of soft semi-topological groups based on the notion of soft element which is defined by Wardovski in [28]. Also some theoretical results are given. Thus soft topological group theory become open to improvement with these new definitions and results.

1. Introduction

Molodtsov defined the soft set theory to describe phenomena and concepts of an ambiguous, undefined vague and imprecise meaning in [16]. So, many research work have been done in the field of soft set theory. Some operations on soft sets were defined by Maji et al. [15]. Although some operations on soft set theory were defined by Maji et al. more related operations to soft set theory are given by many researchers [3, 4, 6, 7, 22, 30]. Especially in [3], Ali et al. gave some new notions such as the restricted intersection, the restricted union, the restricted difference and the extended intersection of two soft sets. Also, Pei et al. [19] studied several soft operations as well and they discussed the relation between soft sets and information systems in [19].

Soft algebraic structures such as soft groups were given by Çağman et al., in [2] as a parameterized family of subgroups. After that Acar et al. defined initial concepts of soft rings in [1]. Sezer et al. introduced soft intersection semigroups, soft intersection left (right, two-sided) ideals and bi-ideals of semigroups [23]. Also some other soft algebraic structures presented in [5, 9, 14, 29].

Çağman et. al. [8] defined the soft topology by modifying the definition of soft set. After this study, Roy and Samantha [20] strengthen the definition of soft topological spaces. Beside these studies Shabir et al. introduced soft topological spaces which are defined over an initial universe with a fixed set of parameters in [24]. Many other researchers have contributed towards the topological structure of soft set theory in [13, 18, 27]. Following these papers, notions on soft semi-topological groups and soft topological groups are discussed by many authors [11, 17, 25, 26].

Wardowski [28] approached soft sets as classical mathematics by giving definition of soft element. By giving this definition, he redefined the soft mapping and gave the continuity of soft mappings. Using the

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notion of soft element Ghosh et. al.,[10], introduced a binary operation on the set of all nonempty soft elements of a given soft set and then they gave a new definition of soft groups and discussed on some algebraic results related to this soft group definition.

Topological group, see [12],is simply a combination of fundamental mathematical concepts; Group and Topological space. After having the definition of soft topological space and soft group, the axiomatization of the concept of soft topological group is a natural procedure. In this paper, as a first step of this purpose, we will introduce the notion of soft semi-topological group and some theoretical results with illustrated examples.

2. Preliminaries

Throughout this paper unless otherwise stated, the universal set, set of parameters and the power set of universal set will be denoted by U, E and $P(U)$ respectively. In this section, we recall some basic notions in soft set theory.

Definition 2.1. ([16]) A soft set F_A on the universe U defined by the set of ordered pairs $F_A = \{(x, f_A(e)) : e \in E, f_A(e) \in P(U)\}$, where $f_A : E \to P(U)$ such that $f_A(e) = \emptyset$ if $e \notin A$.

Here $f_A(e)$ is called an approximate function of the set F_A. Note that the set of all soft sets over U will be denoted by S(U).

Definition 2.2. ([7]) Let $F_A, F_B \in S(U)$. Then, F_A is a soft subset of F_B, denoted by $F_A \widetilde{\subset} F_B$, if $f_A(e) \subseteq f_B(e)$, for all $e \in E$

Definition 2.3. ([7]) Let $F_A, F_B \in S(U)$. Then the soft union $F_A \widetilde{\cup} F_B$ the soft intersection$F_A \widetilde{\cap} F_B$, and the soft difference $F_A \widetilde{\setminus} F_B$ of F_A and F_B are defined by the approximate functions

$f_{A\widetilde{\cup}B}(e) = f_A(e) \cup f_B(e)$, $f_{A\widetilde{\cap}B}(e) = f_A(e) \cap f_B(e)$, $f_{A\widetilde{\setminus}B}(e) = f_A(e) \setminus f_B(e)$, respectively.

Definition 2.4. ([3]) Let $F_A \in S(U)$. The relative complement(we will use it shortly as complement) of a soft set $F_A^{\widetilde{C}}$ of F_A is defined by the approximate function $f_{A^{\widetilde{C}}}(e) = f_A^C(e)$, where $f_A^C(e)$ is the complement of the set $f_A(e)$; that is, $f_A^C(e) = U \setminus f_A(e)$ for all $e \in E$.

Note 2.5. *Let $A, B \subseteq E$ and $F_A, G_B \in S(U)$ such that $F_A \widetilde{\subset} G_B$. Then $F_A \widetilde{\cap} G_B = F_A$ and $F_A \widetilde{\cup} G_B = G_B$.*

Definition 2.6. ([1]) For a soft set F_A, the set $SuppF_A = \{e \in E : f_A(e) \neq \varnothing\}$ is called the support of the soft set F_A

Definition 2.7. ([10]) A soft set F_A is said to be full soft set if $SuppF_A = A$. The collection of all full soft sets on U will be denoted by $S_f(U)$

Definition 2.8. ([7]) Let $F_A \in S(U)$. If $f_A(e) = U$ for all $e \in A$, then F_A is called an A-universal soft set, denoted by $\widetilde{F_A}$.

3. Soft Group and Soft Topological Space Structures Based on Soft Element

3.1. *Soft Group*

The notion of soft group and some related structures were firstly defined by Aktas et al. in [2], 2007 as a parametrized family of subgroups as following:

Definition 3.1. ([2]) Let G be a group and F_A be a soft set over G. The F_A is said to be a soft group over G if and only if $f_A(x)$ is a subgroup of G, for all $e \in A$.

Example 3.2. ([2]) Suppose that $G = A = \mathbb{Z}_6 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}\}$ and the set valued function is defined by;

$$F(\overline{0}) = F(\overline{3}) = \{\overline{0}\}, F(\overline{1}) = F(\overline{5}) = \{\overline{0}, \overline{3}\}, F(\overline{2}) = F(\overline{4}) = \{\overline{0}, \overline{2}, \overline{4}\}.$$

Since for all $x \in A$, $F(x)$ is subgroup of G then F_A is a soft group over G.

Definition 3.3. ([7]) Let $F_A \in S(U)$. If $f_A(x) = \emptyset$ for all $x \in E$, then F_A is called a empty soft set, denoted by $\widetilde{\emptyset}$.

Definition 3.4. ([28]) Let $A \subseteq E$ and $F_A \in S(U)$. We say that $(e, \{u\})$ is a nonempty soft element of F_A if $e \in E$ and $u \in F(e)$. The pair $(e, \emptyset)$, where $e \in E$, will be called an empty soft element of F_A. The fact that $\alpha = (e, \{u\})$ is a soft element of F_A will be denoted by $\alpha \widetilde{\in} F_A$ or $(e, \{u\}) \widetilde{\in} F_A$.

Note 3.5. ([10]) The set of all nonempty soft elements of F_A will be denoted by $F_A^\bullet$. Also note that a soft element $(e, \{u\})$ belongs to $F_A^\bullet$ will be denoted by $(e, \{u\}) \widetilde{\in} F_A^\bullet$.

Proposition 3.6. ([28]) *For each $F_A \in S(U)$, the following holds:*
$F_A = \widetilde{\bigcup}_{(e_i, \{u_j\}) \widetilde{\in} F_A} \{(e_i, \{u_j\})\}$.

Note 3.7. ([10]) For each $F_A \in S_f(U)$, the following also holds:
$F_A = \widetilde{\bigcup}_{(e_i, \{u_j\}) \widetilde{\in} F_A^\bullet} \{(e_i, \{u_j\})\}$.

Note 3.8. An empty soft set contains no nonempty soft element i.e., it consists of only empty soft elements.

Definition 3.9. ([28]) Let $F_A \in S(U)$ and $F_B \in S(U)$. The soft Cartesian product of F_A, G_B, denoted by $F_A \times G_B$ is a soft set on $U \times U$ of the form

$$F_A \widetilde{\times} G_B = \{((a_1, b_1), F(a_1) \times F(b_1)) : a_1, b_1 \in E\}.$$

Definition 3.10. ([28]) Let $F_A \in S(U), G_B \in S(U)$. The soft relation from F_A to G_B, if $R \widetilde{\subseteq} F_A \widetilde{\times} G_B$, i.e., R is a soft set of the form

$$R = \{((a, b), U_p \times U_q) : p, q \in E, U_p \subseteq F_A(p), U_q \subseteq G_B(q)\}.$$

Definition 3.11. ([28]) Let $F_A, G_B \in S(U)$. A soft set relation $T \widetilde{\subseteq} F_A \times G_B$ is called a soft mapping from F to G, which is denoted by $T : F_A \widetilde{\rightarrow} G_B$, if the following two conditions are satisfied:

(SM1) for each soft element $\alpha \widetilde{\in} F_A$, there exists only one soft element $\beta \widetilde{\in} G_B$ such that $\alpha T \beta$ (which will be denoted as $T(\alpha) = \beta$);

(SM2) for each empty soft element $\alpha \widetilde{\in} F_A$, $T(\alpha)$ is an empty soft element of G_B.

After these consequences, the following new definition for the concept of "Soft Group" is proposed by Ghosh and et al. [10] in 2016, which is independent from the Definition [2] and seems more natural.

Definition 3.12. ([10]) Let $(E, \circ)$ and $(U, \star)$ be two groupoids, $A \subseteq E$ and $F_A \in S_f(U)$.The binary operation $\widetilde{\star}$ on $F_A^\bullet$ is defined by

$$(e_i, \{u_k\}) \widetilde{\star} (e_j, \{u_l\}) = (e_i \circ e_j, \{u_k \star u_l\})$$

for all $(e_i, \{u_k\}), (e_j, \{u_l\}) \widetilde{\in} F_A^\bullet$.

$F_A^\bullet$ is said to be closed under the binary composition $\widetilde{\star}$ if and only if $(e_i \circ e_j, \{u_k \star u_l\}) \widetilde{\in} F_A^\bullet$ for all $(e_i, \{u_k\}), (e_j, \{u_l\}) \widetilde{\in} F_A^\bullet$ i.e., if and only if $e_i \circ e_j \in A$ and $u_k \star u_l \in F(e_i \circ e_j)$ for all $(e_i, \{u_k\}), (e_j, \{u_l\}) \widetilde{\in} F_A^\bullet$.

Definition 3.13. ([10]) If $F_A^\bullet$ is closed under the binary composition $\widetilde{\star}$, then the algebraic system $(F_A^\bullet, \widetilde{\star})$ is said to be a soft groupoid over (E, U).

Definition 3.14. ([10]) Let $(F^{\bullet}_{A}, \widetilde{\star})$ be a soft groupoid over (E, U), the binary composition $\widetilde{\star}$ is said to be:
i) commutative if $(e_i, \{u_j\})\widetilde{\star}(e_k, \{u_l\} = (e_k, \{u_l\})\widetilde{\star}(e_i, \{u_j\}$, for all $(e_i, \{u_j\}), (e_k, \{u_l\}), (e_m, \{u_n\})\widetilde{\in}F^{\bullet}_{A}$,
ii) associative if $[(e_i, \{u_j\})\widetilde{\star}(e_k, \{u_l\}]\widetilde{\star}(e_m, \{u_n) = (e_i, \{u_j\})\widetilde{\star}[(e_k, \{u_l\})\widetilde{\star}(e_m, \{u_n\}]$.

Definition 3.15. ([10]) A soft element $(e, \{u\})\widetilde{\in}F^{\bullet}_{A}$ is said to be a soft identity element in a soft groupoid $(F^{\bullet}_{A}, \widetilde{\star})$ if for all $(e_i, \{u_j\})\widetilde{\in}F^{\bullet}_{A}$,

$$(e, \{u\})\widetilde{\star}(e_i, \{u_j\}) = (e_i, \{u_j\}) = (e_i, \{u_j\})\widetilde{\star}(e, \{u\}).$$

Definition 3.16. ([10]) Let $(F^{\bullet}_{A}, \widetilde{\star})$ be a soft groupoid with soft identity element $(e, \{u\})$. A soft element $(e_i, \{u_j\})\widetilde{\in}F^{\bullet}_{A}$ is said to be invertible if there exists a soft element $(e^{'}_{i}, \{u^{'}_{j}\})\widetilde{\in}F^{\bullet}_{A}$ such that

$$(e_i, \{u_j\})\widetilde{\star}(e^{'}_{i}, \{u^{'}_{j}\}) = (e, \{u\}) = (e^{'}_{i}, \{u^{'}_{j}\})\widetilde{\star}(e_i, \{u_j\}).$$

Then $(e^{'}_{i}, \{u^{'}_{j}\})$ is called the soft inverse of $(e_i, \{u_j\})$ and denoted by $(e_i, \{u_j\})^{-1}$.

Definition 3.17. ([10]) Let $(E, \circ)$ and $(U, \star)$ be two groups, $A, B \subseteq E$ and $F_A \in S_f(U)$. A soft groupoid $(F^{\bullet}_{A}, \widetilde{\star})$ is said to be a soft group over (E, U) if
i)$\widetilde{\star}$ is associative,
ii) there exist a soft element $(e, \{u\})\widetilde{\in}F^{\bullet}_{A}$ such that

$$(e, \{u\})\widetilde{\star}(e_i, \{u_j\}) = (e_i, \{u_j\}) = (e_i, \{u_j\})\widetilde{\star}(e, \{u\})$$

for all $(e_i, \{u_j\}\widetilde{\in}F^{\bullet}_{A}$,
iii) for each soft element $(e_i, \{u_j\}\widetilde{\in}F^{\bullet}_{A}$, there exists a soft element $(e^{'}_{i}, \{u^{'}_{j}\})\widetilde{\in}F^{\bullet}_{A}$ such that

$$(e_i, \{u_j\})\widetilde{\star}(e^{'}_{i}, \{u^{'}_{j}\}) = (e, \{u\}) = (e^{'}_{i}, \{u^{'}_{j}\})\widetilde{\star}(e_i, \{u_j\}).$$

We often refer to a soft group $F^{\bullet}_{A}$, rather than use the binary structure notion $(F^{\bullet}_{A}, \widetilde{\star})$, with the understanding that there is of course a binary operation on the set $F^{\bullet}_{A}$.

Example 3.18. Suppose that $(U, \star) = (\mathbb{Z}_6, +)$, $(E, \circ) = (\mathbb{Z}_6, +)$ and soft set $F_A = \{(\bar{0}, \{\bar{0}\}), (\bar{1}, \{\bar{1}\}), (\bar{2}, \{\bar{2}\}), (\bar{3}, \{\bar{3}\}), (\bar{4}, \{\bar{4}\}), (\bar{5}, \{\bar{5}\})\}$. The table of the operation $\widetilde{\star}$ is given as;

$\widetilde{\star}$	$(\bar{0}, \{\bar{0}\})$	$(\bar{1}, \{\bar{1}\})$	$(\bar{2}, \{\bar{2}\})$	$(\bar{3}, \{\bar{3}\})$	$(\bar{4}, \{\bar{4}\})$	$(\bar{5}, \{\bar{5}\})$
$(\bar{0}, \{\bar{0}\})$	$(\bar{0}, \{\bar{0}\})$	$(\bar{1}, \{\bar{1}\})$	$(\bar{2}, \{\bar{2}\})$	$(\bar{3}, \{\bar{3}\})$	$(\bar{4}, \{\bar{4}\})$	$(\bar{5}, \{\bar{5}\})$
$(\bar{1}, \{\bar{1}\})$	$(\bar{1}, \{\bar{1}\})$	$(\bar{2}, \{\bar{2}\})$	$(\bar{3}, \{\bar{3}\})$	$(\bar{4}, \{\bar{4}\})$	$(\bar{5}, \{\bar{5}\})$	$(\bar{0}, \{\bar{0}\})$
$(\bar{2}, \{\bar{2}\})$	$(\bar{2}, \{\bar{2}\})$	$(\bar{3}, \{\bar{3}\})$	$(\bar{4}, \{\bar{4}\})$	$(\bar{5}, \{\bar{5}\})$	$(\bar{0}, \{\bar{0}\})$	$(\bar{1}, \{\bar{1}\})$
$(\bar{3}, \{\bar{3}\})$	$(\bar{3}, \{\bar{3}\})$	$(\bar{4}, \{\bar{4}\})$	$(\bar{5}, \{\bar{5}\})$	$(\bar{0}, \{\bar{0}\})$	$(\bar{1}, \{\bar{1}\})$	$(\bar{2}, \{\bar{2}\})$
$(\bar{4}, \{\bar{4}\})$	$(\bar{4}, \{\bar{4}\})$	$(\bar{5}, \{\bar{5}\})$	$(\bar{0}, \{\bar{0}\})$	$(\bar{1}, \{\bar{1}\})$	$(\bar{2}, \{\bar{2}\})$	$(\bar{2}, \{\bar{2}\})$
$(\bar{5}, \{\bar{5}\})$	$(\bar{0}, \{\bar{0}\})$	$(\bar{1}, \{\bar{1}\})$	$(\bar{2}, \{\bar{2}\})$	$(\bar{3}, \{\bar{3}\})$	$(\bar{4}, \{\bar{4}\})$	$(\bar{5}, \{\bar{5}\})$

One can observe from the table above that $F^{\bullet}_{A}$ is soft group with the binary operation $\widetilde{\star}$ according to Definition 3.17 while it is not a soft group according to the Definition 3.1.

Example 3.19. Let F_A be the soft set given in Example 3.2 where the parameter set and universal sets of the soft set F_A can be regarded as group $(\mathbb{Z}_6, +)$. The soft element-wise writing of $F^{\bullet}_{A}$ is as follows:
$F_A = \{(\bar{0}, \{\bar{0}\}), (\bar{1}, \{\bar{0}\}), (\bar{1}, \{\bar{3}\}), (\bar{2}, \{\bar{0}\}), (\bar{2}, \{\bar{2}\}), (\bar{2}, \{\bar{4}\}), (\bar{3}, \{\bar{0}\}), (\bar{4}, \{\bar{0}\}), (\bar{4}, \{\bar{2}\}), (\bar{4}, \{\bar{4}\}), (\bar{5}, \{\bar{0}\}), (\bar{5}, \{\bar{3}\})\}$. The binary operation $\widetilde{\star}$ obtained from the Definition 3.12 is not closed on $F^{\bullet}_{A}$, since $(\bar{1}, \{\bar{0}\}) + (\bar{1}, \{\bar{3}\}) = (\bar{1} + \bar{1}, \{\bar{3} + \bar{0}\}) = (\bar{2}, \{\bar{3}\})$ and $(\bar{2}, \{\bar{3}\}) \notin F_A$. So F_A is not a soft group according to Definition 3.17.

One can observe from the examples above that the definitions given in of soft group Definition 3.1 and 3.17 do not imply each other.

Definition 3.20. If a subset $G_B^\bullet$ of a soft group $F_A^\bullet$ is closed under the binary operation of $F_A^\bullet$ and if $G_B^\bullet$ with the induced operation from $F_A^\bullet$ is itself a soft group, then $G_B^\bullet$ is called a soft subgroup of $F_A^\bullet$. We will use the notion $G_B^\bullet \widetilde{\leq} F_A^\bullet$ to denote that $G_B^\bullet$ is a soft subgroup of $F_A^\bullet$.

Theorem 3.21. *Let $(F_A^\bullet, \widetilde{\star})$ be a soft group and $(G_B, \widetilde{\star})$ be a subgroup of $(F_A^\bullet, \widetilde{\star})$.*

i) For $\alpha, \beta, \gamma \widetilde{\in} F_A^\bullet$, $\alpha \widetilde{\star} \gamma = \beta \widetilde{\star} \gamma$, then $\alpha = \beta$ (right cancellation law).

ii) For $\alpha, \beta, \gamma \widetilde{\in} F_A^\bullet$, $\gamma \widetilde{\star} \alpha = \gamma \widetilde{\star} \beta$, then $\alpha = \beta$ (left cancellation law).

iii) The soft identity element of $(F_A^\bullet, \widetilde{\star})$ and $(G_B, \widetilde{\star})$ are the same.

iv) If $\alpha \widetilde{\in} G_B$, then the soft inverse of α in F_A is the same.

Proof. i) Let $\alpha = (e_1, \{u_1\}), \beta = (e_2, \{u_2\}), \gamma = (e_3, \{u_3\})$ and the soft identity element of $(F_A^\bullet, \widetilde{\star})$ is $(e, \{u\})$ suppose that $\alpha \widetilde{\star} \gamma = \beta \widetilde{\star} \gamma$. Then $(\alpha \widetilde{\star} \gamma) \widetilde{\star} \gamma^{-1} = (\beta \widetilde{\star} \gamma) \widetilde{\star} \gamma^{-1}$ implies $\alpha \widetilde{\star} (\gamma \widetilde{\star} \gamma^{-1}) = \beta \widetilde{\star} (\gamma \widetilde{\star} \gamma^{-1})$. Hence $\alpha \widetilde{\star} (e, \{u\}) = \beta \widetilde{\star} (e, \{u\})$, then $\alpha = \beta$.

ii) Similar to i).

iii) Let $(e, \{u\})_G$ denote the soft identity of G_B. Note that $(e, \{u\})_G \widetilde{\star})(e, \{u\})_G = (e, \{u\})_G = (e, \{u\})_G \widetilde{\star})(e, \{u\})_F$. Hence, by the cancellation property, $(e, \{u\})_F = (e, \{u\})_G$. This implies that the soft identity elements of G_B and $F_A$are the same.

iv) Let $(e_i, \{u_j\}) \widetilde{\in} G_B$. Let $(e_i, \{u_j\})^{-1}$ denote the inverse of $(e_i, \{u_j\})$ in G_B and $(e_i, \{u_j\})'$ denote the inverse of $(e_i, \{u_j\})$ in F_A. Then $(e_i, \{u_j\}) \widetilde{\star} (e_i, \{u_j\})^{-1} = (e, \{u\}) = (e_i, \{u_j\})^{-1} \widetilde{\star} (e_i, \{u_j\})$ and $(e_i, \{u_j\}) \widetilde{\star} (e_i, \{u_j\})' = (e, \{u\}) = (e_i, \{u_j\})' \widetilde{\star} (e_i, \{u_j\})$.
Now $(e_i, \{u_j\})^{-1} = (e_i, \{u_j\})^{-1} \widetilde{\star} (e, \{u\}) = (e_i, \{u_j\})^{-1} \widetilde{\star} ((e_i, \{u_j\}) \widetilde{\star} (e_i, \{u_j\})') = ((e_i, \{u_j\})^{-1} \widetilde{\star} (e_i, \{u_j\})) \widetilde{\star} (e_i, \{u_j\})' = (e, \{u\}) \widetilde{\star} (e_i, \{u_j\})' = (e_i, \{u_j\})'$.
This implies that the inverse of $(e_i, \{u_j\})$ in G_B and the inverse of $(e_i, \{u_j\})$ in F_A are the same. □

Remark 3.22. If $(F_A^\bullet, \widetilde{\star})$ is a soft group, then the soft set $G_B = \{(e, \{u\})\}$ which consists of only soft identity element of F_A with the operation $\widetilde{\star}$ and $(F_A^\bullet, \widetilde{\star})$ are soft subgroups of $(F_A^\bullet, \widetilde{\star})$. These soft subgroups are called trivial.

Theorem 3.23. *A subset $G_B^\bullet$ of a soft group $F_A^\bullet$ is a soft subgroup if and only if*

1) $G_B^\bullet$ is closed under the binary operation of $F_A^\bullet$,

2) the soft identity element $(e, \{u\})$ of $F_A^\bullet$ is in $G_B^\bullet$,

3) for all $(e_i, \{u_j\}) \widetilde{\in} G_B^\bullet$ it is true that $(e_i, \{u_j\}^{-1}) = (e_i^{-1}, \{u_j^{-1}\}) \widetilde{\in} G_B^\bullet$ also.

Proof. It is obvious that if $G_B^\bullet$ is a soft subgroup of $F_A^\bullet$ then Conditions 1), 2) and 3) are hold.

Conversely, suppose $G_B^\bullet \widetilde{\subseteq} F_A^\bullet$ such that Conditions 1), 2) and 3) are hold. By 2) we have at once Condition ii) in Definition 3.17. Also, Condition iii) in Definition 3.17 is satisfied by 3) and the fact that associative axiom is satisfied can be observed easily. □

Definition 3.24. Let $(F_A^\bullet, \widetilde{\star})$ be a soft group over (E, U) and $G_B^\bullet \widetilde{\subseteq} F_A^\bullet$, $(e_i, \{u_l\}) \widetilde{\in} F_A^\bullet$. For all $(e_j, \{u_k\}) \widetilde{\in} F_A^\bullet$, the soft element $(e_i, \{u_l\}) \widetilde{\star} (e_j, \{u_k\}) \widetilde{\star} (e_i, \{u_l\})^{-1}$ is called conjugate of $(e_j, \{u_k\})$ and the soft set $(e_i, \{u_l\}) \widetilde{\star} G_B^\bullet \widetilde{\star} (e_i, \{u_l\})^{-1} = \{(e_i, \{u_l\}) \widetilde{\star} (e_1, \{u_1\}) \widetilde{\star} (e_i, \{u_l\})^{-1} : (e_1, \{u_1\}) \widetilde{\in} G_B\}$ is called conjugate of $G_B^\bullet$.

Note 3.25. It can be observed that $(e_i, \{u_l\}) \widetilde{\star} G_B^\bullet \widetilde{\star} (e_i, \{u_l\})^{-1}$ is soft subgroup of $F_A^\bullet$.

Definition 3.26. Let $(F_A^\bullet, \widetilde{\star})$ be a soft group over (E, U) and $G_B^\bullet$ be soft subgroup of $F_A^\bullet$. If for all $(e_j, \{u_k\}) \widetilde{\in} F_A^\bullet$, $(e_i, \{u_l\}) \widetilde{\star} G_B^\bullet \widetilde{\star} (e_i, \{u_l\})^{-1} = G_B$ then $G_B^\bullet$ is called normal soft subgroup of $F_A^\bullet$.

Definition 3.27. Let $(F^\bullet_A, \widetilde{\star})$ be a soft group over (E, U) and $G^\bullet_B, H^\bullet_C \widetilde{\subseteq} F^\bullet_A$. Then

i) the set of all soft elements $(e_i, \{u_l\})\widetilde{\star}(e_j, \{u_k\})$ where $(e_i, \{u_l\})\widetilde{\in} G_B$, $(e_j, \{u_k\})\widetilde{\in} H_C$ will denote by $G^\bullet_B \widetilde{\star} H^\bullet_C$.

ii) the set of all soft elements $(e_i, \{u_l\})^{-1}$ where $(e_i, \{u_l\})\widetilde{\in} G^\bullet_B$ will denote by $G^{\bullet\,-1}_B$

Example 3.28. Let $F^\bullet_A$ be soft group given in Example 3.18, $G^\bullet_B = \{(\overline{0}, \{\overline{0}\}), (\overline{1}, \{\overline{1}\}), (\overline{2}, \{\overline{2}\})\}$ and $H^\bullet_C = \{(\overline{0}, \{\overline{0}\}), (\overline{3}, \{\overline{3}\})\}$ be subsets of F_A. Then $G^\bullet_B \widetilde{\star} H^\bullet_C = F^\bullet_A$ and $G^{\bullet\,-1}_B = \{(\overline{0}, \{\overline{0}\}), (\overline{5}, \{\overline{5}\}), (\overline{4}, \{\overline{4}\})\}$, $H^{-1}_C = \{(\overline{0}, \{\overline{0}\}), (\overline{3}, \{\overline{3}\})\}$

3.2. *Soft Topological Spaces*

In this section we introduce the definitions and basic properties concerning soft topological spaces, which will be useful in the next sections. The foundations of the theory of soft topological spaces are given by Roy et al. [21].

Definition 3.29. ([21]) A soft topology on $F_A \in S(U)$ is a collection $\widetilde{\tau}$ of soft subsets F_A satisfying:

1) $\widetilde{\emptyset}, F_{\widetilde{A}} \in \widetilde{\tau}$,

2) For all $i \in I$, I is an index set, $\{F^i_{A_i}\}_{i\in I} \subseteq \widetilde{\tau} \Longrightarrow \widetilde{\bigcup_{i\in I}} F^i_{A_i} \in \widetilde{\tau}$,

3) $G_B, H_C \in \widetilde{\tau} \Longrightarrow G_B \widetilde{\cap} H_C \in \widetilde{\tau}$.

If $\widetilde{\tau}$ is a soft topology on F, then the pair $(F_A, \widetilde{\tau})$ is called a soft topological space.

Definition 3.30. ([21]) Let $(F_A, \widetilde{\tau})$ be a soft topological space. Then every element of $\widetilde{\tau}$ is called a soft open set.

Definition 3.31. ([8]) Let $(F_A, \widetilde{\tau})$ be a soft topological space and $G_B \widetilde{\subseteq} F_A$. Then G_B is said to be soft closed if the soft set $G^{\widetilde{C}}_B$ is soft open.

Definition 3.32. ([8]) Let $(F_A, \widetilde{\tau})$ be a soft topological space and $\widetilde{\beta} \subseteq \widetilde{\tau}$. If every element of $\widetilde{\tau}$ can be written as the union of some elements of $\widetilde{\beta}$, then $\widetilde{\beta}$ is called a soft basis for the soft topology $\widetilde{\tau}$. Each element of $\widetilde{\beta}$ is called a soft basis element.

Definition 3.33. ([21]) A collection Ω of members of a soft topology $\widetilde{\tau}$ is said to be subbase for $\widetilde{\tau}$ if and only if the collection of all finite intersections of members of Ω is a base for $\widetilde{\tau}$.

Example 3.34. ([8]) Let $U = \{u_1, u_2, u_3\}$, $A = \{x_1, x_2\}$ and $F_A = \{(x_1, \{u_1, u_2\}), (x_2, \{u_2, u_3\})\}$. Then all soft subsets of F_A are listed below.

$F_{A_1} = \{(x_1, \{u_1\})\}$,
$F_{A_2} = \{(x_1, \{u_2\})\}$,
$F_{A_3} = \{(x_1, \{u_1, u_2\})\}$,
$F_{A_4} = \{(x_2, \{u_2\})\}$,
$F_{A_5} = \{(x_2, \{u_3\})\}$,
$F_{A_6} = \{(x_2, \{u_2, u_3\})\}$,
$F_{A_7} = \{(x_1, \{u_1\}), (x_2, \{u_2\})\}\}$,
$F_{A_8} = \{(x_1, \{u_1\}), (x_2, \{u_3\})\}$,
$F_{A_9} = \{(x_1, \{u_1\}), (x_2, \{u_2, u_3\})\}$,
$F_{A_{10}} = \{(x_1, \{u_2\}), (x_2, \{u_2\})\}$,
$F_{A_{11}} = \{(x_1, \{u_2\}), (x_2, \{u_3\})\}$,
$F_{A_{12}} = \{(x_1, \{u_2\}), (x_2, \{u_2, u_3\})\}$,
$F_{A_{13}} = \{(x_1, \{u_1, u_2\}), (x_2, \{u_2\})\}$,
$F_{A_{14}} = \{(x_1, \{u_1, u_2\}), (x_2, \{u_3\})\}$,
$F_{A_{15}} = F_A$,
$F_{A_{16}} = F_\emptyset$.

Then $\widetilde{\tau_1} = \{F_\emptyset, F_A\}$, $\widetilde{\tau_2} = \widetilde{P}(F_A)$, and $\widetilde{\tau_3} = \{F_\emptyset, F_A, F_{A_2}, F_{A_{11}}, F_{A_{13}}\}$ are soft topologies on F_A.

Definition 3.35. Let $(F^\bullet_A, \widetilde{\tau})$ be a soft topological space and $(e_j, \{u_l\})\widetilde{\in}F^\bullet_A$. A soft subset $G^\bullet_B$ of $F^\bullet_A$ is said to be a soft neighborhood of soft element $(e_j, \{u_l\})$, if there exist an open set $H^\bullet_C$ such that $(e_j, \{u_l\})\widetilde{\in}H^\bullet_C\widetilde{\subseteq}G^\bullet_B$. The family of all soft neighborhoods of the soft point $(e_j, \{u_l\})$ is denoted by $\mathcal{N}_{(e_j,\{u_l\})}$.

Example 3.36. Let F_A be the soft set and $\widetilde{\tau_3}$ be the soft topology on F_A given in Example 3.34. The set of all nonempty soft element of F_A is $F^\bullet_A = \{(x_1, \{u_1\}), (x_1, \{u_2\}), (x_2, \{u_2\}), (x_2, \{u_3\})\}$.

For the soft element $(x_1, \{u_1\})\widetilde{\in}F_A$, the soft sets containing $(x_1, \{u_1\})$ are $F_A, F_{A_1}, F_{A_3}, F_{A_7}, F_{A_8}, F_{A_9}, F_{A_{13}}, F_{A_{14}}$ and $F_{A_{15}}$.

The set of soft neighborhoods of $(x_1, \{u_1\})$ is $\mathcal{N}_{(x_1,\{u_1\})} = \{F_A, F_{A_{13}}\}$.

The set of soft neighborhoods of $(x_1, \{u_2\})$ is $\mathcal{N}_{(x_1,\{u_2\})} = \{F_{A_2}, F_{A_3}, F_{A_{10}}, F_{A_{11}}, F_{A_{12}}, F_{A_{13}}, F_{A_{14}}, F_A\}$.

The set of soft neighborhoods of $(x_2, \{u_2\})$ is $\mathcal{N}_{(x_2,\{u_2\})} = \{F_A, F_{A_{13}}\}$.

The set of soft neighborhoods of $(x_2, \{u_3\})$ is $\mathcal{N}_{(x_2,\{u_3\})} = \{F_{A_{11}}, F_{A_{12}}, F_{A_{14}}, F_A\}$.

Proposition 3.37. ([28]) *Let $(F_A, \widetilde{\tau})$ be a soft topological space. A soft set $G_B\widetilde{\subseteq}F_A$ is soft open if and only if for each soft element $\alpha\widetilde{\in}G_B$ there exists a soft set $H_C \in \widetilde{\tau}$ such that $\alpha\widetilde{\in}W\widetilde{\subseteq}G_B$.*

Definition 3.38. ([28]) Let $(F_A, \widetilde{\tau})$ be a soft topological space, and let $G_B\widetilde{\subseteq}F_A$. The soft topology on G_B induced by the soft topology $\widetilde{\tau}$ is the family $\widetilde{\tau_{G_B}}$ of the soft subsets of G_B of the form $\widetilde{\tau_{G_B}} = \{V\widetilde{\cap}G_B : V\widetilde{\in}\widetilde{\tau}\}$.

It is easy to verify that the family $\widetilde{\tau_{G_B}}$ is a soft topology on G_B. The soft topological space $(G_B, \widetilde{\tau_{G_B}})$ is called a soft topological subspace of $(F_A, \widetilde{\tau})$.

Definition 3.39. Let $(F_A, \widetilde{\tau_1})$ and $(G_B, \widetilde{\tau_2})$ be soft topological spaces and $\beta = \{F_{A_i} \times G_{B_j} : F_{A_i} \in \widetilde{\tau_1},\ G_{B_j} \in \widetilde{\tau_1}\}$. The collection $\widetilde{\tau}$ of all arbitrary union of elements of β is called the soft product topology over $F_a \times G_B$.

4. Soft Semi-Topological Group Based on Soft Element

Throughout this section, let $(E, \circ)$ and $(U, \star)$ be two groups, $A \subseteq E$ and $F_A \in S_f(U)$.

Definition 4.1. Let $(F^\bullet_A, \widetilde{\star})$ be a soft group and $(F_A, \widetilde{\tau})$ be a soft topological space. Then $(F_A, \widetilde{\star}, \widetilde{\tau})$ is called a soft semi-topological group if for each soft neighborhood F_B of $(e_i, \{u_j\})\widetilde{\star}(e'_i, \{u'_j\})$, there exists a soft neighborhood F_C of $(e_i, \{u_j\})$ and a soft neighborhood F_D of $(e'_i, \{u'_j\})$ such that $F_C\widetilde{\star}F_D\widetilde{\subseteq}F_B$.

Example 4.2. Let $E = \{e_1, e_2\}$, $U = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$ the classes of residues of integers module 4. The composition table of $\circ$ on E is given by

$\circ$	e_1	e_2
e_1	e_1	e_2
e_2	e_2	e_1

Then E is commutative group. Take $A = E$ and define soft set $F : E \longrightarrow P(U)$ by $F_E = \{(e_1, \{\overline{0}, \overline{2}\}), (e_2, \{\overline{1}, \overline{3}\})\}$ and thus, $F^\bullet_E = \{(e_1, \{\overline{0}\}), (e_1, \{\overline{2}\}), (e_2, \{\overline{1}\}), (e_2, \{\overline{3}\})\}$. The table of the operation $\widetilde{\star}$ on F_E is given as;

$\widetilde{\star}$	$(e_1, \{\overline{0}\})$	$(e_1, \{\overline{2}\})$	$(e_2, \{\overline{1}\})$	$(e_2, \{\overline{3}\})$
$(e_1, \{\overline{0}\})$	$(e_1, \{\overline{0}\})$	$(e_1, \{\overline{2}\})$	$(e_2, \{\overline{1}\})$	$(e_2, \{\overline{3}\})$
$(e_1, \{\overline{2}\})$	$(e_2, \{\overline{2}\})$	$(e_1, \{\overline{0}\})$	$(e_2, \{\overline{3}\})$	$(e_2, \{\overline{1}\})$
$(e_2, \{\overline{1}\})$	$(e_2, \{\overline{1}\})$	$(e_2, \{\overline{3}\})$	$(e_1, \{\overline{2}\})$	$(e_1, \{\overline{0}\})$
$(e_2, \{\overline{3}\})$	$(e_2, \{\overline{3}\})$	$(e_2, \{\overline{1}\})$	$(e_1, \{\overline{0}\})$	$(e_1, \{\overline{2}\})$

Here it is easy to verify that $(F^\bullet_E, \widetilde{\star})$ is commutative soft group with soft identity $(e_1, \{\overline{0}\})$. $\widetilde{\tau} = \{F_\emptyset, F_E, F_{E_1}, F_{E_2}\}$ is a soft topological space, where $F_{E_1} = \{(e_1, \{\overline{0}, \overline{2}\})\}$, $F_{E_2} = \{(e_2, \{\overline{1}, \overline{3}\})\}$. Then $(F_E, \widetilde{\star}, \widetilde{\tau})$ is a soft semi-topological group.

Theorem 4.3. *Let $\alpha = (e_i, \{h_j\})$ be a fixed element of a soft semi-topological group $(F_A, \widetilde{\star}, \tau)$. Then the mappings*

$$r_\alpha : (e_k, \{h_l\}) \longrightarrow (e_k, \{h_l\})\widetilde{\star}\alpha$$
$$l_\alpha : (e_k, \{h_l\}) \longrightarrow \alpha\widetilde{\star}(e_k, \{h_l\})$$

of F_A onto F_A are soft homeomorphisms of F_A.

Proof. It is clear that r_α is a 1-1 and onto mapping. Let W be a neighborhood of $(e_i, \{h_j\})\widetilde{\star}(e_k, \{h_l\})$. Since F_A is soft semi-topological group there exist neighborhood U of $(e_i, \{h_j\})$ such that $U\widetilde{\star}(e_k, \{h_l\})\widetilde{\subseteq}W$. This show that $r_{(e_i,\{h_j\})}$ is continuous.

Moreover it is easy to see that the inverse $r^{-1}_{(e_i,\{h_j\})}$ of $r_{(e_i,\{h_j\})}$ is the mapping

$$r^{-1}_{(\alpha} : (e_k, \{h_l\}) \longrightarrow (e_k, \{h_l\})\widetilde{\star}\alpha^{-1}$$

which is continuous by the same argument as above. Hence $r_{(e_i,\{h_j\})}$ is homeomorphism. The fact that $l_{(e_i,\{h_j\})}$ is a homeomorphism follows similarly. □

Corollary 4.4. *Let F_{A_1} be a soft closed set, F_{A_2} be a soft open set, F_{A_3} be an any soft set of a soft semi-topological group F_A be a soft closed set and $(e_i, \{h_j\}) \in F_A$. Then:*
i) $F_{A_1}\widetilde{\star}(e_i, \{h_j\})$, $(e_i, \{h_j\})\widetilde{\star}F_{A_1}$ are soft closed.
ii) $F_{A_2}\widetilde{\star}(e_i, \{h_j\})$, $(e_i, \{h_j\})\widetilde{\star}F_{A_2}$, $F_{A_2}\widetilde{\star}F_{A_3}$ and $F_{A_3}\widetilde{\star}F_{A_2}$ are soft open sets.

Proof. i) is obvious since the mappings in Theorem 4.3 are homeomorphisms. By the same argument the sets $F_{A_2}\widetilde{\star}(e_i, \{h_j\})$ and $(e_i, \{h_j\})\widetilde{\star}F_{A_2}$ in ii) are open. Also the rest of ii) is established since $F_{A_2}F_{A_3} = \bigcup_{(e_l,\{h_k\})\in F_{A_3}}(e_l, \{h_k\})F_{A_2}$, $F_{A_3}F_{A_2} = \bigcup_{(e_l,\{h_k\})\in F_{A_3}} F_{A_2}(e_l, \{h_k\})$, and the union of open sets is open. □

Corollary 4.5. *Let $(F_A, \widetilde{\star}, \widetilde{\tau})$ be a soft semi-topological group. For any $(e_i, \{h_j\}), (e_l, \{h_k\}) \in F_A$, there exists a homeomorphism f of F_A such that $f((e_i, \{h_j\})) = (e_l, \{h_k\})$.*

Proof. Let $(e_1, \{h_1\})^{-1}\widetilde{\star}(e_2, \{h_2\}) = (e_3, \{h_3\}) \in F_A$ and consider the mapping $f : (e_i, \{h_j\}) \longrightarrow (e_i, \{h_j\})\widetilde{\star}(e_3, \{h_3\})$. Then f is a homeomorphism by Theorem 4.3 and $f((e_1, \{h_1\})) = (e_2, \{h_2\})$. □

4.1. Soft Subgroups of Soft Semi-Topological Group

Theorem 4.6. *Let $(F_A, \widetilde{\star}, \widetilde{\tau})$ be a soft semi-topological group and G_B be a soft subgroup of F_A. Then $(G_B, \widetilde{\star}, \widetilde{\tau_{G_B}})$ is a soft semi-topological subgroup of $(F_A, \widetilde{\star}, \widetilde{\tau})$ and called soft semi-topological subgroup of F_A.*

Proof. Proof is straightforward. □

Example 4.7. Let $(F_E, \widetilde{\star}, \widetilde{\tau})$ be soft semi-topological group given in Example 4.2. $F_{E_1}\widetilde{\subseteq}F_E$ a soft semi-topological subgroup of $(F_A, \widetilde{\star}, \widetilde{\tau})$ with the induced operation from F_E.

Theorem 4.8. *Every soft open subgroup G_B of a semi-topological group F_A is soft closed.*

Proof. For each $(e_i, \{h_j\})$, $(e_i, \{h_j\})G_B$ is open by Corollary 4.4 condition ii). Hence $G_B = F_A \setminus \widetilde{\bigcup}(e_i, \{u_j\})G_B$ is closed, because $\widetilde{\bigcup}(e_i, \{u_j\})G_B$ is open, where the union is taken over all disjoint cosets different from G_B. □

5. Conclusion

Soft set theory has a rich potential for applications in several directions. Until today the theory of soft set has been developed successfully by many researchers. In this paper to contribute the improvement of soft set theory we introduce the soft semi-topological groups via soft element. Based on these results one can study further on soft topological algebraic structures.

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