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Lie Rough Groups

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Abstract. This paper introduces the definition of a Lie rough group as a natural development of the concepts of a smooth manifold and a rough group on an approximation space. Furthermore, the properties of Lie rough groups are discussed. It is shown that every Lie rough group is a topological rough group, and that the product of two Lie rough groups is again a Lie rough group. We define the concepts of Lie rough subgroups and Lie rough normal subgroups. Finally, our aim is to give an example by using definition of Lie rough homomorphism sets G and H.

1. Introduction

Rough set theory introduced by Z. Pawlak is one of the latest mathematical approaches for modeling uncertainty and vagueness [15]. In his paper [15], he presented the some basic definitions and the results of this new theory. Later, this theory attracted the attention of many researchers and created a wide field of application in various branches of science. Some of these application fields are economics, finance, engineering, environmental science, biology, chemistry, psychology, and other fields can be also found in [1, 7, 16, 18].

Algebra and topology are two of the most important branches of mathematics. For this reason, many mathematicians have been interested in the algebraic and topological aspects of rough set theory. Algebraically, rough sets have been studied by Biswas and Nanda [3], Kuroki and Mordeson [6], Davvaz and Mahdavipour [5], also see [4]. Biswas and Nanda, in [6], described the concepts of rough group and rough subgroup. Kuroki introduced the concept of a rough ideal in a semigroup [8]. Further, Davvaz defined the concept of rough subring considering the relation between rough set theory and ring theory [4]. On the other hand, some mathematicians have done some studies by combining topology and rough set theory, such as [9, 12, 17]. Also, the definition of topological rough group was given and their topological properties were examined in [2].

This paper is organized as follows. Section 2 is devoted to basic definitions and properties of rough set theory proposed by Pawlak. In addition, basic concepts and theorems of rough group and topological rough group are given in this section. In Section 3, we introduce the definition of a Lie rough group as a new concept and present some characterizations related to Lie rough groups. Also, we define the concepts of compact and connected Lie rough group. Section 4 deals with the subgroups of a Lie rough group. Moreover, the concept of Lie rough homomorphism sets is described in this section and an example of this

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is given. It can be said briefly that this article shows the beginning of a new construction for the expansion of both theoretical and practical studies on the rough set theory without a long history.

2. Preliminaries

In this section, we first recall several fundamental concepts and properties of rough sets, rough groups and topological rough groups for the sake of completeness. For more details, we refer to [2, 3, 6, 13, 15].

Definition 2.1. ([15]) Let U be a certain set called the universe with an equivalence relation R on itself. The pair (U, R) is called an approximation space.

In the approximation space (U, R), we will denote the equivalence classes of R containing an object x by $R(x)$.

Definition 2.2. ([15]) Let X be a certain subset of U. The lower approximation of X denoted by $\underline{R}(X)$ is the union of all the equivalence classes which are completely included in X. Namely,

$$\underline{R}(X) = \bigcup_{x \in U} \{R(x) : R(x) \subseteq X\}$$

Definition 2.3. ([15]) Let X be a certain subset of U. The upper approximation of X denoted by $\overline{R}(X)$ is the union of all the equivalence classes which have non-empty intersection with the set. Namely,

$$\overline{R}(X) = \bigcup_{x \in U} \{R(x) : R(x) \cap X \neq \emptyset\}$$

Definition 2.4. ([15]) Let X be a certain subset of U. The boundary region of X denoted by $B_N R(X)$ is the set $B_N R(X) = \overline{R}(X) - \underline{R}(X)$ in U.

Definition 2.5. ([15]) The X subset is called rough set if the boundary region is different from the empty set, otherwise it is called crisp(exact) set.

Corollary 2.6. ([15]) *Let X and Y be two subsets of U in the approximation space (U, R). Then, we have the followings about the lower and upper approximations:*

1. $\underline{R}(X) \subset X \subset \overline{R}(X)$,
2. $\underline{R}(\emptyset) = \overline{R}(\emptyset) = \emptyset$, $\underline{R}(U) = \overline{R}(U) = U$,
3. $\underline{R}(X \cap Y) = \underline{R}(X) \cap \underline{R}(Y)$,
4. $\underline{R}(X \cup Y) \subset \underline{R}(X) \cup \underline{R}(Y)$,
5. $\overline{R}(X \cup Y) = \overline{R}(X) \cup \overline{R}(Y)$,
6. $\overline{R}(X) \cap \overline{R}(Y) = \overline{R}(X \cap Y)$,
7. $X \subset Y \Longrightarrow \underline{R}(X) \subset \underline{R}(Y)$ *and* $\overline{R}(X) \subset \overline{R}(Y)$,
8. $\underline{R}(\underline{R}(X)) = \overline{R}\underline{R}(X) = \underline{R}(X)$,
9. $\overline{R}(\overline{R}(X)) = \underline{R}\overline{R}(X) = \overline{R}(X)$.

The concept of rough set is defined in topological spaces as follows:

Definition 2.7. ([9]) Let the approximation space (U, R) be a topological space and let $X \subset U$. Let $\overline{X}$, X° and $\delta(X)$ denote closure, interior and boundary points, respectively. Then the following equations can be written

$$\overline{R}(X) = \overline{X}, \qquad \underline{R}(X) = X^\circ, \qquad B_N R(X) = \delta(X)$$

such that X is called crisp(exact) if $\delta(X) = \emptyset$, otherwise X is called rough.

Definition 2.8. ([3]) Let (U, R) be an approximation space with a binary operation "$\cdot$" on U. Then, a subset G of U that provides the following conditions is called a rough group:

1. For all $x, y \in G$, $x \cdot y \in \overline{R}(G)$,
2. For all $x, y, z \in G$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ is provided in $\overline{R}(G)$,
3. There exists an element e in $\overline{R}(G)$ such that $x \cdot e = e \cdot x = x$, for all $x \in G$,
4. For all $x \in G$, there exists $y \in G$ such that $x \cdot y = y \cdot x = e$.

Definition 2.9. ([14]) Let (U_1, R_1) and (U_2, R_2) be two approximation spaces with the binary operations '$\cdot$' on U_1 and '$\star$' on U_2, respectively. Let $G \subset U_1$ and $H \subset U_2$ be two rough groups. If there is a surjection $\phi : \overline{R_1}(G) \longrightarrow \overline{R_2}(H)$, which $\phi(x \cdot y) = \phi(x) \star \phi(y)$ for all $x, y \in \overline{R}(G)$, then the rough groups G and H are called rough homomorphism sets.

Proposition 2.10. ([14]) *Let $G \subset U_1$ and $H \subset U_2$ be two rough groups and $\overline{R_2}(\phi(G)) = \overline{R_2}(H)$. Then $\phi(G)$ is a rough group.*

Definition 2.11. ([3]) Let H be a nonempty subset of rough group G. If H is a rough group with the operation in G, then H is called its rough subgroup.

Definition 2.12. ([2]) Let $(G, \cdot)$ be a rough group together with a topology τ on $\overline{R}(G)$. Then it is called a topological rough group, if the following conditions are satisfied:

1. the map $f : G \times G \longrightarrow \overline{R}(G)$, $f(x, y) = x \cdot y$ is continuous according to the product topology on $G \times G$, obtained from the topology τ_G on G induced by τ,
2. the inversion $g : G \longrightarrow G$, $g(x) = x^{-1}$ is continuous according to the topology τ_G on G induced by τ.

Definition 2.13. ([2]) Let H be a subgroup of the topological rough group G. Then, H is called its topological rough subgroup if:

1. the map $f_H : H \times H \longrightarrow \overline{R}(H)$, $f_H(x, y) = x \cdot y$ is continuous, where $\overline{R}(H)$ is considered to have the topology induced by $\overline{R}(G)$.
2. the inversion $g_H : H \longrightarrow H$, $g_H(x) = x^{-1}$ is continuous.

Proposition 2.14. ([2]) *Every rough subgroup of a topological rough group G is a topological rough subgroup of G by relative topology.*

Let us recall the concept of Lie group. For properties of Lie group and smooth manifold, we refer the reader to [10, 11].

Definition 2.15. ([11]) A Lie group G is a smooth manifold with a group structure such that the group operations are smooth. More precisely, the maps

$$\begin{aligned} m &: G \times G \longrightarrow G \text{ (multiplication)}, \\ i &: G \longrightarrow G \text{ (inversion)} \end{aligned}$$

are smooth.

Example 2.16. ([11]) Take $G = \mathbb{R}$ with $m(a, b) = a + b$, $i(a) = -a$ for all $a, b \in \mathbb{R}$. Then G is an abelian Lie group.

3. Lie Rough Groups

This section introduces the concept of Lie rough grup. Throughout this section, the approximation space (U, R) stands for a topological space.

Definition 3.1. Let $G \subseteq U$ be a rough group with the approximation space (U, R) and $\overline{R}(G)$ be a smooth manifold. A rough group G that satisfies the following conditions is called a Lie rough group:

1. the multiplication $f : G \times G \longrightarrow \overline{R}(G)$, $f(x, y) = x \cdot y$ is smooth,
2. the inversion $g : G \longrightarrow G$, $g(x) = x^{-1}$ is smooth.

Remark 3.2. In the definition above, the inversion g is in fact a diffeomorphism with the inverse $g^{-1} : G \longrightarrow G$, $x \mapsto g^{-1}(x) = x^{-1}$.

In what follows, the underlying manifold of a Lie rough grup G will be taken as the submanifold G induced by $\overline{R}(G)$ smooth manifold.

Example 3.3. Suppose that $G = S^1$, the unit circle. The unit circle S^1 is defined as

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

The topology on S^1 is considered as the relative topology induced from $\mathbb{R}^2$. According to this topology, S^1 is closed so that $\overline{S^1} = S^1$. Then, $\overline{R}(S^1) = S^1$ is obtained from Definition 2.7 such that S^1 is a smooth manifold. It is also clear that S^1 is a rough group according to the multiplication

$$\begin{array}{rcl} \cdot : S^1 \times S^1 & \longrightarrow & \overline{R}(S^1) = S^1 \\ ((x, y), (x', y')) & \mapsto & (x, y) \cdot (x', y') = (xx' - yy', xy' + x'y) \end{array}$$

and the inversion

$$\begin{array}{rcl} S^1 & \longrightarrow & S^1 \\ (x, y) & \mapsto & (x, -y) \end{array}$$

Moreover, the multiplication and the inversion given above are smooth maps. So S^1 is a Lie rough group because it satisfies the requirements in the Definition 3.1.

Proposition 3.4. *Each Lie rough group is a topological rough group with the topological space structure existing in the manifold itself.*

Proof. Let G be a Lie rough group on the approximation space (U, R). Then $\overline{R}(G)$ is a smooth manifold and following maps are smooth.

$$\begin{array}{rcl} f & : & G \times G \longrightarrow \overline{R}(G) \;\; \text{(multiplication)} \\ g & : & G \longrightarrow G \;\; \text{(inversion)} \end{array}$$

Since $\overline{R}(G)$ is a smooth manifold, it is a topological space. Also, the maps f and g are continuous , since every smooth map is continuous. So it is clear from Definition 2.12 that G is a topological rough group. Thus, the proof is completed. □

Let us give two propositions explaining the relation between the Lie group and the Lie rough group as follows:

Proposition 3.5. *Let G be a Lie rough group on the approximation space (U, R). If $G = \overline{R}(G)$, then G is a Lie group.*

Proof. Suppose $G = \overline{R}(G)$. From the definition of a Lie rough group , we can write that $G = \overline{R}(G)$ is a smooth manifold and:

i. the multiplication $f : G \times G \longrightarrow \overline{R}(G) = G$,
ii. the inversion $g : G \longrightarrow G$

are smooth. Therefore, it is obvious that G is a Lie group. □

Proposition 3.6. *Let G be a Lie group and let $\overline{R}(G)$ be a manifold. If the submanifold structure on G induced from $\overline{R}(G)$ and the manifold structure of G are the same, then G is a Lie rough group.*

Proof. The proof is easily obtained from the definitions of Lie rough group and Lie group. □

Proposition 3.7. *The product of any two Lie rough groups is also a Lie rough group.*

Proof. Let G and H be two Lie rough groups on the same approximation space (U, R) with the binary operations $'\cdot'$ and $'\star'$, respectively. Then $\overline{R}(G)$ and $\overline{R}(H)$ are smooth manifolds and the following maps are smooth.

$$\begin{aligned} f : G \times G &\longrightarrow \overline{R}(G) \\ (x, y) &\mapsto f(x, y) = x \cdot y \end{aligned}$$

$$\begin{aligned} g : G &\longrightarrow G \\ x &\mapsto g(x) = x^{-1} \end{aligned}$$

$$\begin{aligned} f_1 : H \times H &\longrightarrow \overline{R}(H) \\ (x_1, y_1) &\mapsto f_1(x_1, y_1) = x_1 \star y_1 \end{aligned}$$

$$\begin{aligned} g_1 : H &\longrightarrow H \\ x_1 &\mapsto g_1(x_1) = x_1^{-1} \end{aligned}$$

Since both $\overline{R}(G)$ and $\overline{R}(H)$ are smooth manifolds, $\overline{R}(G) \times \overline{R}(H)$ is also a smooth manifold with the structure of product manifold. Moreover, we can write the following equation by the Definition 2.7:

$$\overline{R}(G) \times \overline{R}(H) = \overline{G} \times \overline{H} = \overline{G \times H} = \overline{R}(G \times H)$$

So $\overline{R}(G \times H)$ has a smooth manifold structure and the maps

$$\begin{aligned} f \times f_1 : (G \times H) \times (G \times H) &\longrightarrow \overline{R}(G \times H) \\ ((x, x_1), (y, y_1)) &\mapsto (f \times f_1)((x, x_1), (y, y_1)) = (f(x, y), f_1(x_1, y_1)) = (x \cdot y, x_1 \star y_1) \end{aligned}$$

$$\begin{aligned} g \times g_1 : G \times H &\longrightarrow G \times H \\ (x, x_1) &\mapsto (g \times g_1)(x, x_1) = (g(x), g_1(x_1)) = (x^{-1}, x_1^{-1}) \end{aligned}$$

are smooth. Thus, $G \times H$ is a Lie rough group. □

Example 3.8. Since S^1 is a Lie rough group, the product $S^1 \times S^1$ is also a Lie rough group. Namely, torus is a Lie rough group.

Now let us define some notions about Lie rough groups using the topological properties of the underlying manifold of a Lie rough group.

Definition 3.9. If the underlying manifold of a Lie rough group is compact, it is called a compact Lie rough group.

Example 3.10. S^1 is compact, since it is closed and bounded. So S^1 is a compact Lie rough group.

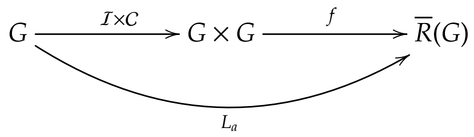
Definition 3.11. If the underlying manifold of a Lie rough group is connected, it is called a connected Lie rough group.

Example 3.12. S^1 is a connected Lie rough group, because it is connected.

From Lie group theory, we know that right and left translations are diffeomorphisms. This statement is not true in Lie rough groups. However, we have

Proposition 3.13. *Suppose G is a Lie rough group and an element a of G is fixed. Then for all $x \in G$:*
1. The left translation $L_a : G \longrightarrow \overline{R}(G)$ defined by $L_a(x) = a \cdot x$ is smooth and one-to-one,
2. The right translation $R_a : G \longrightarrow \overline{R}(G)$ defined by $R_a(x) = x \cdot a$ is smooth and one-to-one.

Proof. 1. The identity map $\mathcal{I}$ and the constant map C are smooth. Then, $\mathcal{I} \times C$ is smooth.



Since f is smooth, the composition map $f \circ (\mathcal{I} \times C)$ is also smooth. Thus, L_a is a smooth map.

For the condition of one-to-one, assume that $L_a(x) = L_a(y)$ for all $x, y \in G$. In this case, we have $a \cdot x = a \cdot y$. Since a^{-1} is in G with respect to the inversion $g : G \longrightarrow G$ defined by $g(x) = x^{-1}$, it follows that $a^{-1} \cdot (a \cdot x) = a^{-1} \cdot (a \cdot y)$. Therefore, it holds that $x = y$, which proves that L_a is one-to-one .

2. The proof is similar to **1**. □

Remark 3.14. If a Lie rough group G is abelian, the translations L_g and R_g are coincide.

4. Lie Rough Subgroups and Lie Rough Normal Subgroups

The aim of this final section is to give the definitions of Lie rough subgroup and Lie rough normal subgroup and to investigate some properties of them.

Definition 4.1. Let $G \subseteq U$ be a Lie rough group with the approximation space (U, R) and $H \subset G$ be a rough subgroup of G. Then, the rough subgroup H is called a Lie rough subgroup of G if the following conditions are satisfied:
1. H is a Lie rough group,
2. the inclusion mapping $i : \overline{R}(H) \hookrightarrow \overline{R}(G)$ is a smooth immersion.

Example 4.2. Every Lie rough group is a Lie rough subgroup of itself.

We know that the rough subgroup of a topological rough group is also a topological rough group [2]. On the other hand, this is not always true for a Lie rough group. However, we have

Proposition 4.3. *Any closed rough subgroup of a Lie rough group G is a Lie rough subgroup of G.*

Proof. Since H is a closed rough subgroup of the Lie rough group G, it follows that H has a smooth structure induced by the smooth structure of G. Hence, H is hold the first condition of Definition 4.1. Also, since H is closed, we have $H = \overline{H} = \overline{R}(H) \subset G \subset \overline{R}(G)$. So there is a smooth immersion $i : \overline{R}(H) \hookrightarrow \overline{R}(G)$. Therefore, H is a Lie rough subgroup of G. □

The notion of normal subgroup is important in Lie group theory. We introduce its rough version here:

Definition 4.4. Let N be a Lie rough subgroup of a Lie rough group G. If $a \cdot N = N \cdot a$ for all a in G, then N is called a Lie rough normal subgroup.

Proposition 4.5. *If G is an abelian Lie rough group, then each Lie rough subgroup of G is a Lie rough normal subgroup.*

Proof. Suppose that G is an abelian Lie rough group and N is a Lie rough subgroup of G. Since G is abelian, we have $a \cdot N = N \cdot a$ for all $a \in G$. Hence, N is a Lie rough normal subgroup of G. □

4.1. Lie Rough Homomorphisms

Let (U_1, R_1) and (U_2, R_2) be two approximation spaces with the binary operations $'\cdot'$ and $'\star'$ on them, respectively.

Definition 4.6. Let $G \subset U_1$ and $H \subset U_2$ be Lie rough groups. If there is a smooth map $\phi : \overline{R_1}(G) \longrightarrow \overline{R_2}(H)$, which allows the rough groups G and H to be rough homomorphisms sets, then the Lie rough groups G and H are called Lie rough homomorphism sets.

In addition, if ϕ^{-1} is also smooth, the Lie rough groups G and H are called Lie rough isomorphism sets.

Example 4.7. Let G be a Lie rough group on the approximation space (U, R). Then, the Lie rough group G is the Lie rough homomorphism set itself with the identity map $\mathcal{I} : \overline{R}(G) \longrightarrow \overline{R}(G)$. Moreover, it is a Lie rough isomorphism set itself.

Proposition 4.8. *Let G and H be Lie rough homomorphism sets with the map $\phi : \overline{R_1}(G) \longrightarrow \overline{R_2}(H)$. Suppose that $\overline{R_2}(\phi(G)) = \overline{R_2}(H)$ and $\phi(G) \subset H$. Then, $\phi(G)$ is a Lie rough group.*

Proof. Let G be a Lie rough group. Since G and H are Lie rough homomorphism sets with the map $\phi : \overline{R_1}(G) \longrightarrow \overline{R_2}(H)$, the Proposition 2.10 states that $\phi(G)$ is a rough group. Let us show that $\phi(G)$ is a Lie rough group. Since H is a Lie rough group, then $\overline{R_2}(H)$ is a smooth manifold such that

$$\begin{aligned} f : H \times H &\longrightarrow \overline{R_2}(H) \\ (x, y) &\mapsto f(x, y) = x \star y \end{aligned}$$

and

$$\begin{aligned} g : H &\longrightarrow H \\ x &\mapsto g(x) = x^{-1} \end{aligned}$$

are smooth. Since $\overline{R_2}(\phi(G)) = \overline{R_2}(H)$, $\overline{R_2}(\phi(G))$ is a smooth manifold. Since $\phi(G) \subset H$, restrictions of f and g to $\phi(G)$ are also smooth. Thus, $\phi(G)$ is a Lie rough group. □

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