Filomat 32:16 (2018), 5665–5677
https://doi.org/10.2298/FIL1816665G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Convergence Theorems of a Faster Iteration Process Including Multivalued Mappings with Analytical and Numerical Examples

Birol Gunduz[a], Osman Alagoz[b], Sezgin Akbulut[c]

[a]*Department of Mathematics, Faculty of Science and Art, Erzincan University, Erzincan, 24000, Turkey*
[b]*Department of Mathematics, Faculty of Science and Art, Bilecik Seyh Edebali University, Bilecik, 11000, Turkey*
[c]*Department of Mathematics, Faculty of Science and Art, Ataturk University, Erzurum, 25240, Turkey*

Abstract. In this paper, we first give the modified version of the iteration process of Thakur et al. [15] which is faster than Picard, Mann, Ishikawa, Noor, Agarwal et al. [2] and Abbas et al. [1] processes. Secondly, we prove weak and strong convergence theorems of this iteration process for multivalued quasi nonexpansive mappings in uniformly convex Banach spaces. Thirdly, we support our theorems with analytical examples. Finally, we compare rates of convergence for multivalued version of iteration processes mentioned above via a numerical example.

1. Introduction

Fixed point theory has an important role in modern mathematics because any problem can be turned into a fixed point problem. Banach fixed-point theorem is the basis of this theory; it guarantees the existence and uniqueness of fixed points of a contraction mapping defined on complete metric spaces and gives a constructive method to find those fixed points. The multivalued version of Banach contraction principle was first proved by Nadler [8] in 1969. Moreover, Lim [6] constructed the existence theorem for fixed point of multivalued nonexpansive mappings in uniformly convex Banach spaces. Actually, the fixed point theory for multivalued nonexpansive mappings is more difficult than the corresponding theory for single valued nonexpansive mappings. But because of their applications in control theory, convex optimization, differential inclusion and economics, many authors have been studied not only existence and uniqueness of fixed point of multivalued nonexpansive mappings but also approximated fixed point of multivalued nonexpansive mappings. Different iterative algorithms have been used to approximate the fixed points of single or multivalued nonexpansive mappings in different spaces. Most known iterative algorithms are Mann [7], Ishikawa [3] and Noor [9] iteration.

In 2009, Shahzad and Zegeye [13] studied convergence of the Mann and the Ishikawa iteration processes for multivalued mappings in a nonempty closed convex subset of a uniformly convex Banach space. They defined $P_T(x) = \{y \in Tx : \|x - y\| = d(x, Tx)\}$ for a multivalued mapping to do it well defined.

In 2007, Agarwal et al. [2] introduced a new iteration process which converges at a rate that is the same as that of the Picard iteration and faster than the Ishikawa iteration for contractions. In [5], Khan and

2010 *Mathematics Subject Classification*. Primary 47H10; Secondary 49M05, 54H25
Keywords. Faster iteration, Banach spaces, quasi-nonexpansive multivalued mapping
Received: 19 June 2017; Revised 09 December 2017; Accepted: 10 December 2017
Communicated by Ljubiša D.R. Kočinac
Email addresses: birolgndz@gmail.com (Birol Gunduz), osman.alagoz@bilecik.edu.tr (Osman Alagoz), sezginakbulut@atauni.edu.tr (Sezgin Akbulut)

Yildirim modified this iteration process for multivalued nonexpansive mappings and proved strong and weak convergence results for this iteration process.

Recently, Abbas et al. [1] introduced a new iteration process and proved some weak and strong convergence theorems for two nonexpansive mappings. They showed that this process converges faster than the Agarwal et al. [2] iteration process. Moreover, they found solutions of constrained minimization problems and feasibility problems as applications of their results. In [4], Khan et al. used an iteration to compute fixed points of multivalued quasi-nonexpansive mappings in Banach spaces. Motivated by Agarwal et al. [2] and Abbas et al. [1], Thakur et al. [15] introduced a new three step iteration process for computing fixed point of nonexpansive mappings in a uniformly convex Banach space. They also showed that their iteration process converges faster than the iteration processes of the Agarwal et al. [2] and the Abbas et al. [1].

Now, we will construct the above iteration processes for single valued mappings and multivalued mappings in the sense of Shahzad and Zegeye [13] using $P_T(x) = \{y \in Tx : \|x - y\| = d(x, Tx)\}$.

Let K be a nonempty closed convex subset of a uniformly convex Banach space E and T be a single valued or multivalued mapping. The above iteration processes generated by the following relation for arbitrary chosen $x_1 \in K$ and sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0, 1)$.

Iteration	Singlevalued Version	Multivalued Version
Mann	$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n$	$x_{n+1} = (1-\alpha_n)x_n + \alpha_n u_n$
Ishikawa	$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n$ $y_n = (1-\beta_n)x_n + \beta_n T x_n$	$x_{n+1} = (1-\alpha_n)x_n + \alpha_n v_n$ $y_n = (1-\beta_n)x_n + \beta_n u_n$
Noor	$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n$ $y_n = (1-\beta_n)x_n + \beta_n T z_n$ $z_n = (1-\gamma_n)x_n + \gamma_n T x_n$	$x_{n+1} = (1-\alpha_n)x_n + \alpha_n v_n$ $y_n = (1-\beta_n)x_n + \beta_n w_n$ $z_n = (1-\gamma_n)x_n + \gamma_n u_n$
Agarwal et al.	$x_{n+1} = (1-\alpha_n)T x_n + \alpha_n T y_n$ $y_n = (1-\beta_n)x_n + \beta_n T x_n$	$x_{n+1} = (1-\alpha_n)u_n + \alpha_n v_n$ $y_n = (1-\beta_n)x_n + \beta_n u_n$
Abbas et al.	$x_{n+1} = (1-\alpha_n)T y_n + \alpha_n T z_n$ $y_n = (1-\beta_n)T x_n + \beta_n T z_n$ $z_n = (1-\gamma_n)x_n + \gamma_n T x_n$	$x_{n+1} = (1-\alpha_n)v_n + \alpha_n w_n$ $y_n = (1-\beta_n)u_n + \beta_n w_n$ $z_n = (1-\gamma_n)x_n + \gamma_n u_n$
Thakur et al.	$x_{n+1} = (1-\alpha_n)T z_n + \alpha_n T y_n$ $y_n = (1-\beta_n)z_n + \beta_n T z_n$ $z_n = (1-\gamma_n)x_n + \gamma_n T x_n$	$x_{n+1} = (1-\alpha_n)w_n + \alpha_n v_n$ $y_n = (1-\beta_n)z_n + \beta_n w_n$ $z_n = (1-\gamma_n)x_n + \gamma_n u_n$

where $u_n \in P_T(x_n)$, $v_n \in P_T(y_n)$ and $w_n \in P_T(z_n)$.

Our aim in this paper is four fold. First is to show that multivalued version of the Thakur et al. [15] process converges strongly and weakly to a fixed point of a multivalued nonexpansive mapping in Banach spaces. Second is to provide validity of our theorem by giving some examples. Third is to see that multivalued version of iteration processes given in the above table converges fixed point of a multivalued mapping using a numerical example. Fourth is to compare rate of convergence of multivalued version of iteration processes given in the above table by considering the numerical example.

Motivated by the previous iteration, we introduce a new modified iteration process to approximate fixed points of multivalued quasi-nonexpansive mappings where the sequence $\{x_n\}$ is generated by $x_1 \in K$ and

$$\begin{cases} x_{n+1} &= (1-\alpha_n)w_n + \alpha_n v_n \\ y_n &= (1-\beta_n)z_n + \beta_n w_n \\ z_n &= (1-\gamma_n)x_n + \gamma_n u_n \end{cases} \tag{1}$$

where $\alpha_n, \beta_n, \gamma_n \in [0, 1]$ for all $n \in \mathbb{N}$, $u_n \in P_T(x_n)$, $v_n \in P_T(y_n)$ and $w_n \in P_T(z_n)$.

2. Preliminaries

Now, we give some concepts, definitions and lemmas which will be used through the proof of our main results

Definition 2.1. [Proximinal] Let E be a real Banach space. A subset K is called proximinal if for each $x \in E$, there exists an element $k \in K$ such that

$$d(x,k) = \inf\{y \in K : \|x-y\| = d(x,K)\}.$$

From now on, we assume that $C(K)$, $CB(K)$ and $P(K)$ show compact subsets, closed and bounded subsets and proximinal bounded subsets of K, respectively.

Definition 2.2. [Hausdorff metric] For every $A, B \in CB(E)$,

$$H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\}$$

denotes the Pompeiu-Hausdorff metric on $CB(E)$ induced by the metric d.

A point $x \in K$ is called a fixed point of a multivalued mapping $T : K \to CB(K)$ if $x \in Tx$. A set of all fixed points of T is denoted by $F(T)$.

Definition 2.3. A multivalued mapping $T : K \to CB(K)$ is said to be

- nonexpansive mapping if $H(Tx,Ty) \leq \|x-y\|$ for all $x, y \in K$,
- quasi-nonexpansive mapping if $F(T) \neq \emptyset$ and $H(Tx,p) \leq \|x-p\|$ for all $x \in K$ and $p \in F(T)$.

It is well known that every nonexpansive mapping with a nonempty fixed point set is quasi-nonexpansive mapping. But converse is not true. This statement is valid not only for single valued mappings but also multivalued mappings.

Definition 2.4. [Opial condition] ([10]) A Banach space E is said to satisfy Opial's condition if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$. Hilbert spaces and all l^p spaces $(1 < p < \infty)$ satisfy Opial condition but, $L^p[0,2\pi]$ space with $1 < p \neq 2$ fails to satisfy Opial condition.

Definition 2.5. [Demiclosed mapping] A multivalued mapping $T : K \to CB(K)$ is called demiclosed at $y \in K$ if for any sequence $\{x_n\}$ in K weakly convergent to x and $y_n \in Tx_n$ strongly convergent to y, we have $y \in Tx$.

The following lemmas have an important role in proof of our theorems.

Lemma 2.6. ([14]) *Let $T : K \to P(K)$ be a multivalued mapping and $P_T(x) = \{y \in Tx : \|x-y\| = d(x,Tx)\}$. Then the following are equivalent*

1. $x \in F(T)$.
2. $P_T(x) = \{x\}$.
3. $x \in F(P_T)$.

Moreover, $F(T) = F(P_T)$.

Lemma 2.7. ([11]) *Let E be a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

3. Main Results

Before we give our main theorems, we give a useful lemma with its proof.

Lemma 3.1. *Let E be a uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $T : K \to P(K)$ be a multivalued mapping such that $F(T) \neq \emptyset$ and P_T is a quasi-nonexpansive mapping. Let $\{x_n\}$ be the sequence as defined in (1). Then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$ and $\lim_{n\to\infty} d(x_n, P_T(x_n)) = 0$.*

Proof. For a given $p \in F(T)$, we have

$$\begin{aligned}
\|z_n - p\| &= \|(1-\gamma_n)x_n + \gamma_n u_n - p\| \\
&\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|u_n - p\| \\
&\leq (1-\gamma_n)\|x_n - p\| + \gamma_n H(P_T(x_n), P_T(p)) \\
&\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|x_n - p\| \\
&\leq \|x_n - p\|. \qquad (2)
\end{aligned}$$

Next

$$\begin{aligned}
\|y_n - p\| &= \|(1-\beta_n)z_n + \beta_n w_n - p\| \\
&\leq (1-\beta_n)\|z_n - p\| + \beta_n\|w_n - p\| \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n H(P_T(z_n), P_T(p)) \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n\|z_n - p\| \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n\|x_n - p\| \\
&\leq \|x_n - p\|, \qquad (3)
\end{aligned}$$

and by using (2) and (3) we get

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1-\alpha_n)w_n + \alpha_n v_n - p\| \\
&\leq (1-\alpha_)\|w_n - p\| + \alpha_n\|v_n - p\| \\
&\leq (1-\alpha_n)H(P_T(z_n), P_T(p)) + \alpha_n H(P_T(y_n), P_T(p)) \\
&\leq (1-\alpha_n)\|z_n - p\| + \alpha_n\|y_n - p\| \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|x_n - p\| \\
&\leq \|x_n - p\|.
\end{aligned}$$

Hence $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$. In order to prove the rest of the Lemma (3.1), we shall prove that $\lim_{n\to\infty} \|x_n - u_n\| = 0$. Assume that $\lim_{n\to\infty} \|x_n - p\| = c$. From (2) and (3) we have

$$\limsup_{n\to\infty} \|z_n - p\| \leq c \qquad (4)$$

and

$$\limsup_{n\to\infty} \|y_n - p\| \leq c. \qquad (5)$$

Moreover, we have

$$\begin{aligned}
&\|u_n - p\| \leq H(P_T(x_n), P_T(p)) \leq \|x_n - p\|, \\
&\|v_n - p\| \leq H(P_T(y_n), P_T(p)) \leq \|y_n - p\|
\end{aligned}$$

and

$\|w_n - p\| \leq H(P_T(z_n), P_T(p)) \leq \|z_n - p\|$.

Taking limsup on both sides of above inequalities, we obtain

$$\begin{cases} \limsup_{n\to\infty} \|u_n - p\| \leq c \\ \limsup_{n\to\infty} \|v_n - p\| \leq c \\ \limsup_{n\to\infty} \|w_n - p\| \leq c \end{cases} \tag{6}$$

and so

$$c = \lim_{n\to\infty} \|x_{n+1} - p\| = \lim_{n\to\infty} \|(1 - \alpha_n)(w_n - p) + \alpha_n(v_n - p)\|.$$

From Lemma (2.7), we have

$$\lim_{n\to\infty} \|w_n - v_n\| = 0.$$

Now

$$\|x_{n+1} - p\| = \|(1 - \alpha_n)w_n + \alpha_n v_n - p\| \leq \|w_n - p\| + \alpha_n \|w_n - v_n\|$$

yields that

$$c \leq \liminf_{n\to\infty} \|w_n - p\|. \tag{7}$$

Next (6) and (7) give

$$\lim_{n\to\infty} \|w_n - p\| = c.$$

On the other hand, we have

$$\begin{aligned} \|w_n - p\| =& \|w_n - p + v_n - v_n\| \leq \|w_n - v_n\| + \|v_n - p\| \\ \leq& \|w_n - v_n\| + \|y_n - p\| \end{aligned}$$

and this yields that

$$c \leq \liminf_{n\to\infty} \|y_n - p\|. \tag{8}$$

From (5) and (8), we get

$$c = \lim_{n\to\infty} \|y_n - p\| = \lim_{n\to\infty} \|(1 - \beta_n)(z_n - p) + \beta_n(w_n - p)\|.$$

Using Lemma (2.7), from (4) and (6) we get

$$\lim_{n\to\infty} \|z_n - w_n\| = 0.$$

Since

$$\begin{aligned} \|y_n - p\| =& \|(1 - \beta_n)z_n + \beta_n w_n - p\| \\ \leq& \|z_n - p\| + \beta_n \|w_n - z_n\|, \end{aligned}$$

we obtain

$$c \leq \liminf_{n\to\infty} \|z_n - p\|. \tag{9}$$

So, from (4) and (9) we get

$$\lim_{n\to\infty} \|z_n - p\| = c.$$

Consequently,

$$\begin{aligned} c &= \lim_{n\to\infty} \|z_n - p\| \\ &= \lim_{n\to\infty} \|(1-\alpha_n)x_n + \alpha_n u_n - p\| \\ &= \lim_{n\to\infty} \|(1-\alpha_n)(x_n - p) + \alpha_n(u_n - p)\|. \end{aligned}$$

Now by using Lemma (2.7), (6) and because of $\lim_{n\to\infty} \|x_n - p\| = c$, we obtain

$$\lim_{n\to\infty} \|x_n - u_n\| = 0 \tag{10}$$

which yields that $\lim_{n\to\infty} d(x_n, P_T(x_n)) = 0$ as desired. □

Now we give our convergence theorems with examples which are satisfied.

Theorem 3.2. *Let E be a uniformly convex Banach space and K be a nonempty compact convex subset of E. Let $T : K \to P(K)$ be a multivalued mapping such that $F(T) \neq \emptyset$ and P_T be a quasi-nonexpansive mapping. Let $\{x_n\}$ be the sequence as defined in (1). Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. As we proved that $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$. Because of the compactness of K there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} \|x_{n_k} - q\| = 0$ for some $q \in K$. Then

$$\begin{aligned} d(q, P_T(q)) &\le d(x_{n_k}, q) + d(x_{n_k}, P_T(x_{n_k})) + H(P_T(x_{n_k}), P_T(q)) \\ &\le \|x_{n_k} - q\| + \|x_{n_k} - u_{n_k}\| + \|x_{n_k} - q\| \to 0 \end{aligned}$$

as $n \to \infty$. From Lemma (3.1) we have $\lim_{n\to\infty} \|x_{n_k} - u_{n_k}\| = 0$.

So we get $d(q, P_T(q)) = 0$. Hence q is a fixed point of P_T. Since $F(T) = F(P_T)$ by Lemma (2.6), $\{x_n\}$ converges strongly to a fixed point of T. □

Now we give an example to satisfy the Theorem (3.2).

Example 3.3. Let $(\mathbb{R}, \|.\|)$ be a normed space with usual norm and $K = [0, 2]$. Define $T : K \to P(K)$ as:

$$Tx = \Big[0, \frac{x+1}{2}\Big].$$

It is clear that K is a compact convex subset of $\mathbb{R}$. Moreover $F(T) = [0, 1] \neq \emptyset$. Let $\alpha_n = \beta_n = \gamma_n = \dfrac{1}{2^n}$. If $x \in [0, 1]$ observe that $P_T(x) = \{x\}$. If $x \notin [0, 1]$, then

$$\begin{aligned} P_T(x) &= \Big\{y \in Tx : |y - x| = d(x, [0, \frac{x+1}{2}])\Big\} \\ &= \Big\{y \in Tx : |y - x| = \Big|x - \frac{x+1}{2}\Big| = \Big|\frac{x-1}{2}\Big|\Big\} \\ &= \Big\{y \in Tx : |y - x| = \frac{x-1}{2}\Big\} \\ &= \Big\{y = \frac{x+1}{2}\Big\}. \end{aligned}$$

Now we show that $P_T(x)$ is a quasi-nonexpansive mapping for all $x \in K$. If $x \in [0, 1]$, it is clear since $P_T(x) = \{x\}$. If $x \notin [0, 1]$, we get $H(P_T(x), P_T(p)) = H(\frac{x+1}{2}, p) = \Big|\frac{x+1}{2} - p\Big| \le |x - p|$. So $P_T(x)$ is a quasi-nonexpansive mapping. Thus T satisfies conditions of above theorem. We generate a sequence $\{x_n\}$ as defined in (1).

Choose $x_1 = \frac{3}{2} \in K = [0,2]$. Then $P_T(x_1) = \left\{\frac{x_1+1}{2}\right\} = \left\{\frac{\frac{3}{2}+1}{2}\right\} = \left\{1+\frac{1}{4}\right\}$ and $u_1 \in P_T(x_1) = \{1+\frac{1}{4}\}$. That is, $u_1 = 1+\frac{1}{4}$. Then

$$z_1 = (1-\gamma_1)x_1 + \gamma_1 u_1 = (1-\frac{1}{2}).\frac{3}{2} + \frac{1}{2}.(1+\frac{1}{4}) = 1+\frac{3}{8}$$

and

$$P_T(z_1) = \left\{\frac{z_1+1}{2}\right\} = \left\{\frac{1+\frac{3}{8}+1}{2}\right\} = \{1+\frac{3}{16}\}.$$

Choose $w_1 \in P_T(z_1) = \{1+\frac{3}{16}\}$, that is, $w_1 = 1+\frac{3}{16}$. Then

$$y_1 = (1-\beta_1)z_1 + \beta_1 w_1 = (1-\frac{1}{2}).(1+\frac{3}{8}) + \frac{1}{2}.(1+\frac{3}{16}) = 1+\frac{9}{32}$$

and

$$P_T(y_1) = \{\frac{y_1+1}{2}\} = \left\{\frac{1+\frac{9}{32}+1}{2}\right\} = \{1+\frac{9}{64}\}.$$

Choose $v_1 \in P_T(y_1) = \{1+\frac{9}{64}\}$, $v_1 = 1+\frac{9}{64}$. Then

$$x_2 = (1-\alpha_1)w_1 + \alpha_1 v_1 = (1-\frac{1}{2}).(1+\frac{3}{16}) + \frac{1}{2}.(1+\frac{9}{64}) = 1+\frac{21}{128} < 1+\frac{1}{4}$$

and

$$P_T(x_2) = \{\frac{x_2+1}{2}\} = \left\{\frac{1+\frac{21}{128}+1}{2}\right\} = \{1+\frac{21}{256}\}.$$

Next, take $u_2 \in P_T(x_2) = \{1+\frac{21}{256}\}$, that is $u_2 = 1+\frac{21}{256}$. Then

$$z_2 = (1-\gamma_2)x_2 + \gamma_2 u_2 = (1-\frac{1}{4}).(1+\frac{21}{128}) + \frac{1}{4}.(1+\frac{21}{256}) = 1+\frac{147}{1024}.$$

Choose $w_2 \in P_T(z_2) = \{1+\frac{147}{2048}\}$, that is $w_2 = 1+\frac{147}{2048}$. Then

$$y_2 = (1-\beta_2)z_2 + \beta_2 w_2 = (1-\frac{1}{4}).(1+\frac{147}{1024}) + \frac{1}{4}.(1+\frac{147}{2048}) = 1+\frac{1029}{8192}$$

and

$$P_T(y_2) = \{\frac{y_2+1}{2}\} = \left\{\frac{1+\frac{1029}{8192}+1}{2}\right\} = \{1+\frac{1029}{16384}\}.$$

Choose $v_2 \in P_T(y_2) = \{1+\frac{1029}{16384}\}$, $v_2 = 1+\frac{1029}{16384}$. Then

$$x_3 = (1-\alpha_2)w_2 + \alpha_2 v_2 = (1-\frac{1}{4}).(1+\frac{147}{2048}) + \frac{1}{4}.(1+\frac{1029}{16384}) = 1+\frac{4557}{65536} < 1+\frac{1}{6}.$$

In a similar way, $x_4 < 1+\frac{1}{8}$, $x_5 < 1+\frac{1}{10}, \ldots, x_n < 1+\frac{1}{2n}$. This shows that $\{x_n\}$ converges strongly to a point of $F_T = [0,1]$.

It is known that the condition (I) is weaker than compactness of K. So we now prove our strong convergence theorem by using condition (I) which was originally given by Senter and Dotson [12].

Definition 3.4. [Condition (I)] ([12]) A multivalued nonexpansive mapping $T : K \to CB(K)$ is said to satisfy Condition (I), if there exists a continuous non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $d(x, Tx) \geq f(d(x, F(T))$ for all $x \in K$.

Theorem 3.5. *Let E be a uniformly convex Banach space, let K be a non-empty closed and convex subset of E and let $T : K \to P(K)$ be a multivalued mapping satisfying condition (I) such that $F(T) \neq \emptyset$ and let P_T be a quasi-nonexpansive mapping. Then the sequence $\{x_n\}$ as defined in (1) converges strongly to a fixed point of T.*

Proof. We show that $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T) = F(P_T)$ in Lemma (3.1). In case $\lim_{n\to\infty} \|x_n - p\| = 0$ the proof is over. So we take $\lim_{n\to\infty} \|x_n - p\| = c > 0$. In Lemma (3.1), we also prove that $\|x_{n+1} - p\| \leq \|x_n - p\|$. Thus

$$d(x_{n+1}, F(T)) \leq d(x_n, F(T)).$$

This inequality goes to $\lim_{n\to\infty} d(x_{n+1}, F(T))$ exists.

Now, we focus to prove that $\lim_{n\to\infty} d(x_{n+1}, F(T)) = 0$. Assume that $\lim_{n\to\infty} d(x_{n+1}, F(T)) = b > 0$. For all $n \in \mathbb{N}$, choose

$$t_n = \frac{u_n - p}{\|x_n - p\|}, \quad t'_n = \frac{x_n - p}{\|x_n - p\|}$$

Since $\|u_n - p\| \leq H(P_T(x_n), P_T(p)) \leq \|x_n - p\|$, then $\|t_n| \leq 1$ and

$$\|t'_n\| = \left\| \frac{x_n - p}{\|x_n - p\|} \right\| = 1.$$

Now from condition (I), we have

$$\begin{aligned}
\|t'_n - t_n\| &= \left\| \frac{x_n - p}{\|x_n - p\|} - \frac{u_n - p}{\|x_n - p\|} \right\| \\
&= \left\| \frac{x_n - u_n}{\|x_n - p\|} \right\| \\
&= \frac{\|x_n - u_n\|}{\|x_n - p\|} \\
&\geq \frac{d(x_n, Tx_n)}{\|x_n - p\|} \\
&\geq \frac{f(d(x_n, F(T))}{\|x_n - p\|}.
\end{aligned}$$

Since f is a continuous function

$$\liminf_{n\to\infty} \|t'_n - t_n\| \geq \frac{f(b)}{c} > 0$$

for all $n \in \mathbb{N}$. In Lemma (3.1), we showed that $\lim_{n\to\infty} \|x_n - p\| = c$ and $\lim_{n\to\infty} \|z_n - p\| = c$. Next,

$$\begin{aligned}
\lim_{n\to\infty} \|(1 - \gamma_n)t'_n + \gamma_n t_n\| &= \lim_{n\to\infty} \left\| (1 - \gamma_n). \frac{x_n - p}{\|x_n - p\|} + \gamma_n \frac{u_n - p}{\|x_n - p\|} \right\| \\
&= \lim_{n\to\infty} \left\| \frac{(1 - \gamma_n)x_n + \gamma_n u_n - p}{\|x_n - p\|} \right\| \\
&= \frac{\lim_{n\to\infty} \|z_n - p\|}{\lim_{n\to\infty} \|x_n - p\|} = \frac{c}{c} = 1.
\end{aligned}$$

So, from Lemma (2.7), $\|t_n\| \leq 1$ and $\|t'(n)\| = 1$, we get $\lim_{n\to\infty} \|t'_n - t_n\| = 0$. But we just established $\liminf_{n\to\infty} \|t'_n - t_n\| > 0$, so this is a contradiction. Now we have $\lim_{n\to\infty} d(x_{n+1}, F(T)) = 0$ and this follows that

$$\lim_{n\to\infty} \|x_n - p\| = 0.$$

Thus the sequence $\{x_n\}$ converges to a fixed point p of T. □

Now we give an example to support Theorem 3.5.

Example 3.6. Choose $K = [a, b]$; for $a, b \in \mathbb{R}^+$. It is clear that K is a non-empty closed and convex subset of $\mathbb{R}$. Define $T : K \to P(K)$ as

$$Tx = \left[a, a + \frac{x}{b}\right]$$

for $0 < a < b - 1$. Then $F_T = [a, \frac{ab}{b-1}]$ and $F_T \neq \emptyset$. Let $\alpha_n = \beta_n = \gamma_n = \frac{1}{2}$. Define a continuous and non-decreasing function $f : [0, \infty) \to [0, \infty)$ by $f(r) = \frac{r}{2b/(b-1)}$. First we show that $d(x, Tx) \geq f(d(x, F_T))$ for all $x \in K$. Obviously if $x \in F_T = \left[a, \frac{ab}{b-1}\right]$, then $d(x, Tx) = 0 = f(d(x, F_T))$. When $x \in \left[\frac{ab}{b-1}, b\right]$, we have

$$d(x, Tx) = d(x, [a, a + \frac{x}{b}]) = |x - (a + \frac{x}{b})| = \frac{|(b-1)x - ab|}{b}$$

and so

$$f(d(x, F_T)) = f\Big(d(x, \Big[a, \frac{ab}{b-1}\Big]\Big) = f\Big(\Big|x - \frac{ab}{b-1}\Big|\Big) = \frac{|(b-1)x - ab|}{2b}.$$

Thus $d(x, Tx) \geq f(d(x, F_T))$ for all $x \in K$. As we show in the previous example, $P_T(x) = \{x\}$ when $x \in \left[a, \frac{ab}{b-1}\right]$. If $x \in \left[\frac{ab}{b-1}, b\right]$, then

$$\begin{aligned} P_T(x) =& \Big\{y \in Tx : |y - x| = d(x, \Big[a, a + \frac{x}{b}\Big])\Big\} \\ & \Big\{y \in Tx : |y - x| = |x - (a + \frac{x}{b})|\Big\} \\ & \Big\{y \in Tx : |y - x| = \frac{|(b-1)x - ab|}{b}\Big\} \\ & \Big\{y = a + \frac{x}{b}\Big\}. \end{aligned}$$

Now, we show that P_T is a quasi-nonexpansive mapping for all $x \in K$. If $x \in F_T$, then $P_T(x) = \{x\}$. So,

$$H(P_T(x), p) = H(\{x\}, p) = |x - p|.$$

If we take $x \in \left[\frac{ab}{b-1}, b\right]$, then

$$H(P_T(x), P_T(p)) = H(\{a + \frac{x}{b}\}, \{p\}) = |a + \frac{x}{b} - p| \leq |x - p|.$$

Finally, we generate a sequence $\{x_n\}$ as defined in (1) and show that it converges strongly to a fixed point of T.
Choose $a = 1$ and $b = 3$ thus $Tx = \left[1, 1 + \frac{x}{3}\right]$ and $F_T = \left[1, \frac{3}{2}\right] \neq \emptyset$.
Take $x_1 = \frac{9}{4} \in K = [1, 3]$, $P_T(x_1) = \{1 + \frac{x_1}{3}\} = \{\frac{7}{4}\}$. That is, $u_1 = \frac{7}{4}$. Then

$$z_1 = (1 - \gamma_1)x_1 + \gamma_1 u_1 = \frac{1}{2}.\frac{9}{4} + \frac{1}{2}.\frac{7}{4} = 2$$

and

$$P_T(z_1) = \{1 + \frac{z_1}{3}\} = \{1 + \frac{2}{3}\} = \{\frac{5}{3}\}.$$

Choose $w_1 \in P_T(z_1) = \{\frac{5}{3}\}$. That is, $w_1 = \frac{5}{3}$. Then

$$y_1 = (1 - \beta_1)z_1 + \beta_1 w_1 = \frac{1}{2}.2 + \frac{1}{2}.\frac{5}{3} = \frac{11}{6}$$

and

$$P_T(y_1) = \{1 + \frac{y_1}{3}\} = \{1 + \frac{11}{18}\} = \{\frac{29}{18}\}.$$

Choose $v_1 \in P_T(y_1) = \{\frac{29}{18}\}$, $v_1 = \frac{29}{18}$. Then

$$x_2 = (1 - \alpha_1)w_1 + \alpha_1 v_1 = \frac{1}{2}.\frac{5}{3} + \frac{1}{2}.\frac{29}{18} = \frac{54}{36} + \frac{5}{36} < \frac{3}{2} + \frac{1}{4}.$$

Continuing in this way, we get $x_n < \frac{3}{2} + \frac{1}{2n}$. This shows that $\{x_n\}$ converges strongly to a fixed point of T in $F_T = [1, \frac{3}{2}]$.

Now we also show that our new iteration process is faster than the Mann [7], the Ishikawa [3], the Noor [9], the Agarwal [2] and the Abbas and Nazir [1] iteration processes, by giving a numerical example.

Example 3.7. Let $K = [0, 3]$ and let $T : K \to P(K)$ as

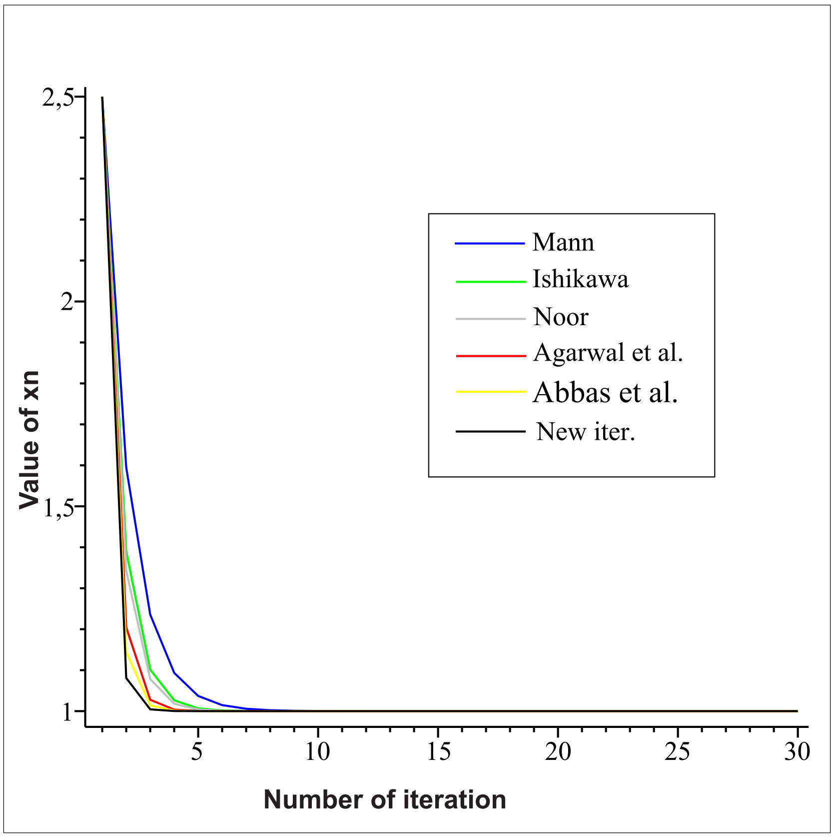
$$Tx = [0, \frac{3x + 8}{11}]$$

for all $x \in K$. It is clear that T is a multivalued mapping and P_T is a quasi-nonexpansive multivalued mapping with $F_T = [0, 1] \neq \emptyset$. Choose $\alpha_n = \beta_n = \gamma_n = 0.83$, with the initial value $x_1 = 2.5$. We give the first 30 terms of multivalued version of iteration processes mentioned above. Then all sequences converges to $p = 1 \in F_T = [0, 1]$.

Step	Mann	Ishikawa	Noor	Agarwal	Abbas	New iter.
1	2.500000000	2.500000000	2.500000000	2.500000000	2.500000000	2.500000000
2	1.594545454	1.389583471	1.343187532	1.204128925	1.144047630	1.080909283
3	1.235656198	1.101183521	1.078518455	1.027779079	1.013833147	1.004364208
4	1.093405548	1.026279618	1.017964370	1.003780342	1.001328422	1.000235404
5	1.037022562	1.006825403	1.004110099	1.000514452	1.000127570	1.000012697
6	1.014674397	1.001772710	1.000940356	1.000070010	1.000012251	1.000000685
7	1.005816397	1.000460413	1.000215146	1.000009527	1.000001176	1.000000038
8	1.002305408	1.000119579	1.000049224	1.000001297	1.000000113	1.000000002
9	1.000913780	1.000031057	1.000011262	1.000000176	1.000000011	1.000000000
10	1.000362189	1.000008066	1.000002577	1.000000024	1.000000001	1.000000000
11	1.000143559	1.000002095	1.000000590	1.000000004	1.000000000	1.000000000
12	1.000056901	1.000000544	1.000000135	1.000000001	1.000000000	1.000000000
13	1.000022553	1.000000142	1.000000031	1.000000000	1.000000000	1.000000000
14	1.000008939	1.000000036	1.000000007	1.000000000	1.000000000	1.000000000
15	1.000003543	1.000000009	1.000000002	1.000000000	1.000000000	1.000000000
16	1.000001404	1.000000002	1.000000000	1.000000000	1.000000000	1.000000000
17	1.000000557	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
18	1.000000221	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
19	1.000000087	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
20	1.000000035	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
21	1.000000014	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
22	1.000000006	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
23	1.000000003	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
24	1.000000001	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
25	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
26	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
27	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
28	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
29	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
30	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000

Note that convergence of all iteration processes given in above table are proved in [7], [3], [9], [2] and [1]. Also the mapping $P_T(x) = \{y \in Tx : \|x - y\| = \|x - Tx\|\}$ defined by Shahzad and Zegeye [13] guaranteed all iteration processes converges same fixed point of T.

We now demonstrate the convergence behaviours of the Mann [7], the Ishikawa [3], the Noor [9], the Agarwal et al. [2], the Abbas et al. [1] and the new iteration for the mapping T which is given in Example 3.7.



Finally, we give weak convergence theorem of the sequence as defined in (1) by using Opial's condition.

Theorem 3.8. *Let E be a uniformly convex Banach space satisfying Opial's condition and K be a non-empty closed convex subset of E. Let $T : K \to P(K)$ be a multivalued mapping such that $F(T) \neq \emptyset$ and P_T is a quasi-nonexpansive mapping. Let $\{x_n\}$ be the sequence as defined in (1). Let $I - P_T$ be demiclosed with respect to zero, then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Let $p \in F(T) = F(P_T)$. We established that $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$ in Lemma (3.1). Choose z_1 and z_2 to be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. From (10), there exists $\lim_{n\to\infty} \|x_n - u_n\| = 0$ satisfying $u_n \in Tx_n$. Since $I - P_T$ is demiclosed with respect to zero, thus we get $z_1 \in F(P_T) = F(T)$. In a similar way, we can see that $z_2 \in F(T)$.

Now, we prove $z_1 = z_2$. Suppose on contrary that $z_1 \neq z_2$. Then using Opial's condition, we get

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - z_1\| &= \lim_{n_i\to\infty} \|x_{n_i} - z_1\| \\
&< \lim_{n_i\to\infty} \|x_{n_i} - z_2\| \\
&= \lim_{n\to\infty} \|x_n - z_2\| \\
&= \lim_{n_j\to\infty} \|x_{n_j} - z_2\| \\
&< \lim_{n_j\to\infty} \|x_{n_j} - z_1\| \\
&= \lim_{n\to\infty} \|x_n - z_1\|.
\end{aligned}$$

So this is a contradiction. Thus $z_1 = z_2$ and $\{x_n\}$ converges to a point in $F(T)$. $\square$

References

[1] M. Abbas, T. Nazir, A new faster iteration process applied to constrained minimization and feasibility problems, Mat. Vesn. 66 (2014) 223–234.
[2] R.P. Agarwal, D. O'Regan, D.R. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, J. Nonlinear Convex Anal. 8 (2007) 61–79.
[3] S. Ishikawa, Fixed points by a new iteration method, Proc. Amer. Math. Soc. 44 (1974) 147–150.
[4] S.H. Khan, M. Abbas, S. Ali, Fixed points of multivalued quasi-nonexpansive mappings using a faster iterative process, Acta Math. Sin. Engl. Ser. 30 (2014) 1231–1241.
[5] S.H. Khan, I. Yildirim, Fixed points of multivalued nonexpansive mappings in Banach spaces, Fixed Point Theory Appl. (2012) 2012:73.
[6] T.C. Lim, A fixed point theorem for multivalued nonexpansive mappings in a uniformly convex Banach spaces, Bull. Amer. Math. Soc. 80 (1974) 1123–1126.
[7] W.R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. 4 (1953) 506–510.
[8] S.B. Nadler, Multivalued contraction mappings, Pacific J. Math. 30 (1969) 475–488.
[9] M.A. Noor, New approximation schemes for general variational inequalities, J. Math. Anal. Appl. 251 (2000) 217–229.
[10] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967) 591–597
[11] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bull. Aust. Math. Soc. 43 (1991) 153–159.
[12] H.F. Senter, W.G. Dotson, Approximating fixed points of nonexpansive mappings, Proc. Amer. Math. Soc. 44 (1974) 375–380
[13] N. Shahzad, H. Zegeye, On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces, Nonlinear Anal. 71 (2009) 838–844.
[14] Y. Song, Y.J. Cho, Some notes on Ishikawa iteration for multivalued mappings, Bull. Korean. Math. Soc. 48 (2011) 575–584.
[15] B.S. Thakur, D. Thakur, M. Postolache, A new iteration scheme for approximating fixed points of nonexpansive mappings, Filomat 30 (2016) 2711–2720.