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Analysis of Keller-Segel Model with Atangana-Baleanu Fractional Derivative

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Abstract. The new definition of the fractional derivative was defined by Atangana and Baleanu in 2016. They used the generalized Mittag-Leffler function with the non-singular and non-local kernel. Further, their version provides all properties of fractional derivatives. Our aim is to analyse the Keller-Segel model with Caputo and Atangana-Baleanu fractional derivative in Caputo sense. Using fixed point theory, we first show the existence of coupled solutions. We then examine the uniqueness of these solutions. Finally, we compare our results numerically by modifying our model according to both definitions, and we demonstrate these results on the graphs in detail. All computations were done using *Mathematica*.

1. Introduction

The fractional derivative has been known for a long time. During the last four decades, it has taken an important place in practice as much as it has in the theoretical mathematics. There have been significant developments in fractional derivatives throughout this fourty years. After this important development, fractional derivative and integral have been applied in many fields, such as engineering, chemistry, physics etc. [8, 9, 28, 32]. Caputo fractional derivative [18] was used in many applications [22, 24, 30]. However, this definition has a considerable weakness because of the singularity of kernels. We see that this issue has been addressed by the change of the kernel in the work of Atangana and Baleanu [6]. It is stated that some physics problems including initial value give better results than the Caputo's definition and have great advantages. One of the most important advantages of the new definition is that their derivation has a non-singular and non-local kernel [6]. After the new definition, a lot of important work was done. Some of these works are shown below. Atangana and Koca contributed to the existence and uniqueness of the system solutions of the fractional system. [7]. Chua's circuit model was analyzed with Atangana-Baleanu fractional derivative by Alkahtani [1]. A new analysis of the zika model was made for non-singular and non-local fractional operators [3]. Baskonus and Bulut also contributed to some articles on fractional derivatives [11–14, 23]. The important research on the Keller-Segel model in recent years was made by Atangana and Badr. They

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used the Caputo-Fabrizio derivative to analyze the Keller-Segel model [5]. Besides, there are some research about fractional Keller-Segel model [4, 15, 17, 21].

The Keller-Segel model is one of the most widely used models of chemotaxis. It was developed to focus on the cellular slime mould *Dictyostelium discoideum*[35]. We denote $a(x,t)$ as the gradient in an attractant, which prompts a movement. The flux can be written [29],

$$J = \rho\chi(a)\nabla a,$$

where J is chemotactic flux and $\chi(a)$ is the concentration of the attractant [31]. We can write the equation generally for $\rho(x,t)$:

$$\rho_t = D\rho_{xx} - (\rho\chi(a)a_x)_x + f(n),$$

The above equation is called the *reaction-diffusion-chemotaxis* equation. It is known that the function $a(x,t)$ is a chemical term, and in general we may write $a(x,t)$:

$$a_t = D_a a_{xx} + g(a,\rho),$$

where $g(a,\rho)$ is the kinetics term and D_a is the diffusion coefficient of a. This term depend upon ρ and a [16].

$$\rho_t = D\rho_{xx} - \chi(\rho a_x)_x$$
$$a_t = D_a a_{xx} + g(a,\rho).$$

2. Preliminaries

We give in this section some fundamental definitions [6, 18, 27, 33] on fractional derivative.

Definition 2.1. The Caputo derivative of fractional derivative is defined as [18]:

$${}_a^C D_t^\nu f(t)) = \frac{1}{\Gamma(n-\nu)} \int_a^t \frac{f^{(n)}(r)}{(t-r)^{\nu+1-n}} dr, \quad n-1 < \nu < n \in \mathbb{N}. \tag{1}$$

Definition 2.2. The Riemann-Liouville fractional integral is defined as [27]:

$$J^\nu f(t) = \frac{1}{\Gamma(\nu)} \int_a^t f(r)(t-r)^{\nu-1} dr. \tag{2}$$

Definition 2.3. The Riemann-Liouville fractional derivative is defined as [27]:

$${}_a^R D_t^\nu f(t) = \frac{1}{\Gamma(n-\nu)} \frac{d^n}{dt^n} \int_a^t \frac{f(r)}{(t-r)^{\nu+1-n}} dr, \quad n-1 < \nu < n \in \mathbb{N}. \tag{3}$$

Definition 2.4. The Sobolev space of order 1 in (a,b) is defined as [33]:

$$H^1(a,b) = \{u \in L^2(a,b) : u' \in L^2(a,b)\}.$$

Definition 2.5. Let a function $u \in H^1(a,b)$ and $\nu \in (0,1)$. The AB fractional derivative in Caputo sense of order ν of u with a based point a is defined as [6]:

$${}_a^{ABC} D_t^\nu u(t)) = \frac{B(\nu)}{1-\nu} \int_a^t u'(s) E_\nu\Big[-\frac{\nu}{1-\nu}(t-s)^\nu\Big] ds, \tag{4}$$

where $B(\nu)$ has the same properties as in Caputo and Fabrizio case, and is defined as

$$B(\nu) = 1 - \nu + \frac{\nu}{\Gamma(\nu)},$$

$E_{\nu,\beta}(\lambda^\nu)$ is the Mittag-Leffler function, defined in terms of a series as the following entire function

$$E_{\nu,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\lambda^\nu)^k}{\Gamma(\nu k + \beta)}, \qquad \nu > 0, \quad \lambda < \infty \quad and \quad \beta > 0, \quad \lambda = -\nu(1-\nu)^{-1}. \tag{5}$$

Definition 2.6. Let a function $u \in H^1(a,b)$ and $\nu \in (0,1)$. The AB fractional derivative in Riemann-Liouville sense of order ν of u with a based point a is defined as [6]:

$$ {}_{a}^{ABR}D_t^{\nu}u(t)) = \frac{B(\nu)}{1-\nu}\frac{d}{dt}\int_a^t u(s)E_{\nu}\Big[-\frac{\nu}{1-\nu}(t-s)^{\nu}\Big]ds, \tag{6} $$

when the function u is constant, we get zero.

Definition 2.7. The Atangana-Baleanu fractional integral of order ν with base point a is defined as [6]:

$$ {}^{AB}I_t^{\nu}u(t)) = \frac{1-\nu}{B(\nu)}u(t) + \frac{\nu}{B(\nu)\Gamma(\nu)}\int_a^t u(s)(t-s)^{\nu-1}ds, \tag{7} $$

when the function u is constant, we get zero.

3. Keller-Segel Model with Caputo Derivative

The best approach in terms of the technique to be employed to research on this topic is to visit the Keller and Segel model depicted in the ground-breaking paper (1970). It predates the formal structure Keller-Segel though it is probably the first edition. They [25, 26] presented the illustration of the aggregation behaviour of cellular slim mould which they said is caused by instability. L.R. Litter also came up with four species important to the approach in the Keller and Segel [32].

One dimensional Keller-Segel model is given by

$$ \begin{cases} \rho_t = D\rho_{xx} - \chi\Big(\rho a_x\Big)_x \\ a_t = D_a a_{xx} + h\rho - ka. \end{cases} \tag{8} $$

The parameters d, D_a, χ are constants. D and D_a are the diffusion coefficient and a,respectively, h and k are positive constants. The first term in the first equation in (8) involves a Laplacian, representing the random spatial motion of the cells. The second term models the chemotactic motion of the cells. In the second equation in (8) the first term represent diffusion of the chemoattractant. The second term models the production of the chemoattractant by the cells, and the third term represents linear decay. The initial conditions are $\rho(x,0) = \rho_0(x)$ and $a(x,0) = a_0(x)$ for the system. The system (8) with Caputo derivative is given as below

$$ \begin{cases} {}_0^C D_t^{\nu}\rho = D\rho_{xx} - \chi\Big(\rho a_x\Big)_x \\ {}_0^C D_t^{\nu}a = D_a a_{xx} + h\rho - ka. \end{cases} \tag{9} $$

3.1. Existence and Uniqueness Solutions

We will give in this chapter the existence and uniqueness of the solutions. We also will present the uniqueness of the positive solutions. Let us present every continuous functions $G = C[a,b]$ in the Banach space defined in the closed set $[a,b]$ and consider $Z = \{\rho, a \in G, \rho(x,t) \geq 0 \quad and \quad a(x,t) \geq 0, a \leq t \leq b\}$

Definition 3.1. ([36]) Let X be a Banach space with a cone H. H initiates a restricted order $\leq$ in E in the succeeding approach.

$$ y \geq x \Rightarrow y - x \in H $$

Now applying the fractional integral in equation (10), we obtain the following,

$$ \begin{cases} \rho(x,t) - \rho(x,0) = \frac{1}{\Gamma(\nu)}\int_0^t (t-r)^{\nu-1}\Big[D\rho(x,t)_{xx} - \chi\Big(\rho(x,t)a(x,t)_x\Big)_x\Big]dr \\ a(x,t) - a(x,0) = \frac{1}{\Gamma(\nu)}\int_0^t (t-r)^{\nu-1}\Big[D_a a(x,t)_{xx} + h\rho(x,t) - ka(x,t)\Big]dr. \end{cases} \tag{10} $$

Now we can use system (11) to show the existence of equation (8). Necessary lemma for the existence of the solutions are given as Lemma 3.2. We now need to define an operator which $T : K \to K$.

$$\begin{cases} T\rho(x,t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-r)^{\nu-1} s(x,r,\rho(x,r))dr \\ Ta(x,t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-r)^{\nu-1} s(x,r,a(x,r))dr. \end{cases} \tag{11}$$

To be dealt with more easily, let us consider below

$$\begin{cases} s(x,r,\rho) = D\rho_{xx} - \chi\rho_x a_{xx} \\ s(x,r,a) = D_a a_{xx} + h\rho - ka. \end{cases} \tag{12}$$

Lemma 3.2. *The mapping $T : K \to K$ is completely continuous.*

Proof. Let $M \subset K$ be bounded. There exists a constants $l, m > 0$ such that $\|p\| < l$, $\|a\| < m$. Let,

$$L_1 = max_{\substack{0\le t\le 1\\ 0\le\rho\le l}} s(x,t,\rho(x,t)) \quad and \quad L_2 = max_{\substack{0\le t\le 1\\ 0\le a\le m}} s(x,t,a(x,t))$$

$\forall \rho, a \in M$, we have,

$$\begin{aligned} \|T\rho(x,t)\| &\le \frac{1}{\Gamma(\nu)}\int_0^t (t-r)^{\nu-1}\|s(x,r,\rho(x,r))\|dr \\ &\le \frac{L_1}{\Gamma(\nu)}\int_0^t (t-r)^{\nu-1}dr \\ &= \frac{L_1}{\Gamma(\nu+1)}t^\nu \end{aligned} \tag{13}$$

So that, we can write as below,

$$\|T\rho\| \le \frac{L_1}{\Gamma(\nu+1)}$$

Similarly,

$$\begin{aligned} \|Ta(x,t)\| &\le \frac{1}{\Gamma(\nu)}\int_0^t (t-r)^{\nu-1}\|s(x,r,a(x,r))\|dr \\ &\le \frac{L_2}{\Gamma(\nu)}\int_0^t (t-r)^{\nu-1}dr \\ &= \frac{L_2}{\Gamma(\nu+1)}t^\nu \end{aligned} \tag{14}$$

So that, we can write as below,

$$\|Ta\| \le \frac{L_2}{\Gamma(\nu+1)}$$

Hence $T(M)$ is bounded.

Now in the following part, we will consider $t_1 < t_2$ and $\rho(x,t), a(x,t) \in M$ and ; then for a given $\epsilon > 0$ if $|t_2 - t_1| < \delta$. We have,

$$
\begin{aligned}
\|T\rho(x,t_2) - T\rho(x,t_1)\| =& \|\frac{1}{\Gamma(\nu)}\int_0^{t_2}(t_2-r)^{\nu-1}\|s(x,r,\rho(x,r))\|dr \\
&-\frac{1}{\Gamma(\nu)}\int_0^{t_1}(t_1-r)^{\nu-1}\|s(x,r,\rho(x,r))dr\| \\
=& \|\frac{1}{\Gamma(\nu)}\int_0^{t_2}(t_2-r)^{\nu-1}\|s(x,r,\rho(x,r))\|dr \\
&-\frac{1}{\Gamma(\nu)}\int_0^{t_2}(t_1-r)^{\nu-1}\|s(x,r,\rho(x,r))\|dr \\
&-\frac{1}{\Gamma(\nu)}\int_{t_1}^{t_2}(t_1-r)^{\nu-1}\|s(x,r,\rho(x,r))\|dr \\
\leq& \frac{1}{\Gamma(\nu)}\int_0^{t_2}\|(t_2-r)^{\nu-1}-(t_1-r)^{\nu-1}\|\,\|s(x,r,\rho(x,r))\|dr \\
&+\frac{1}{\Gamma(\nu)}\int_{t_1}^{t_2}\|(t_1-r)^{\nu-1}\|\,\|s(x,r,\rho(x,r))\|dr \\
\leq& \frac{L_1}{\Gamma(\nu)}\int_0^{t_2}\Big((t_2-r)^{\nu-1}-(t_1-r)^{\nu-1}\Big)dr+\frac{L_1}{\Gamma(\nu)}\int_{t_1}^{t_2}(t_1-r)^{\nu-1}dr \\
=& \frac{L_1}{\Gamma(\nu)}\Big(\int_0^{t_2}(t_2-r)^{\nu-1}dr-\int_0^{t_2}(t_1-r)^{\nu-1}dr+\int_{t_1}^{t_2}(t_1-r)^{\nu-1}dr\Big) \\
=& \frac{L_1}{\Gamma(1+\nu)}\Big(t_2^{\nu}+(t_1-t_2)^{\nu}-t_1^{\nu}+(t_1-t_2)^{\nu}\Big) \\
\leq& \frac{2L_1}{\Gamma(1+\nu)}(t_1-t_2)^{\nu}+\frac{L_1}{\Gamma(1+\nu)}(t_1-t_1)^{\nu} \\
=& \frac{2L_1}{\Gamma(1+\nu)}(t_1-t_2)^{\nu} \\
<& \frac{2L_1}{\Gamma(1+\nu)}\delta^{\nu} \\
=& \epsilon
\end{aligned} \tag{15}
$$

It is clear seen that, when the same steps are applied to the $a(x,t)$ function, we get same situation. Finally,

$$|T\rho(x,t_2) - T\rho(x,t_1)| \leq \epsilon \qquad \text{and} \qquad |Ta(x,t_2) - Ta(x,t_1)| \leq \epsilon$$

are satisfied. Where $\delta = \Big(\epsilon\Gamma(1+\nu/2L)\Big)^{1/\nu}$.Therefore $T(M)$ is equicontinuous. So that $\overline{T(M)}$ is compact via The Arzela-Ascoli theorem. □

Theorem 3.3. *Let $S : [\rho_1, \rho_2] \times [0,\infty) \to [0,\infty)$, then $S(x,t,.)$ is non-decreasing for each t in $[\rho_1,\rho_2]$. there exists a positive constants v_1 and v_2 such that $B(n)v_1 \leq S(x,t,v_1)$, $B(n)v_2 \geq S(x,t,v_2)$, $0 \leq v_1(x,t) \leq v_2(x,t)$, $\rho_1 \leq t \leq \rho_2$. This means that the new equation has a positive solution.*

Proof. We only need to consider the fixed point for operator of T. With framework of Lemma 3.2, the considered operator $T : H \to H$ is completely continuous. Let us take two arbitrary ρ_1 and ρ_2,

$$
\begin{aligned}
T\rho_1(x,t) =& \frac{1}{\Gamma(\nu)}\int_0^{t}(t-r)^{\nu-1}s(x,r,\rho_1(x,r))dr \\
\leq& \frac{1}{\Gamma(\nu)}\int_0^{t}(t-r)^{\nu-1}s(x,r,\rho_2(x,r))dr \\
=& Tp_2(x,t)
\end{aligned} \tag{16}
$$

Hence T is a non-decreasing operator. So that the operator $T : \langle v_1, v_2\rangle \to \langle v_1, v_2\rangle$ is compact and continuous via Lemma 1. In that case, H is a normal cone of T. □

3.2. Uniqueness of Solution

The aim of this chapter is to prove the uniqueness of solutions to the system (10). So the uniqueness of the solution is presented as below,

$$\begin{aligned} \|T\rho_1(x,t) - T\rho_2(x,t)\| &= \|\frac{1}{\Gamma(\nu)}\int_0^t (t-r)^{\nu-1}\Big(s(x,r,\rho_1(x,r)) - s(x,r,\rho_2(x,r))\Big)dr\| \\ &\leq \frac{1}{\Gamma(\nu)}C_1\int_0^t (t-r)^{\nu-1}\|\rho_1(x,r) - \rho_2(x,r)\|dr \end{aligned} \tag{17}$$

So that,

$$\|T\rho_1(x,t) - T\rho_2(x,t)\| \leq \left\{\frac{C_1 t^\nu}{\Gamma(\nu+1)}\right\}\|\rho_1(x,r) - \rho_2(x,r)\|$$

Similarly,

$$\begin{aligned} \|Ta_1(x,t) - Ta_2(x,t)\| &= \|\frac{1}{\Gamma(\nu)}\int_0^t (t-r)^{\nu-1}\Big(s(x,r,a_1(x,r)) - s(x,r,a_2(x,r))\Big)dr\| \\ &\leq \frac{1}{\Gamma(\nu)}C_2\int_0^t (t-r)^{\nu-1}\|a_1(x,r) - a_2(x,r)\|dr \end{aligned} \tag{18}$$

So that,

$$\|Ta_1(x,t) - Ta_2(x,t)\| \leq \left\{\frac{C_2 t^\nu}{\Gamma(\nu+1)}\right\}\|a_1(x,r) - a_2(x,r)\|$$

Therefore, if the following conditions hold,

$$\left\{\frac{C_1 t^\nu}{\Gamma(\nu+1)}\right\} < 1 \text{ and } \left\{\frac{C_2 t^\nu}{\Gamma(\nu+1)}\right\} < 1$$

Then mapping T is a contraction, which implies fixed point, and thus the model has a unique positive solution.

4. Keller-Segel Model with AB Derivative in Caputo Sense

We present in this chapter the existence and uniqueness of solutions of the Keller-Segel model using the Atangana-Baleanu derivative. Let $\Omega = (a, b)$ be an open and bounded subset of R^n. For a given $\nu \in (0, 1)$ and functions $\rho(x,t), a(x,t) \in H^1(\Omega)\times[0,T]$. Here $\rho(x,t)$ represent the concentration of the chemical substance and the function $a(x,t)$ represent concentration of amoebae. We apply the system (8) to the Atangana-Baleanu fractional derivative,

$$\begin{cases} {}_0^{ABC}D_t^\nu \rho = \sigma_1(x,t,\rho) \\ {}_0^{ABC}D_t^\nu a = \sigma_2(x,t,a). \end{cases} \tag{19}$$

where

$$\begin{cases} \sigma_1(x,t,\rho) = D\rho_{xx} - \chi\rho_x a_{xx} \\ \sigma_2(x,t,a) = D_a a_{xx} + h\rho - ka. \end{cases} \tag{20}$$

Using the Atangana-Baleanu integral to (21) it yields

$$\begin{cases} \rho(x,t) = \rho(x,0) + \frac{1-\nu}{B(\nu)}\sigma_1(x,t,\rho(x,t)) + \frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^t \sigma_1(x,r,\rho(x,r))(t-r)^{\nu-1}dr \\ a(x,t) = a(x,0) + \frac{1-\nu}{B(\nu)}\sigma_2(x,t,a(x,t)) + \frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^t \sigma_2(x,r,a(x,r))(t-r)^{\nu-1}dr. \end{cases} \tag{21}$$

for all $t \in [0,T]$.

Theorem 4.1. *If the inequality (23) hold, σ_1 and σ_2 satisfy Lipshitz condition and contraction.*

$$0 < D\gamma_1^2 + \chi\gamma_2\left\|\frac{\partial^2 a(x,y)}{\partial x^2}\right\| \leq 1 \tag{22}$$

Proof. We would like to start with the kernel σ_1. Let κ_1 and κ_2 are two functions, the following equation is written:

$$\|\sigma_1(x,t,\kappa_1) - \sigma_1(x,t,\kappa_2)\| = \left\|D\Big(\kappa_1(x,t)_{xx} - \kappa_2(x,t)_{xx}\Big) - \chi(\kappa_1(x,t)_x - \kappa_2(x,t)_x)\frac{\partial^2 a(x,t)}{\partial x^2}\right\|$$

When we convert the above equation via triangular inequality, then we get

$$\|\sigma_1(x,t,\kappa_1) - \sigma_1(x,t,\kappa_2)\| \leq D\left\|\Big(\kappa_1(x,t)_{xx} - \kappa_2(x,t)_{xx}\Big)\right\| + \chi\left\| - (\kappa_1(x,t)_x - \kappa_2(x,t)_x))\frac{\partial^2 a(x,t)}{\partial x^2}\right\|$$

Using the operator derivative, we can find two constants such as γ_1 and γ_2 :

$$\begin{cases} D\left\|\Big(\kappa_1(x,t)_{xx} - \kappa_2(x,t)_{xx}\Big)\right\| \leq D\gamma_1^2\|\kappa_1(x,t) - \kappa_2(x,t)\| \\ \chi\left\| - (\kappa_1(x,t)_x - \kappa_2(x,t)_x)\frac{\partial^2 a(x,t)}{\partial x^2}\right\| \leq \chi\gamma_2\left\|\frac{\partial^2 a(x,t)}{\partial x^2}\right\|\|(\kappa_1(x,t) - \kappa_2(x,t)\| \end{cases} \tag{23}$$

When we substitute equation (24) in equation (21), we get:

$$\|\sigma_1(x,t,\kappa_1) - \sigma_1(x,t,\kappa_2)\| \leq K\|(\kappa_1(x,t) - \kappa_2(x,t)\|. \tag{24}$$

where

$$K = \left(D\gamma_1^2 + \chi\gamma_2\left\|\frac{\partial^2 a(x,t)}{\partial x^2}\right\|\right)$$

Therefore σ_1 satisfies the Lipschitz condition. Then we can say that it is a contraction.
In the another case, the following inequality can be written because our kernel is linear,

$$\sigma_2(x,t,v_1) - \sigma_2(x,t,v_2) \leq (c\vartheta_1^2 + d)\|v_1(x,t) - v_2(x,t)\|$$

Hence, the proof is complete. We can now show that the uniqueness of the solution. □

4.1. *Uniqueness of Solution*

The uniqueness solution for system (20) is presented as below. Let $\rho_1, \rho_2 \in H^1$ be two solutions of (20). Let $\rho = \rho_1 - \rho_2$. the following equation can be written,

$$\rho = \frac{1-\nu}{B(\nu)}\Big(\sigma_1(x,t,\rho_1(x,t)) - \sigma_1(x,t,\rho_2(x,t))\Big) + \frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^t \Big(\sigma_1(x,r,\rho_1(x,r)) - \sigma_1(x,r,\rho_2(x,r))\Big)dr,$$

If the norms of both sides are taken, by the Gronwall inequality [20],

$$\begin{aligned} \|\rho\| &\leq \frac{1-\nu}{B(\nu)}\|\sigma_1(x,t,\rho_1(x,t)) - \sigma_1(x,t,\rho_2(x,t))\| + \frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^t \|\sigma_1(x,r,\rho_1(x,r)) - \sigma_1(x,r,\rho_2(x,r))\|dr \\ &\leq K_1\int_0^t \|\sigma_1(x,t,\rho_1(x,t))\|_{H^1}dr. \end{aligned}$$

Similarly, let $a_1, a_2 \in H^1$ be two solutions of (20). Let $a = a_1 - a_2$. the following equation can be written,

$$\begin{aligned}\|a\| &\leq \frac{1-\nu}{B(\nu)}\|\sigma_2(x,t,a_1(x,t)) - \sigma_2(x,t,a_2(x,t))\| + \frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^t \|\sigma_2(x,r,a_1(x,r)) - \sigma_2(x,r,a_2(x,r))\|dr \\ &\leq K_2\int_0^t \|\sigma_2(x,t,a_1(x,t))\|_{H^1}dr.\end{aligned}$$

Finally, the system (20) has a unique solution for the equations ρ and a.

5. Conclusion

In this work, we compare the Caputo and Atanagana-Baleanu fractional derivative in Caputo sense for functions ρ and a. The results show that the answers are similar for both derivatives in Table 1.

Approximate result				
t	x	ρ_{exact}	ρ_{Caputo}	$\rho_{Atangana-Baleanu}$
0.5	0.1	3.3516	3.35167	3.35167
	0.3	2.5816	2.58167	2.58167
	0.5	1.89161	1.89167	1.89167
	0.7	1.28162	1.28167	1.28167
	0.9	0.75166	0.751667	0.751667
t	x	a_{exact}	a_{Caputo}	$a_{Atangana-Baleanu}$
0.5	0.1	2.67694	2.676944	2.676944
	0.3	2.6549	2.65494	2.65494
	0.5	2.7459	2.74594	2.74594
	0.7	2.9524	2.95244	2.95244
	0.9	3.2744	3.27444	3.27444

Table 1: Numerical result of concentration of amoebae and chemical when $\nu = 1$ for system (8)

It can be clearly seen that, the results are fairly close for both derivatives but Atangana-Baleanu derivative is quite fast compared to Caputo in derivation calculations in Table 2 .

	Caputo	Atangana-Baleanu
$\rho_0(x,t)$	0	0
$\rho_5(x,t)$	2.13375	0.17875
$\rho_{10}(x,t)$	11.52625	2.84125
a_0(x,t)	0	0
$a_5(x,t)$	0.94500	0.058625
$a_{10}(x,t)$	3.12250	0.92375

Table 2: CPU speed in calculation

In figure 1 and 2, we compared the approximate solutions for the ρ and a function with Caputo and Atangana-Baleanu derivative, respectively. As can be seen from the graphs 1 and 2, we have very similar results. It shows that both definitions works correctly.

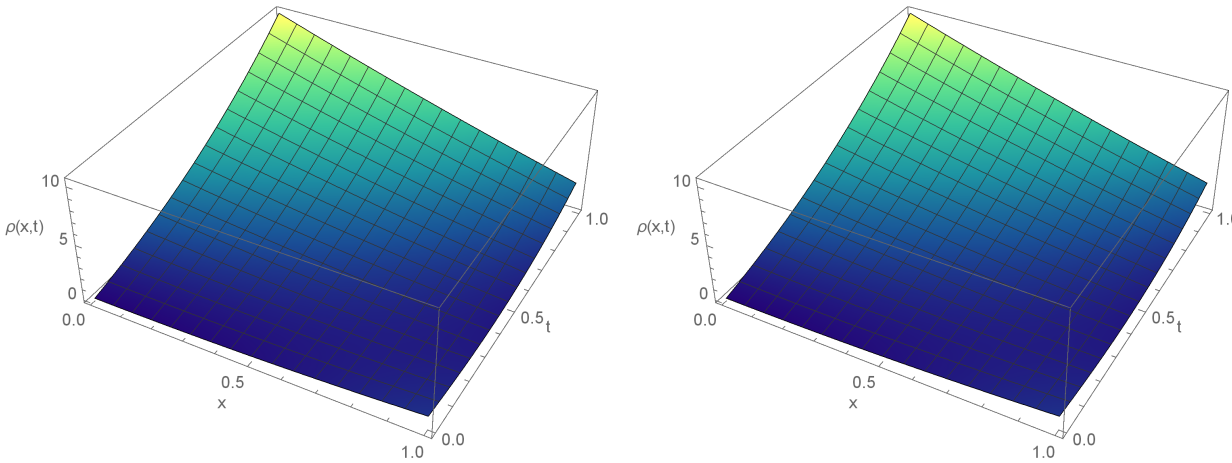

Figure 1: The approximate answer for Caputo and Atangana-Baleanu derivative of concentration of amoebae,respectively

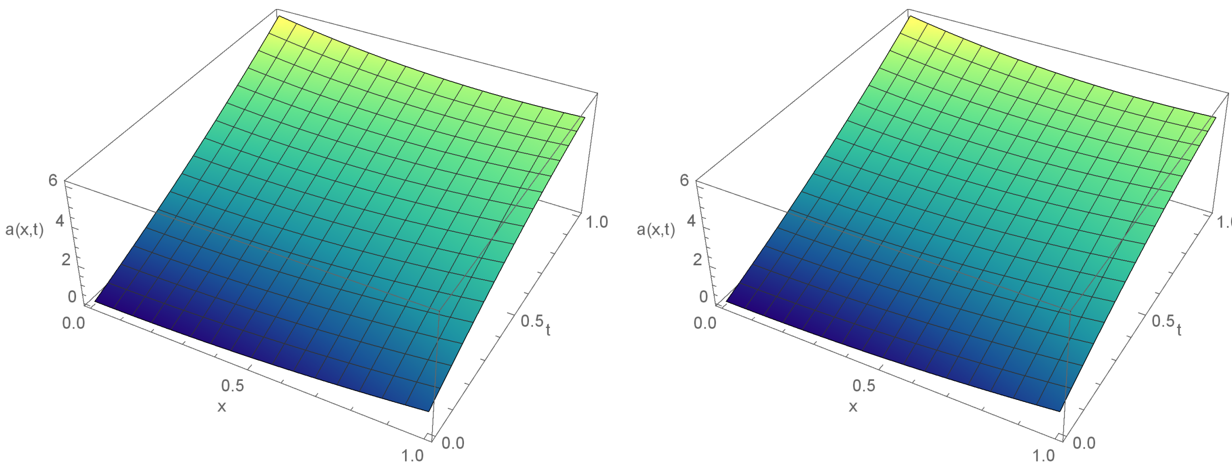

Figure 2: The approximate answer for Caputo and Atangana-Baleanu derivative of concentration of chemical, respectively

We also compared the approximate results for $\nu = 0,5$ and $\nu = 0.7$ with Atangana-Baleanu fractional derivative in Caputo sense. Figure 3 and Figure 4 show that the system can be work to control the behaviour of the solution for the different scale.

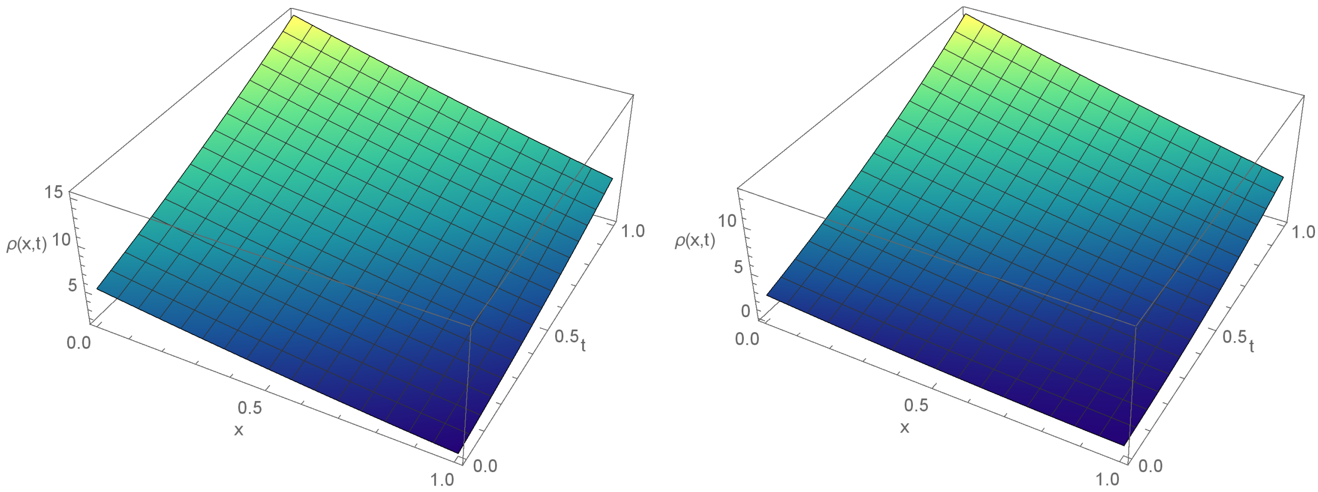

Figure 3: The approximate answer Atangana-Baleanu derivative of concentration of chemical for $\nu = 0.5$ and $\nu = 0.7$ respectively.

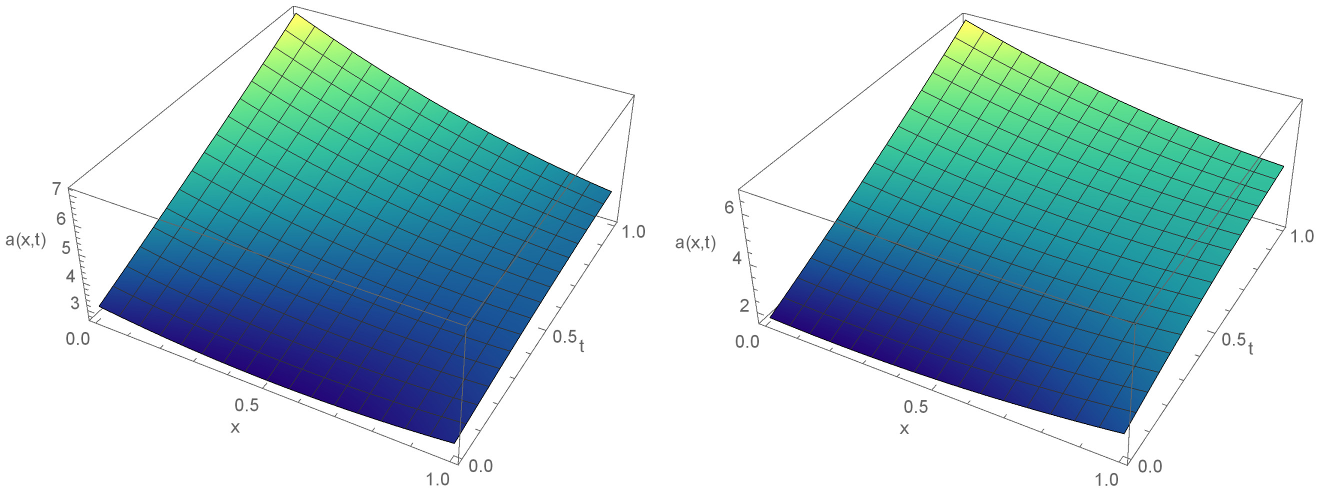

Figure 4: The approximate answer Atangana-Baleanu derivative of concentration of amoebae for $\nu = 0.5$ and $\nu = 0.7$ respectively.

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