Filomat 32:16 (2018), 5585–5593
https://doi.org/10.2298/FIL1816585K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Asymptotically Deferred f-Statistical Equivalence of Sequences

Şükran Konca[a]

[a]*Department of Mathematics, Bitlis Eren University, 13000, Turkey*

Abstract. In this work, we obtain a generalization of asymptotically deferred statistical equivalence of non-negative real-valued sequences with the aid of a modulus function. Further, we examine some of main properties of asymptotically deferred f-statistical equivalence and obtain some equivalence results and inclusion relations.

1. Introduction

Asymptotically regular matrices which preserve the asymptotic equivalence of two non-negative number sequences has been investigated by Pobyvanets in [29]. Fridy [16] generalized this idea and suggested a new way to compare convergence rate of non-negative sequences. After the work of Fridy, Marouf [23] continued this subject and gave some necessary and sufficient conditions to be asymptotic regular matrices. The same problem in [23] is considered and under weak conditions some further results are given by Jinlu Li in [18]. The last twenty years, by considering different kind of convergence, etc. this subject is studied by many different names in [8–13, 17, 27, 28].

The concept of statistical convergence was defined by Steinhaus [34] and Fast [15] and later reintroduced by Schoenberg [32] independently. Although statistical convergence was introduced over nearly the last sixty years, it has become an active area of research in recent years. For further results we may recommend, for example, see, [9, 11, 12, 14].

In 1932, Agnew [1] defined the deferred Cesaro mean of a sequence and later Kosar et al. [20] gave mainly a typical generalization of definition of asymptotically statistical equivalence of non-negative sequences by considering deferred statistical density which is given and studied in [3, 8, 31] (see also [21, 35]).

The notion of a modulus function was introduced by Nakano [25]. Ruckle [30] and Maddox [22] introduced and discussed some properties of sequence spaces defined by using a modulus function. In 2014, Aizpuru et al. [2] defined a new concept of density with the help of an unbounded modulus function and, as a consequence, they obtained a new concept of nonmatrix convergence, namely, f-statistical convergence, which is intermediate between the ordinary convergence and the statistical convergence and agrees with the statistical convergence when the modulus function is the identity mapping. Quite recently, Bhardwaj and Dhawan [4], and Bhardwaj et al. [5], have introduced and studied the concepts of f-statistical convergence of order α and f-statistical boundedness, respectively, by using the approach of Aizpuru et al. [2] (see also [6, 7]).

2010 *Mathematics Subject Classification*. Primary 40A05; Secondary 40A35, 40A99, 40C05, 40G15
Keywords. Asymptotically equivalent sequences, deferred statistical convergence, strongly deferred Cesaro summability, modulus function
Received: 04 June 2017; Revised: 17 December 2017; Accepted: 25 December 2017
Communicated by Ljubiša D.R. Kočinac
Email address: skonca@beu.edu.tr (Şükran Konca)

With the aid of modulus functions, we obtain a generalization of asymptotically deferred statistical equivalence of sequences. Further, we examine some of main properties of asymptotically deferred f-statistical equivalence and obtain some equivalence results and inclusion relations.

2. Definitions and Preliminaries

The following definitions and preliminaries are essential to our work. By $\mathbb{N}$ and $\mathbb{R}$, we mean the set of all natural and real numbers respectively. We use the notation $\lim_k x_k$, shortly, for $\lim_{k\to\infty} x_k$.

Definition 2.1. ([26]) A number sequence $x = (x_k)$ is said to be *statistically convergent to* ℓ if for every $\varepsilon > 0$ the set $\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\}$ has natural density zero, where the natural density of a subset $K \subset \mathbb{N}$ is defined by $d(K) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in K\}|$, where $|\{k \leq n : k \in K\}|$ denotes the number of elements of K not exceeding n. Obviously, we have $d(K) = 0$ provided that K is a finite set of positive integers. If a sequence is statistically convergent to ℓ, then it is written as S-$\lim_k x_k = \ell$ or $x_k \to \ell(S)$. The set of all statistically convergent sequences is denoted by S.

Definition 2.2. ([23]) Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be *asymptotically equivalent* if

$$lim_k \frac{x_k}{y_k} = 1. \tag{1}$$

It is denoted by $x \sim y$. If the limit in (1) is ℓ, then it will be denoted by $x \overset{\ell}{\sim} y$.

Definition 2.3. ([27]) Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be *asymptotically statistically equivalent of multiple* ℓ provided that for every $\varepsilon > 0$ $\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{x_k}{y_k} - \ell \right| \geq \varepsilon \right\} \right| = 0$ and it is denoted by $x \overset{S_\ell}{\sim} y$. If $\ell = 1$ then the sequences x and y are simply called asymptotically statistical equivalent and it is denoted by $x \overset{S}{\sim} y$.

Recall that a *modulus function* f is a function from $[0, \infty)$ to $[0, \infty)$ such that

1. $f(x) = 0 \Leftrightarrow x = 0$,
2. $f(x + y) \leq f(x) + f(y)$ for $x \geq 0$, $y \geq 0$,
3. f is increasing,
4. f is continuous from the right at 0.

It is said that a modulus $f : [0, \infty) \to [0, \infty)$ is *slowly varying* if the limit relation $\lim_{x\to\infty} \frac{f(ax)}{x} = 1$ holds for every $a > 0$. It is clear that all bounded modulus are slowly varying. The function $f(x) = log(x + 1)$ is an example of unbounded, slowly varying modulus (see, chapter 1 in [33]).

Lemma 2.4. ([22]) *Let $f : [0, \infty) \to [0, \infty)$ be a modulus. Then there is a finite $lim_{t\to\infty} \frac{f(t)}{t}$ and equality*

$$\lim_{t\to\infty} \frac{f(t)}{t} = \inf \left\{ \frac{f(t)}{t} : t \in (0, \infty) \right\}$$

holds.

Definition 2.5. ([2]) Let f be an unbounded modulus function. The *f-density* of a set $K \subset \mathbb{N}$ is defined by

$$d^f(K) = \lim_{n\to\infty} \frac{f(|\{k \leq n : k \in A\}|)}{f(n)}$$

in case this limit exists. Clearly, finite sets have zero f-density and $d^f(\mathbb{N} - K) = 1 - d^f(K)$ does not hold, in general. But if $d^f(K) = 0$ then $d^f(\mathbb{N} - K) = 1$.

For example, if we take $f(x) = log(x+1)$ and $K = \{2n : n \in \mathbb{N}\}$, then $d^f(K) = d^f(\mathbb{N} - K) = 1$. For any unbounded modulus f and $K \subset \mathbb{N}$, $d^f(K) = 0$ implies that $d(K) = 0$. But converse need not be true in the sense that a set having zero natural density may have non-zero f-density with respect to some unbounded modulus f. For example, if we take $f(x) = log(x+1)$ and $K = \{1, 4, 9, ...\}$, then $d(K) = 0$ but $d^f(K) = 1/2$. However, $d(K) = 0$ implies $d^f(K) = 0$ is always true in case of any finite set $K \subset \mathbb{N}$, irrespective of the choice of unbounded modulus f (see, [2]).

Definition 2.6. ([2]) Let f be an unbounded modulus function. A number sequence $x = (x_k)$ is said to be *f-statistically convergent to ℓ* or *S^f-convergent to ℓ*, if for each $\varepsilon > 0$ $d^f(\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\}) = 0$, that is,

$$\lim_{n\to\infty} \frac{f(|\{k \leq n : |x_k - \ell| \geq \varepsilon\}|)}{f(n)} = 0,$$

and one writes it as $S^f - \lim_k x_k = l$ or $x_k \to l\left(S^f\right)$. The set of all f-statistically convergent sequences is denoted by S^f.

Agnew [1] defined the *deferred Cesaro mean $D_{p,q}$* of a sequence $x = (x_n)$ by

$$\left(D_{p,q}x\right)_n := \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} x_k,$$

where $p_n = \{p(n)\}_{n\in\mathbb{N}}$ and $q_n = \{q(n)\}_{n\in\mathbb{N}}$ are sequences of non-negative integers satisfying

$$p_n < q_n \text{ and } \lim_n q_n = \infty. \tag{2}$$

Let K be an arbitrary subset of $\mathbb{N}$ and for given sequences p_n and q_n

$$K_{p,q}(n) := \{p_n < k \leq q_n : k \in K\}, \tag{3}$$

be an associated set of K.

Definition 2.7. ([20]) Let K be an arbitrary subset of $\mathbb{N}$. If the following limit

$$K = \delta_{p,q}(K) := \lim_n \frac{1}{q_n - p_n} \left|K_{p,q}(n)\right|,$$

exists, then the limit $\delta_{p,q}(K)$ is called *deferred density* of K. Two nonnegative sequences $x = (x_n)$ and $y = (y_n)$ are said to be *asymptotically deferred statistical equivalent with multiple ℓ* provided that for every $\varepsilon > 0$ the limit

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} \left|\left\{p_n < k \leq q_n : \left|\frac{x_k}{y_k} - \ell\right| \geq \varepsilon\right\}\right| = 0,$$

holds. It is denoted by $x \overset{DS_\ell}{\sim} y$ and if $\ell = 1$, it is called simply *asymptotically deferred statistical equivalent* and denoted by $x \overset{DS}{\sim} y$.

For some special p and q, $\delta_{p,q}(K)$ coincides with some known notions. When $q(n) = n$ and $p(n) = 0$, then $\delta_{p,q}(K)$ coincides with the asymptotic density of K. If $q(n) = k_n$ and $p(n) = k_{n-1}$, where $\theta = (k_n)$ is a lacunary sequence of nonnegative integers with $k_n - k_{n-1}$ as $n \to \infty$, then deferred density $\delta_{p,q}(K)$ of K coincides with lacunary density of K which is denoted by $\delta_\theta(K)$ and so on (see, [20]).

3. Main Results

In this section, we present our main results. Throughout the paper, let f be an unbounded modulus function from $[0,\infty)$ to $[0,\infty)$.

Definition 3.1. A sequence $x = (x_n)$ is called *f-strongly deferred Cesaro convergent to ℓ* if

$$\lim_n \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} f\left(|x_k - \ell|\right) = 0.$$

Definition 3.2. Let K be an arbitrary subset of $\mathbb{N}$. If the following limit

$$\delta^f_{p,q}(K) := \lim_n \frac{1}{f(q_n - p_n)} f\left(\left|K_{p,q}(n)\right|\right) \tag{4}$$

exists, then the limit $\delta^f_{p,q}(K)$ is called *f-deferred density* of K, where $K_{p,q}(n)$ is defined as in (3).

Two non-negative sequences $x = (x_n)$ and $y = (y_n)$ are said to be *asymptotically deferred f-statistical equivalent with multiple ℓ* provided that for every $\varepsilon > 0$ f-deferred density of the set $\left\{p_n < k \le q_n : \left|\frac{x_k}{y_k} - \ell\right| \ge \varepsilon\right\}$ is zero, that is, the limit

$$\lim_n \frac{1}{f\left(q_n - p_n\right)} f\left(\left|\left\{p_n < k \le q_n : \left|\frac{x_k}{y_k} - \ell\right| \ge \varepsilon\right\}\right|\right) = 0 \tag{5}$$

holds. It is denoted by $x \overset{DS_\ell^f}{\sim} y$ and simply called *asymptotically deferred f-statistical equivalent* if $\ell = 1$ and denoted by $x \overset{DS^f}{\sim} y$.

Under some special choices it is easy to see that our definition coincides with some other known definitions.

Remark 3.3.
1. If $f(x) = x$, $q_n = n$ and $p_n = n - 1$, then (5) coincides with (1).
2. For $q_n = n$ and $p_n = 0$, then Definition 3.2 coincides with the definition of asymptotically f-statistical equivalence [19].
3. If $f(x) = x$, $q_n = n$ and $p_n = 0$, then Definition 3.2 coincides with Definition 2.3.
4. If we consider $f(x) = x$, $q_n = k_n$ and $p_n = k_{n-1}$, where k_n is a lacunary sequence of non-negative integers with $k_n - k_{n-1} \to \infty$ as $n \to \infty$, then Definition 3.2 coincides with the definition of asymptotically lacunary statistical equivalence which is given by Patterson and Savas [28] and Braha [8].
5. For $f(x) = x$, $q_n = \lambda_n$ and $p_n = 0$, when λ_n is a strictly increasing sequence of natural numbers such that $\lim_n \lambda_n = \infty$, then Definition 3.2 coincides with the definition of λ-statistical equivalence of sequences [8].
6. If we consider $f(x) = x$, $q_n = n$ and $p_n = n - \lambda_n$, where (λ_n) is a non-decreasing sequence of natural numbers such that $\lambda_0 = 1$ and $\lambda_{n+1} \le \lambda_n + 1$ satisfied, then (4) coincides with the λ-density $\delta_\lambda(K)$ defined by Mursaleen [24].

Definition 3.4. Two non-negative sequences $x = (x_n)$ and $y = (y_n)$ are said to be *f-strongly asymptotically deferred equivalent with multiple ℓ* if

$$\lim_n \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} f\left(\left|\frac{x_k}{y_k} - \ell\right|\right) = 0$$

holds. It is denoted by $x \overset{D_\ell(f)}{\sim} y$ and simply called *f-strongly asymptotically deferred equivalent*, and denoted by $x \overset{D(f)}{\sim} y$, when $\ell = 1$.

Theorem 3.5. *Let $x = (x_n)$ and $y = (y_n)$ be nonnegative real valued sequences and f be any unbounded modulus function for which $\lim_{n\to\infty} \frac{f(q_n-p_n)}{q_n-p_n} > 0$, and c be a positive constant such that $f(xy) \geq c\ f(x) f(y)$ for all $x \geq 0$, $y \geq 0$. If $x \overset{D_\ell(f)}{\sim} y$, then $x \overset{DS_\ell^f}{\sim} y$.*

Proof. Let f be any unbounded modulus function for which $\lim_{n\to\infty} \frac{f(q_n-p_n)}{q_n-p_n} > 0$ and c be a positive constant such that $f(xy) \geq c\ f(x)\ f(y)$ for all $x \geq 0$, $y \geq 0$. Assume that $x \overset{D_\ell(f)}{\sim} y$, that is,

$$\lim_{n\to\infty} \frac{1}{q_n-p_n} \sum_{k=p_n+1}^{q_n} f\left(\left|\frac{x_k}{y_k} - \ell\right|\right) = 0.$$

For any $\varepsilon > 0$ we have the following by the definition of a modulus function (2) and (3):

$$\begin{aligned}
\tfrac{1}{q_n-p_n} \sum_{k=p_n+1}^{q_n} f\left(\left|\tfrac{x_k}{y_k} - \ell\right|\right) &\geq \tfrac{1}{q_n-p_n} f\left(\sum_{k=p_n+1}^{q_n} \left|\tfrac{x_k}{y_k} - \ell\right|\right) \geq \tfrac{1}{q_n-p_n} f\left(\sum_{\substack{k=p_n+1 \\ \left|\frac{x_k}{y_k}-\ell\right|\geq\varepsilon}}^{q_n} \left|\tfrac{x_k}{y_k} - \ell\right|\right) \\
&\geq \tfrac{1}{q_n-p_n} f\left(\left|\left\{p_n < k \leq q_n : \left|\tfrac{x_k}{y_k} - \ell\right| \geq \varepsilon\right\}\right| \varepsilon\right) \\
&\geq \tfrac{c}{q_n-p_n} f\left(\left|\left\{p_n < k \leq q_n : \left|\tfrac{x_k}{y_k} - \ell\right| \geq \varepsilon\right\}\right|\right) f(\varepsilon) \\
&= \tfrac{c}{f(q_n-p_n)} f\left(\left|\left\{p_n < k \leq q_n : \left|\tfrac{x_k}{y_k} - \ell\right| \geq \varepsilon\right\}\right|\right) \tfrac{f(q_n-p_n)}{q_n-p_n} f(\varepsilon)
\end{aligned}$$

from where it follows that $x \overset{DS_\ell^f}{\sim} y$ as $n \to \infty$. □

Theorem 3.6. *If $x \overset{DS_\ell^f}{\sim} y$, then $x \overset{DS_\ell}{\sim} y$.*

Proof. Suppose that $x \overset{DS_\ell^f}{\sim} y$. Then by the definition of the limit and the fact that f is a subadditive function, for every $m \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ we have

$$\begin{aligned}
f\left(\left|\left\{p_n < k \leq q_n : \left|\tfrac{x_k}{y_k} - \ell\right| \geq \varepsilon\right\}\right|\right) &\leq \tfrac{1}{m} f(q_n - p_n) \\
&= \tfrac{1}{m} f\left(\tfrac{m(q_n-p_n)}{m}\right) \leq \tfrac{1}{m}\left[\overbrace{f\left(\frac{q_n-p_n}{m}\right) + \ldots + f\left(\frac{q_n-p_n}{m}\right)}^{m \text{ times}}\right]. \\
&= \tfrac{1}{m} m f\left(\tfrac{q_n-p_n}{m}\right) = f\left(\tfrac{q_n-p_n}{m}\right).
\end{aligned}$$

We know that a modulus function $f : [0,\infty) \to [0,\infty)$ is an increasing function. Then

$$\left|\left\{p_n < k \leq q_n : \left|\frac{x_k}{y_k} - \ell\right| \geq \varepsilon\right\}\right| \leq \frac{q_n-p_n}{m} \Rightarrow \frac{1}{q_n-p_n}\left|\left\{p_n < k \leq q_n : \left|\frac{x_k}{y_k} - \ell\right| \geq \varepsilon\right\}\right| \leq \frac{1}{m}.$$

Hence $x \overset{DS_\ell}{\sim} y$. □

The following corollary is a result of Theorem 3.5 and Theorem 3.6.

Corollary 3.7. *Let f be a modulus function such that $\lim_{n\to\infty}\frac{f(q_n-p_n)}{q_n-p_n}>0$ and c be a positive constant such that $f(xy)\geq cf(x)f(y)$ for all $x\geq 0$, $y\geq 0$. If $x\overset{D_\ell(f)}{\sim}y$ then $x\overset{DS_\ell}{\sim}y$.*

Theorem 3.8. *If $x=(x_n)$ and $y=(y_n)\in l_\infty$, where l_∞ denotes the set of all bounded sequences, and $x\overset{DS_\ell^f}{\sim}y$ then $x\overset{D_\ell(f)}{\sim}y$ for any unbounded modulus f.*

Proof. Suppose that $x\overset{DS_\ell^f}{\sim}y$ and $x=(x_n)$ and $y=(y_n)\in l_\infty$. Then there exists a $M>0$ such that $\left|\frac{x_k}{y_k}-\ell\right|\leq M$ holds for all $k\in\mathbb{N}$. For any $\varepsilon>0$

$$\begin{aligned}\frac{1}{q_n-p_n}\sum_{k=p_n+1}^{q_n}f\left(\left|\frac{x_k}{y_k}-\ell\right|\right)&=\frac{1}{q_n-p_n}\sum_{\substack{k=p_n+1\\ \left|\frac{x_k}{y_k}-\ell\right|\geq\varepsilon}}^{q_n}f\left(\left|\frac{x_k}{y_k}-\ell\right|\right)+\frac{1}{q_n-p_n}\sum_{\substack{k=p_n+1\\ \left|\frac{x_k}{y_k}-\ell\right|<\varepsilon}}^{q_n}f\left(\left|\frac{x_k}{y_k}-\ell\right|\right)\\&\leq\frac{1}{q_n-p_n}\left|\left\{p_n<k\leq q_n:\left|\frac{x_k}{y_k}-\ell\right|\geq\varepsilon\right\}\right|f(M)\\&\quad+\frac{1}{q_n-p_n}(q_n-p_n)f(\varepsilon)\end{aligned}.$$

Taking limit on both sides as $n\to\infty$, we obtain

$$\lim_{n\to\infty}\frac{1}{q_n-p_n}\sum_{k=p_n+1}^{q_n}f\left(\left|\frac{x_k}{y_k}-\ell\right|\right)=0$$

in view of Theorem 3.6 and the fact that f is increasing. □

3.1. DS_ℓ^f-Equivalence of Sequences

Let $x=(x_n)$ and $y=(y_n)$ be sequences of nonnegative real numbers. The notation "$x\prec y$" will be used if $x_n\leq y_n$ holds for all $n\in\mathbb{N}$.

Theorem 3.9. *Let $x=(x_n)$, $y=(y_n)$ and $z=(z_n)$ be sequences of nonnegative real numbers and f be an unbounded modulus. If $z\prec x$ and $x-z\overset{DS_{\ell'}^f}{\sim}y$ then $x\overset{DS_\ell^f}{\sim}y$ implies $z\overset{DS_{(\ell-\ell')}^f}{\sim}y$.*

Proof. Assume that $x-z\overset{DS_{\ell'}^f}{\sim}y$. We need $z\prec x$ to guarantee the sequence $x-z=(x-z)_k=x_k-z_k$ to be a sequence of nonnegative real numbers. Then

$$\left|\frac{z_k}{y_k}-(\ell-\ell')\right|\leq\left|\frac{x_k}{y_k}-\ell\right|+\left|\frac{x_k-z_k}{y_k}-\ell'\right| \tag{6}$$

holds for all $k\in\mathbb{N}$. Then for a given $\varepsilon>0$ the following inequality

$$\left|\left\{p_n<k\leq q_n:\left|\frac{z_k}{y_k}-(\ell-\ell')\right|\geq\varepsilon\right\}\right|\leq\left|\left\{p_n<k\leq q_n:\left|\frac{x_k}{y_k}-\ell\right|\geq\frac{\varepsilon}{2}\right\}\right|+\left|\left\{p_n<k\leq q_n:\left|\frac{x_k-z_k}{y_k}-\ell'\right|\geq\frac{\varepsilon}{2}\right\}\right|$$

is satisfied. Since f is an unbounded increasing modulus, from the above equation we obtain the following

$$\frac{f\left(\left|\left\{p_n<k\leq q_n:\left|\frac{z_k}{y_k}-(\ell-\ell')\right|\geq\varepsilon\right\}\right|\right)}{f(q_n-p_n)}\leq\frac{f\left(\left|\left\{p_n<k\leq q_n:\left|\frac{x_k}{y_k}-\ell\right|\geq\frac{\varepsilon}{2}\right\}\right|\right)}{f(q_n-p_n)}+\frac{f\left(\left|\left\{p_n<k\leq q_n:\left|\frac{x_k-z_k}{y_k}-\ell'\right|\geq\frac{\varepsilon}{2}\right\}\right|\right)}{f(q_n-p_n)}.$$

Then the desired result is obtained taking the limit for $n\to\infty$. □

Theorem 3.10. *Let $x = (x_n)$, $y = (y_n)$ and $z = (z_n)$ be sequences of nonnegative real numbers and f be an unbounded modulus. If $y \prec z$ and $x \overset{DS^f_{\ell'}}{\sim} z - y$ then $x \overset{DS^f_{\ell}}{\sim} y$ implies $x \overset{DS^f_{1/\ell''}}{\sim} z$ where $\ell'' := 1/\ell + 1/\ell'$.*

Proof. Assume that $x \overset{DS_\ell f}{\sim} y$. Since $y \prec z$ then for all $n \in \mathbb{N}$ $y_n \le z_n$ holds. We need it to guarantee the sequence $y - z = (y - z)_k = y_k - z_k$ to be a sequence of nonnegative real numbers. Then from the following equality

$$\frac{z_k}{x_k} = \frac{z_k - y_k}{x_k} + \frac{y_k}{x_k} = \frac{z_k - y_k}{x_k} - \frac{1}{\ell'} + \frac{1}{\ell'} + \frac{1}{\ell} - \frac{1}{\ell} + \frac{y_k}{x_k}$$

$$\left|\frac{z_k}{x_k} - \ell''\right| \le \left|\frac{z_k - y_k}{x_k} - \frac{1}{\ell'}\right| + \left|\frac{y_k}{x_k} - \frac{1}{\ell}\right| \tag{7}$$

is satisfied for all $k \in \mathbb{N}$. Then for any $\varepsilon > 0$ we have the inclusion

$$\left\{p_n < k \le q_n : \left|\frac{z_k}{x_k} - \ell''\right| \ge \varepsilon\right\} \subseteq \left\{p_n < k \le q_n : \left|\frac{z_k - y_k}{x_k} - \frac{1}{\ell'}\right| \ge \frac{\varepsilon}{2}\right\} \cup \left\{p_n < k \le q_n : \left|\frac{y_k}{x_k} - \frac{1}{\ell}\right| \ge \frac{\varepsilon}{2}\right\}.$$

Since f is an unbounded increasing modulus, from the above inclusion we obtain the inequality

$$\frac{f\left(\left\{p_n < k \le q_n : \left|\frac{z_k}{x_k} - \ell''\right| \ge \varepsilon\right\}\right)}{f(q_n - p_n)} \le \frac{f\left(\left\{p_n < k \le q_n : \left|\frac{z_k - y_k}{x_k} - \frac{1}{\ell'}\right| \ge \frac{\varepsilon}{2}\right\}\right)}{f(q_n - p_n)} + \frac{f\left(\left\{p_n < k \le q_n : \left|\frac{y_k}{x_k} - \frac{1}{\ell}\right| \ge \frac{\varepsilon}{2}\right\}\right)}{f(q_n - p_n)}.$$

By taking the limit for $n \to \infty$, we obtain the result. □

Theorem 3.11. *Let $x = (x_n)$, $y = (y_n)$ and $z = (z_n)$ be sequences of nonnegative real numbers and f be an unbounded modulus. If there exists a dense subsequence $\{z_n{}^*\}$ of $\{z_n\}$ such that $z^* \prec x$ and $x - z^* \overset{DS^f_{\ell'}}{\sim} y$, then $x \overset{DS^f_{\ell}}{\sim} y$ implies $z \overset{DS^f_{\ell-\ell'}}{\sim} y$*

Proof. Let $K = K(n) := \{n : z_n \ne z_n^*\}$, then from the assumption we have $\delta^f_{p,q}(K) = 0$. So from the following inequality

$$\begin{aligned}\frac{z_n}{y_n} &= \frac{x_n - (x_n - z_n)}{y_n} = \frac{x_n}{y_n} - \left(\frac{x_n - z_n{}^*}{y_n}\right)_{n\in K^c} - \left(\frac{x_n - z_n}{y_n}\right)_{n\in K} \\ &= \frac{x_n}{y_n} - \ell + \ell - \ell' + \ell' - \left(\frac{x_n - z_n{}^*}{y_n}\right)_{n\in K^c} - \left(\frac{x_n - z_n}{y_n}\right)_{n\in K}\end{aligned}$$

we have

$$\left|\frac{z_n}{y_n} - (\ell - \ell')\right| \le \left|\frac{x_n}{y_n} - \ell\right| + \left|\frac{x_n - z_n{}^*}{y_n} - \ell'\right|_{n\in K^c} - \left|\frac{x_n - z_n}{y_n}\right|_{n\in K}.$$

Hence

$$\begin{aligned}\left|\left\{p_n < k \le q_n : \left|\frac{z_k}{y_k} - (\ell - \ell')\right| \ge \varepsilon\right\}\right| \le \left|\left\{p_n < k \le q_n : \left|\frac{x_k}{y_k} - \ell\right| \ge \frac{\varepsilon}{3}\right\}\right| + \left|\left\{p_n < k \le q_n \text{ and } k \in K : \left|\frac{x_k - z_k}{y_k}\right| \ge \frac{\varepsilon}{3}\right\}\right| \\ + \left|\left\{p_n < k \le q_n \text{ and } k \in K^c : \left|\frac{x_k - z_k{}^*}{y_k} - \ell'\right| \ge \frac{\varepsilon}{3}\right\}\right|.\end{aligned}$$

Using the subadditivity of modulus and since f is an increasing function, then we have

$$\begin{aligned}\frac{f\left(\left|\left\{p_n < k \le q_n : \left|\frac{z_k}{y_k} - (\ell - \ell')\right| \ge \varepsilon\right\}\right|\right)}{f(q_n - p_n)} \le \frac{f\left(\left|\left\{p_n < k \le q_n : \left|\frac{x_k}{y_k} - \ell\right| \ge \frac{\varepsilon}{3}\right\}\right|\right)}{f(q_n - p_n)} + \frac{f\left(\left|\left\{p_n < k \le q_n \text{ and } k \in K : \left|\frac{x_k - z_k}{y_k}\right| \ge \frac{\varepsilon}{3}\right\}\right|\right)}{f(q_n - p_n)} \\ + \frac{f\left(\left|\left\{p_n < k \le q_n \text{ and } k \in K^c : \left|\frac{x_k - z_k{}^*}{y_k} - \ell'\right| \ge \frac{\varepsilon}{3}\right\}\right|\right)}{f(q_n - p_n)}.\end{aligned}$$

We obtain $z \overset{DS^f_{\ell-\ell'}}{\sim} y$ when $n \to \infty$ since $\lim_{n\to\infty} \frac{f\left(\left|\left\{p_n<k\leq q_n \text{ and } k\in K: \left|\frac{x_k-z_k}{y_k}\right|\geq \frac{\varepsilon}{3}\right\}\right|\right)}{f(q_n-p_n)} \leq \lim_{n\to\infty} \frac{f(|K|)}{f(q_n-p_n)} = \delta^f_{p,q}(K) = 0$, $x - z^* \overset{DS^f_{\ell'}}{\sim} y$ and $x \overset{DS^f_{\ell}}{\sim} y$. □

Theorem 3.12. *Let $x = (x_n)$, $y = (y_n)$ and $z = (z_n)$ be any sequences of nonnegative real numbers and f be an unbounded modulus. If there exists a dense subsequence $\{y_n{}^*\}$ of $\{y_n\}$ such that $y^* \prec z$ and $x \overset{DS^f_{\ell'}}{\sim} z - y^*$, then $x \overset{DS^f_{\ell}}{\sim} y$ implies $x \overset{DS^f_{1/\ell''}}{\sim} z$ where $\ell'' := 1/\ell + 1/\ell'$.*

Proof. Let $K = K(n) := \{n : y_n \neq y_n^*\}$, then from the assumption we have $\delta^f_{p,q}(K) = 0$. So from the following inequality

$$\frac{z_n}{x_n} = \frac{z_n-y_n}{x_n} + \frac{y_n}{x_n} = \frac{y_n}{x_n} + \left(\frac{z_n-y_n{}^*}{x_n}\right)_{n\in K^c} + \left(\frac{z_n-y_n}{x_n}\right)_{n\in K}$$
$$= \frac{y_n}{x_n} - \frac{1}{\ell} + \frac{1}{\ell} + \frac{1}{\ell'} + \left(\frac{z_n-y_n{}^*}{x_n}\right)_{n\in K^c} - \frac{1}{\ell'} + \left(\frac{z_n-y_n}{x_n}\right)_{n\in K}$$

we have

$$\left|\frac{z_n}{x_n} - \left(\frac{1}{\ell} + \frac{1}{\ell'}\right)\right| \leq \left|\frac{y_n}{x_n} - \frac{1}{\ell}\right| + \left|\frac{z_n - y_n{}^*}{x_n} - \frac{1}{\ell'}\right|_{n\in K^c} + \left|\frac{z_n - y_n}{x_n}\right|_{n\in K}.$$

Hence

$$\begin{aligned}\left|\left\{p_n < k \leq q_n : \left|\frac{z_k}{x_k} - \ell''\right| \geq \varepsilon\right\}\right| \leq{}& \left|\left\{p_n < k \leq q_n : \left|\frac{y_k}{x_k} - \frac{1}{\ell}\right| \geq \frac{\varepsilon}{3}\right\}\right| + \left|\left\{p_n < k \leq q_n \text{ and } k \in K : \left|\frac{z_k-y_k}{x_k}\right| \geq \frac{\varepsilon}{3}\right\}\right| \\ &+ \left|\left\{p_n < k \leq q_n \text{ and } k \in K^c : \left|\frac{z_k-y_k{}^*}{x_k} - \frac{1}{\ell'}\right| \geq \frac{\varepsilon}{3}\right\}\right|.\end{aligned}$$

Since f is an increasing, subadditive function, then we have

$$\begin{aligned}\frac{f\left(\left\{p_n<k\leq q_n: \left|\frac{z_k}{x_k}-\ell''\right|\geq\varepsilon\right\}\right)}{f(q_n-p_n)} \leq{}& \frac{f\left(\left\{p_n<k\leq q_n \text{ and } k\in K: \left|\frac{z_k-y_k}{x_k}\right|\geq\frac{\varepsilon}{3}\right\}\right)}{f(q_n-p_n)} \\ &+ \frac{f\left(\left\{p_n<k\leq q_n \text{ and } k\in K^c: \left|\frac{z_k-y_k{}^*}{x_k}-\frac{1}{\ell'}\right|\geq\frac{\varepsilon}{3}\right\}\right)}{f(q_n-p_n)} + \frac{f\left(\left\{p_n<k\leq q_n: \left|\frac{y_k}{x_k}-\frac{1}{\ell}\right|\geq\frac{\varepsilon}{3}\right\}\right)}{f(q_n-p_n)}.\end{aligned}$$

We obtain $x \overset{DS^f_{1/\ell''}}{\sim} z$ when $n \to \infty$ where $\ell'' = 1/\ell + 1/\ell'$ since $x \overset{DS^f_{\ell}}{\sim} y$, $x \overset{DS^f_{\ell'}}{\sim} z - y^*$ and

$$\lim_{n\to\infty} \frac{f\left(\left\{p_n < k \leq q_n \text{ and } k \in K : \left|\frac{z_k-y_k}{x_k}\right| \geq \frac{\varepsilon}{3}\right\}\right)}{f(q_n - p_n)} \leq \lim_{n\to\infty} \frac{f(|K|)}{f(q_n - p_n)} = \delta^f_{p,q}(K) = 0.$$

□

4. Concluding Remarks

In this work, we obtain a generalization of asymptotically deferred statistical equivalence of nonnegative real-valued sequences using a modulus function. Further, we examine some of main properties of asymptotically deferred f-statistical equivalence and obtain some equivalence results and inclusion relations. We have used the partial order "$\prec$" on $\mathbb{R}$ for two non-negative real valued sequences x and y denoted by "$x \prec y$" to represent if $x_n \leq y_n$ holds for all $n \in \mathbb{N}$. We have obtained some nice results related to asymptotically deferred statistical equivalence of nonnegative real-valued sequences with the aids of an unbounded modulus and partial order. For the sequences $x = (x_n)$, $y = (y_n)$ and $z = (z_n)$ of nonnegative real numbers, In Theorem 3.9, we have shown that if $z \prec x$ and $x - z \overset{DS^f_{\ell'}}{\sim} y$ then $x \overset{DS^f_{\ell}}{\sim} y$ implies $z \overset{DS^f_{(\ell-\ell')}}{\sim} y$ and in Theorem 3.10, we have seen that if $y \prec z$ and $x \overset{DS^f_{\ell'}}{\sim} z - y$ then $x \overset{DS^f_{\ell}}{\sim} y$ implies $x \overset{DS^f_{1/\ell''}}{\sim} z$ where $\ell'' := 1/\ell + 1/\ell'$. In Theorem 3.11 and Theorem 3.12 we want to see that under which conditions these theorems are satisfied for dense subsequences $\{z_n{}^*\}$ and $\{y_n{}^*\}$ of $\{z_n\}$ and $\{y_n\}$, respectively.

Acknowledgements

The author would like to thank the anonymous referees for their comments to improve this article.

References

[1] R.P. Agnew, On deferred Cesaro mean, Ann. Math. 33 (1932) 413–421.
[2] A. Aizpuru, M.C. Listan-Garcia, F. Rambla-Barreno, Density by moduli and statistical convergence, Quaest. Math. 37 (2014) 525-530.
[3] M. Başarır, S. Altundağ, On Δ^m-lacunary statistical asymptotically equivalent sequences, Filomat. 22:1 (2008) 161-172.
[4] V.K. Bhardwaj, S. Dhawan, f-statistical convergence of order α and strong Cesaro summability of order α with respect to a modulus, J. Ineq. Appl. 2015 (332) (2015), 14 pages.
[5] V.K. Bhardwaj, S. Dhawan, S. Gupta, Density by moduli and statistical boundedness, Abst. Appl. Analysis. 2016, Article ID 2143018, 6 pages.
[6] V.K. Bhardwaj, S. Dhawan, Density by moduli and lacunary statistical convergence, Abst. Appl. Analysis. 2016 (2016), Article ID 9365037, 11 pages.
[7] T. Bilgin, (σ, f)-Asymptotically lacunary equivalent sequences, Internat. J. Analysis 2014 (2014), Article ID 945902, 8 pages.
[8] N.L. Braha, On asymptotically Δ^m-lacunary statistical equivalent sequences, Appl. Math. Comput. 219 (2012) 280-288.
[9] H. Çakallı, A study on statistical convergence, Funct. Anal. Approx. Comput. 1 (2) (2009) 19–24.
[10] H. Çakallı, A. Sönmez, C.G. Aras, λ-statistically ward continuity, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.) LXIII, f. 2 (2017) 313–321.
[11] A. Caserta, G. Di Maio, Lj.D.R. Kocinac, Statistical convergence in function spaces, Abstr. Appl. Anal. 2011 (2011), Article ID 420419, 11 pages.
[12] A. Caserta, Lj. D.R. Kocinac, On statistical exhaustiveness, Appl. Math. Lett. 25 (10) (2012) 1447–1451.
[13] J. Connor, H. Gümüş, Summability matrices that preserve asymtotic equivalence for ideal convergence, doi: 10.13140/R.G.2.1.3659.7926.
[14] G. Di Maio, Lj.D.R. Kočinac, Statistical convergence in topology, Topology Appl. 156 (2008) 28-45.
[15] H. Fast, Sur la convergence statistique, Colloq. Math. 2 (3-4) (1951) 241–244.
[16] J.A. Fridy, Minimal rates of summability, Can J. Math. XXX 4 (1978) 808-816.
[17] B. Hazarika, A. Eşi, N.L. Braha, On asymptotically Wijsman lacunary σ-statistical convergence of set sequences, J. Math. Anal. 4 (3) (2013) 33-46.
[18] L. Jinlu, Asymptotic equivalence of sequences and summability, Internat. J. Math. Math. Sci. 20 (4) (1997) 749-758.
[19] Ş. Konca, M. Küçükaslan, On asymptotically f-statistical equivalent sequences, J. Indonesian Math. Soc., in press.
[20] C. Koşar, M. Küçükaslan, M. Et, On asymptotically deferred statistical equivalence of sequences, Filomat. 31 (16) (2017) 5139-5150.
[21] M. Küçükaslan, M. Yılmaztürk, On deferred statistical convergence of sequences, Kyungpook Math. J. 56 (2016) 357-366.
[22] I.J. Maddox, Sequence spaces defined by a modulus, Math. Proc. Camb. Philos. Soc. 101 (1987) 523-527.
[23] M. Marouf, Asymptotic equivalence and summability, Internat. J. Math. Math. Sci. 16 (1993) 755-762.
[24] M. Mursaleen, λ-Statistical convergence, Math. Slovaca. 50 (2000) 111-115.
[25] H. Nakano, Concave modulars, J. Math. Soc. Japan. 5 (1953) 29-49.
[26] I. Niven, H.S. Zuckerman, H.L. Montgomery, An Introduction to the Theory of Numbers, Fifth Ed., John Wiley and Sons, New York, 1991.
[27] R.F. Patterson, On asymptotically statistically equivalent sequences, Demonstratio Math. 36 (2003) 149-153.
[28] R.F. Patterson, E. Savaş, On asymptotically lacunary statistically equivalent sequences, Thai J. Math. 4 (2006) 267–272.
[29] J.P. Pobyvanets, Asymptotic equivalence of some linear transformation defined by a non-negative matrix and reduced to generalized equivalence in the sense of Cesaro and Abel, Math. Fiz. 28 (123) (1980) 83-87.
[30] W.H. Ruckle, FK spaces in which the sequence of coordinate vectors is bounded, Can. J. Math. 25 (1973) 973-978.
[31] R. Savaş, M. Başarır, (σ, λ)-asymptotically statistical equivalent sequences, Filomat 20:1 (2006) 35-42.
[32] I.J. Schoenberg, The integrability of certain functions and related summability methods, Amer. Math. Monthly 66:5 (1959) 361–375.
[33] E. Seneta, Regularly Varying Functions, Lecture Notes in Mathematics, 508, Springer-Verlag, Berlin-Heidelberg-New York, 1976.
[34] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, Colloq. Math. 2 (1951) 73–74.
[35] M. Yılmaztürk, M. Küçükaslan, On strongly deferred Cesaro summability and deferred statistical convergence of the sequences, Bitlis Eren Univ. J. Sci. Tech. 3 (2013) 22–25.