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Numerical Solutions of the Improved Boussinesq Equation by the Galerkin Quadratic B-Spline Finite Element Method

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Abstract. In this paper, we are going to obtain numerical solutions of the improved Boussinesq equation with the aid of Galerkin quadratic B-spline finite element method. To test the accuracy and efficiency of the current method, four test problems have been used. These are solitary wave movement, interaction of two solitary waves, wave break-up and blow-up of solutions. Their results have been compared with those available in the literature for different values of space and time steps. Also, the error norms L_2 and L_∞ have been computed and presented in comparison.

1. Introduction

First, Joseph Boussinesq has proposed Boussinesq equation (Bq) in 1873. The equation has the characteristics of being a fourth-order nonlinear partial differential equation belonging to the family of KdV equations and it defines the movement of long waves in shallow water subject to gravity propagation in bi-directional way. The equation has a wide range of usage in diverse fields of science such as physics, chemistry, biology, mechanics, in particular nonlinear wave phenomena including ion-sound in plasma, lattice waves [5, 12]. The common form of the Boussinesq equation can be given as follows

$$u_{tt} = u_{xx} + qu_{xxxx} + \left(u^2\right)_{xx} \qquad x \in [a, b], \quad t \in [0, T] \tag{1}$$

where $q = 1$ or $q = -1$. The original form of the equation utilized by Boussinesq in [2] had $q = 1$ in it. One can find in the literature the Bq equation for $q = -1$ as good Boussinesq (GBq) equation, when $q = 1$ is known as bad Boussinesq (BBq) equation. Bogolubsky [1] has proposed a new form of BBq equation, namely improved Boussinesq (IBq) which is physically stable and also correct in the sense of Cauchy problem by replacing the term u_{xxxx} with u_{xxtt} given in the following form

$$u_{tt} = u_{xx} + u_{xxtt} + \left(u^2\right)_{xx} \qquad x \in [a, b], \quad t \in [0, T]. \tag{2}$$

There exist both analytical and numerical solutions of the IBq equation in the literature. Among others, Bogolubsky [1] has investigated the dynamics of several initial wave packets, moreover he takes into

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consideration a modified version of the IBq equation where the term $\left(u^2\right)_{xx}$ has been changed with $\left(u^3\right)_{xx}$. Iskandar and Jain [9] have solved the problem using three-level finite difference scheme. Zoheiry [6] has proposed a three-level iterative scheme derived from implicit compact difference scheme. Inc and Evans [7] have used Adomian decomposition method to obtain the solutions in terms of convergent power series for IBq. Wazwaz [16] has studied the variants of the equation both having positive and negative exponents and also found out many sorts of exact solutions. Bratsos [3, 4] has utilized implicit finite-difference method depending on rational approximants for numerical solution of the IBq equation. Lin and Wu [11] have used finite element method with linear B-spline basis functions obtain the numerical solution of the equation. Irk and Dag [8] have also obtained numerical solutions by using both finite difference and quintic B-spline finite element collocation method. Shokri and Dehghan [14] by using collocation and radial basis functions have obtained approximate solutions. Zhang and Lu [17] have devised a quadratic finite volume scheme to solve the initial boundary value problem for the IBq.

1.1. Problem Statement

We are going to consider Eq.(2) with the initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \qquad x \in [a,b] \tag{3}$$

and the boundary conditions

$$\begin{aligned} &u(a,t) = u(b,t) = 0, \qquad t \in [0,T] \\ &u_x(a,t) = u_x(b,t) = 0, \qquad t \in [0,T] \end{aligned} \tag{4}$$

in which $f(x)$ and $g(x)$ are any given functions, $u = u(x,t)$ is assumed to be sufficiently differentiable function, here the subscripts t and x denote differentiation in terms of time and space, respectively. One can find in the bibliography [1] that Eq.(2) has the following soliton type solutions where Amp is the amplitude, c is the wave speed and x_0 is the initial position of the solitary wave

$$\begin{aligned} &u(x,t) = Amp\ \mathrm{sech}^2\left(\frac{1}{c}\sqrt{\frac{Amp}{6}}\,(x - ct - x_0)\right) \\ &c = \pm\sqrt{1 + \frac{2Amp}{3}}. \end{aligned} \tag{5}$$

Our aim is apply Galerkin quadratic B-spline finite element method to IBq equation given by Eq.(2) with its corresponding initial (3) and boundary (4) conditions. For this purpose we split the original problem into two different equation by applying $u_t = v$. So Improved Boussinesq equation transform to following form

$$\begin{aligned} &u_t = v \\ &v_t = u_{xx} + v_{xxt} + \left(u^2\right)_{xx}. \end{aligned} \tag{6}$$

2. The Quadratic B-Spline Finite Element Solution

Let us assume that the interval $[a,b]$ of the problem is partitioned into N finite elements having equal lengths defined as h by the nodal points $a = x_o < x_1 < ... < x_{N-1} < x_N = b$. After this assumption, we have the quadratic B-spline $\phi_m(x)$ functions at the nodal points x_m described as [13]

$$\phi_m(x) = \frac{1}{h^2}\begin{cases} (x - x_{m-1})^2, & [x_{m-1}, x_m], \\ (x - x_{m-1})^2 - 3(x - x_m)^2, & [x_m, x_{m+1}], \\ (x - x_{m-1})^2 - 3(x - x_m)^2 + 3(x - x_{m+1})^2, & [x_{m+1}, x_{m+2}], \\ 0, & otherwise, \end{cases} \tag{7}$$

which constitute a basis for all the functions defined over the interval $[a,b]$. Thus, we can now write the global approximations $U_N(x,t)$ and $V_N(x,t)$ to the exact ones $u(x,t)$ and $v(x,t)$ in terms of those newly defined quadratic B-splines as follows

$$U_N(x,t) = \sum_{j=-1}^{N} \delta_j(t)\phi_j(x) \quad V_N = \sum_{j=-1}^{N} \sigma_j(t)\phi_j(x) \tag{8}$$

where δ_j are time dependent parameters which are going to be find out using the initial, boundary and weighted residual conditions of the stated problem.

If we use the local coordinate transformation given as $\xi = x - x_m,\ 0 \le \xi \le h$, then Eq.(7) can also be stated as follows

$$\begin{matrix} \phi_{m-1} \\ \phi_m \\ \phi_{m+1} \end{matrix} = \frac{1}{h^2}\left\{\begin{array}{l} (h-\xi)^2, \\ h^2 + 2h\xi - 2\xi^2, \quad 0 \le \xi \le h \\ \xi^2, \end{array}\right. \tag{9}$$

due to the fact that all the remaining quadratic B-splines are exactly zero on the element defined over $[x_m, x_{m+1}]$, we have the right to write the approximation (8) over this element in terms of the basis functions (9) as follows

$$U_N = \sum_{j=m-1}^{m+1} \delta_j\phi_j, \quad V_N = \sum_{j=m-1}^{m+1} \sigma_j\phi_j. \tag{10}$$

When Eqs. (9) and (10) are used, the nodal values U_m, V_m and U'_m, V'_m and at the knot x_m are easily got in terms of the time-dependent element parameters δ_m as follows

$$\begin{aligned} U(x_m) &= U_m = \delta_{m-1} + \delta_m \\ V(x_m) &= V_m = \sigma_{m-1} + \sigma_m \\ U'(x_m) &= U'_m = \frac{2}{h}(\delta_m - \delta_{m-1}) \\ V'(x_m) &= V'_m = \frac{2}{h}(\sigma_m - \sigma_{m-1}). \end{aligned}$$

Let us note that from now on and throughout the manuscript, the prime is going to denote differentiation with respect to x. The corresponding weak form of the Improved Boussinesq equation can be stated by the following integral equations

$$\begin{array}{l} \int_{x_m}^{x_{m+1}} W(U_t - V)\ dx = 0 \\ \int_{x_m}^{x_{m+1}} (WV_t + W_xU_x + W_xV_{xt} + 2W_xUU_x)\ dx = \{WU_x + WV_{xt} + 2WUU_x\}|_{x_m}^{x_{m+1}} \end{array} \tag{11}$$

where W is the weight function. Because of the method used in the solution process, if we take the weight function $W(x)$ as quadratic B-splines and replace (10) into Eq.(11) we result in

$$\sum_{j=m-1}^{m+1}\left\{\left(\int_0^h \phi_i\phi_j d\xi\right)\dot{\delta}_j^e - \left(\int_0^h \phi_i\phi_j d\xi\right)\sigma_j^e\right\} = 0$$

$$\begin{aligned} &\sum_{j=m-1}^{m+1}\left\{\left(\int_0^h \phi_i\phi_j d\xi\right)\dot{\sigma}_j^e + \left(\int_0^h \phi'_i\phi'_j d\xi\right)\delta_j^e + \left(\int_0^h \phi'_i\phi'_j d\xi\right)\dot{\sigma}_j^e + 2\sum_{k=m-1}^{m+1}\left(\int_0^h \phi'_i\phi'_j\phi_k d\xi\right)\delta_j^e\delta_k^e\right\} \\ &\qquad = \sum_{j=m-1}^{m+1}\left\{\left(\phi_i\phi'_j\right)\delta_j^e + \left(\phi_i\phi'_j\right)\dot{\sigma}_j^e + 2\sum_{k=m-1}^{m+1}\left(\phi_i\phi'_j\phi_k\right)\delta_j^e\delta_k^e\right\}\Big|_0^h \end{aligned}$$

which can also be rearranged in a matrix form as follows

$$\begin{array}{l} \dot{\delta}^e - \sigma^e = 0 \\ A^e\dot{\sigma}^e + B^e\delta^e + B^e\dot{\sigma}^e + C^e(\delta^e)\delta^e - D^e\delta^e - D^e\dot{\sigma}^e - E^e(\delta^e)\delta^e = 0. \end{array} \tag{12}$$

Let us note that from now on and throughout the manuscript, the dot is going to denote differentiation with respect to t, $\delta^e = (\delta_{m-1}, \delta_m, \delta_{m+1})$ and $\sigma^e = (\sigma_{m-1}, \sigma_m, \sigma_{m+1})$ are going to be treated as the element parameters, and A^e, B^e, $C^e(\delta^e)$, D^e and $E^e(\delta^e)$ are the element matrices given by the following integrals:

$$\begin{aligned}
&A^e_{ij} = \int_0^h \phi_i \phi_j d\xi, \\
&B^e_{ij} = \int_0^h \phi'_i \phi'_j d\xi, \\
&C^e_{ijk}(\delta^e) = 2\int_0^h \phi'_i \phi'_j \phi_k d\xi, \\
&D^e_{ij} = \phi_i \phi'_j \Big|_0^h \\
&E^e_{ijk}(\delta^e) = 2\left(\phi_i \phi'_j \phi_k\right)\Big|_0^h
\end{aligned} \tag{13}$$

where $i, j, k = m-1, m, m+1$. After simple arrangements, the element matrices (13) can be found as follows

$$\begin{aligned}
A^e_{ij} &= \int_0^h \phi_i \phi_j d\xi = \frac{h}{30}\begin{bmatrix} 6 & 13 & 1 \\ 13 & 54 & 13 \\ 1 & 13 & 6 \end{bmatrix}, \\
B^e_{ij} &= \int_0^h \phi'_i \phi'_j d\xi = \frac{2}{3h}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \\
D^e_{ij} &= \phi_i \phi'_j \Big|_0^h = \frac{2}{h}\begin{bmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & -1 & 1 \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
C^e_{ijk}(\delta^e) &= \frac{h}{15}\begin{bmatrix} (24,-18,-6)\delta & (52,-24,-28)\delta & (4,2,-6)\delta \\ (-18,16,2)\delta & (-24,48,-24)\delta & (2,16,-18)\delta \\ (-6,2,4)\delta & (-28,-24,52)\delta & (-6,-18,24)\delta \end{bmatrix} \\
E^e_{ijk}(\delta^e) &= \frac{4}{h}\begin{bmatrix} (1,-1,0)\delta & (1,-1,0)\delta & (0,0,0)\delta \\ (1,-1,0)\delta & (1,-2,1)\delta & (0,-1,1)\delta \\ (0,0,0)\delta & (0,-1,1)\delta & (0,-1,1)\delta \end{bmatrix}.
\end{aligned}$$

If we combine all the contributions coming from all those elements lying in the solution domain of the problem, Eq.(12) results in the system of equations

$$\begin{aligned}
&\dot{\delta} - \sigma = 0 \\
&(A + B - D)\dot{\sigma} + (B - D + C(\delta) - E(\delta))\delta = 0
\end{aligned} \tag{14}$$

where vectors of the global element parameters are $\delta = (\delta_{-1}, \delta_0, ..., \delta_{N-1}, \delta_N)^T$, $\sigma = (\sigma_{-1}, \sigma_0, ..., \sigma_{N-1}, \sigma_N)^T$ and $A, B, D, C(\delta), E(\delta)$ are global matrices and each generalized m^{th} row of the global matrices is as follows

$$\begin{aligned}
&A : \tfrac{h}{30}(1, 26, 66, 26, 1), \\
&B : \tfrac{2}{3h}(-1, -2, 6, -2, -1), \\
&D : \tfrac{2}{h}(0, 0, 0, 0, 0) \\
&C(\delta) : \tfrac{h}{15}((-6, 2, 4, 0, 0)\,\delta, (-28, -42, 68, 2, 0)\,\delta, (-6, -42, 96, -42, -6)\,\delta, (0, 2, 68, -42, -28)\,\delta, (0, 0, 4, 2, -6)\,\delta), \\
&E(\delta) : \tfrac{4}{h}((0, 0, 0, 0, 0)\,\delta, (0, 0, 0, 0, 0)\,\delta, (0, 0, 0, 0, 0)\,\delta, (0, 0, 0, 0, 0)\,\delta, (0, 0, 0, 0, 0)\,\delta).
\end{aligned}$$

Finally, if we utilize the boundary conditions in (4) the system results in a $N \times N$ matrix system. Now, the initial conditions of the problem for (14) are obtained by taking (3). In that case,we easily obtain

$$U_N(x,0) = \sum_{j=-1}^{N} \delta_j^0 \phi_j, \ V_N(x,0) = \sum_{j=-1}^{N} \sigma_j^0 \phi_j$$

As a result, we will have the system of ordinary differential equations (14) with extra conditions, which can be obtained from $u'(x_0,0) = U'_N(x_0,0)$ and $v'(x_0,0) = V'_N(x_0,0)$. Thanks to the reality that the first derivative of the approximate initial conditions are going to be in harmony with those of the exact ones, the initial vectors δ_j^0 and σ_j^0 can be respectively obtained from the following matrix equations:

$$\begin{bmatrix} \frac{-2}{h} & \frac{2}{h} & & & & & \\ 1 & 1 & & & & & \\ & 1 & 1 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & 1 & 1 \end{bmatrix} \begin{bmatrix} \delta_{-1} \\ \delta_0 \\ \delta_1 \\ \cdot \\ \cdot \\ \cdot \\ \delta_N \end{bmatrix} = \begin{bmatrix} U'_0 \\ U_0 \\ U_1 \\ \cdot \\ \cdot \\ \cdot \\ U_N \end{bmatrix}$$

and

$$\begin{bmatrix} \frac{-2}{h} & \frac{2}{h} & & & & & \\ 1 & 1 & & & & & \\ & 1 & 1 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & 1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{-1} \\ \sigma_0 \\ \sigma_1 \\ \cdot \\ \cdot \\ \cdot \\ \sigma_N \end{bmatrix} = \begin{bmatrix} V'_0 \\ V_0 \\ V_1 \\ \cdot \\ \cdot \\ \cdot \\ V_N \end{bmatrix}.$$

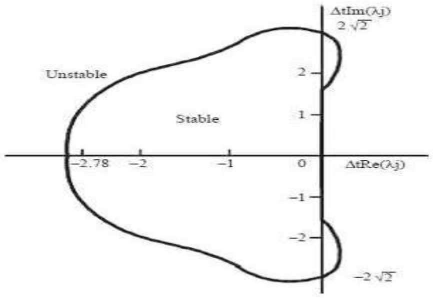
Consequently, the approximate solution functions for $u(x,t)$ and $v(x,t)$ are going to be derived from δ^n and σ^n by utilizing Eq. (14) which can be also be handled out using a classical fourth-order Runge-Kutta method by taking time steps as Δt.

3. Stability Analysis

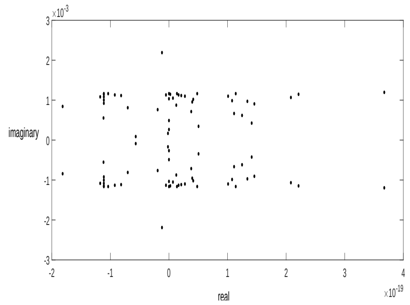
The evolution of dynamical system is a fixed phenomenon for describing the state of system in the future. For determining behaviour of the systems, stability is probably the first and most important problem to dealt with it. Stability plays a significant role for studying on numerical approaches. Especially, for a discreating scheme, numerical stability is the most important concept.For this reason, the purpose of this section is to investigate stability analysis of RK4 which is used to solve IBq. Before consider stability on IBq, let us recall stability theory for a given initial value problem which is expressed in form

$$\frac{dy}{dt} = \tilde{A}y; \qquad y\,(t_0) = y_0 \tag{15}$$

where $\tilde{A}$ is coefficients of right side of the differential equation system expressed as matrix which has N column and N rows. λ_js are eigenvalues of matrix $\tilde{A}$ for $j = 1,2,3,...,N$. The linear stability analysis of time-step methods which applied to (15) is considered in the $\Delta t\lambda_-$ plane. So, the eigenvalues of matrix $\tilde{A}$ determine stability character of the system given in (15). The system is stable if the parameter $\Delta t\lambda$ lies on stability region for $RK4$ illustrated in Fig. 1 (a)[10].



Stability region for the RK4



Eigenvalues for $N = 44$ and $\Delta t = 0.001$

Figure 1: Stability analysis

The concept of linear analysis depend on eigenvalues of linearizing the nonlinear problem. In order to consider linear stability analysis of (14), we linearize the problem by treating the nonlinear term u as a scalar $\alpha = \max\limits_{0\le i\le N-1}\{u_i\}$. So, we get

$$\begin{aligned}\dot{\delta} &= \sigma \\ \dot{\sigma} &= -(A+B-D)^{-1}((1+2\alpha)(B-D))\delta\end{aligned} \tag{16}$$

If the coefficient matrix of (16) is symbolized via a matrix K,

$$K = \begin{bmatrix} 0 & 1 \\ -(A+B-D)^{-1}((1+2\alpha)(B-D)) & 0 \end{bmatrix}_{N\times N}$$

we can rewrite (16) as follows

$$\frac{d}{dt}\left\{\begin{array}{c}\delta\\ \sigma\end{array}\right\} = K\left\{\begin{array}{c}\delta\\ \sigma\end{array}\right\}.$$

Thus, stability of the linearized system are formulated in terms of the eigenvalues of the matrix K . After calculation of eigenvalues of matrix K , we find that big amount of eigenvalues associated with matrix K are complex valued. The place of this eigenvalues illustrated in Fig.1 (b) for $N = 44$ and $\Delta t = 0.001$.

It can be seen from the fig. 1 (b) that complex eigenvalues lie on the stability region of $RK4$. However, some of them have small positive real parts which lie between the stability region and $\Delta t\lambda \le 4.10^{-19}$. In these case, as it referent in [10], if eigenvalues of coefficient matrix have enough small positive real parts, there can be a tolerance for stability. Therefore, the results show that RK4 method which applied to numerical scheme (16) is stable.

4. Numerical Examples and Results

In this section, the equation (2) corresponding to Eqs. (3)-(4) after applying the present method was solved numerically using fourth order Runge-Kutta method for four test problems with various initial and boundary conditions. To be able to demonstrate how good the obtained numerical solutions when compared with the exact ones, we are going to use the error norms L_2 and L_∞ described by

$$\begin{aligned}L_2 &= \sqrt{h\sum_{j=0}^{N}\left|u_j-(U_N)_j\right|^2}, \\ L_\infty &= \|u-U_N\|_\infty = \max_{0\le j\le N}\left|u_j-(U_N)_j\right|.\end{aligned}$$

Table 1: The error norms $L_\infty$and L_2 of obtained numerical solutions of IBq equation for different values of A, Δt and h.

Amp		$h = 0.1$		$h = 0.2$		$h = 0.25$		$h = 0.5$	
	Δt	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$
0.2	0.001	0.002031	0.007553	0.004577	0.011671	0.000991	0.004031	0.001579	0.010850
	0.025	0.002907	0.011037	0.002941	0.008618	0.002871	0.008348	0.002093	0.005628
	0.05	0.011855	0.033356	0.012045	0.032400	0.011965	0.032269	0.011224	0.029730
	0.1	0.048605	0.129915	0.048796	0.129233	0.048826	0.129263	0.047817	0.126512
	0.25	0.305993	0.808817	0.306056	0.808038	0.306006	0.807795	0.304154	0.805242
0.25	0.001	0.002390	0.010282	0.000639	0.001858	0.000980	0.003809	0.002178	0.006369
	0.025	0.005591	0.018199	0.005647	0.015802	0.005557	0.015296	0.004167	0.011082
	0.05	0.022888	0.061957	0.022985	0.060652	0.022962	0.060590	0.021512	0.056025
	0.1	0.092733	0.242742	0.092827	0.241996	0.092767	0.241793	0.090906	0.237292
	0.25	0.580247	1.510986	0.580579	1.511266	0.580728	1.510769	0.576718	1.506435
0.5	0.001	0.003711	0.009214	0.008204	0.052424	0.001400	0.00488	0.015424	0.036146
	0.025	0.042973	0.111140	0.042469	0.106918	0.041959	0.104440	0.034386	0.080297
	0.05	0.169141	0.419592	0.168548	0.415189	0.167993	0.413469	0.160190	0.389052
	0.1	0.674561	1.658423	0.673786	1.654639	0.672754	1.653118	0.664029	1.627687
	0.25	4.186985	10.277500	4.183889	10.273220	4.178461	10.271300	4.170167	10.246360

4.1. Solitary Wave Movement

In this example, we are going to take following solitary type solution of Eq.(2) into consideration

$$u(x,t) = Amp\,\mathrm{sech}^2\left(\frac{1}{c}\sqrt{\frac{Amp}{6}}\,(x-ct-x_0)\right); \quad c = \pm\sqrt{1+\frac{2Amp}{3}} \tag{17}$$

in which Amp is amplitude of the pulse which is initially centered at $x = x_0$ moving with a velocity of c. We have chosen the initial conditions by setting $t = 0$ as follows

$$\begin{aligned} u(x,0) &= Amp\,\mathrm{sech}^2\left(\tfrac{1}{c}\sqrt{\tfrac{Amp}{6}}\,(x-x_0)\right) \\ u_t(x,0) = v(x,0) &= 2Amp\sqrt{\tfrac{Amp}{6}}\,\mathrm{sech}^2\left(\tfrac{1}{c}\sqrt{\tfrac{Amp}{6}}\,(x-x_0)\right)\tanh\left(\tfrac{1}{c}\sqrt{\tfrac{Amp}{6}}\,(x-x_0)\right). \end{aligned}$$

For reasons of comparison with the works in [8] we have taken interval $[-80, 140]$ at which the boundary values have been taken to fit $u \approx 0$. Table 1 shows errors norms L_2 and L_∞ with various amplitudes and mesh sizes at $t = 72$. Figure 2 illustrates the solutions of the solitary wave for $h = 0.5$, $\Delta t = 0.001$ and $Amp = 0.5$ at different time levels. It can clearly be seen from the figure that the solitary wave moves to the right at a constant velocity and almost an unchanged amplitude with increasing time. The amplitude is 0.5 at $t = 0$ which is located at $x = 0$, while it is 0.499398 at $t = 72$ which is located at $x = 83$. The computed velocity of the wave at $t = 72$ is 1.152778, which is in good agreement with the theoretical value 1.154701. Table 2 shows a comparison of the error norms L_∞ obtained by the present method with those of Ref. [8]. From the table, we can observe that error norms L_∞ obtained by the present method are more less than M1, M2 and M3 method and in good agreement with those obtained by M4 in Ref. [8].

4.2. Interaction of Two Solitary Waves

Now, in the second numerical experiment the interaction of two solitary waves is studied by using the following initial conditions;

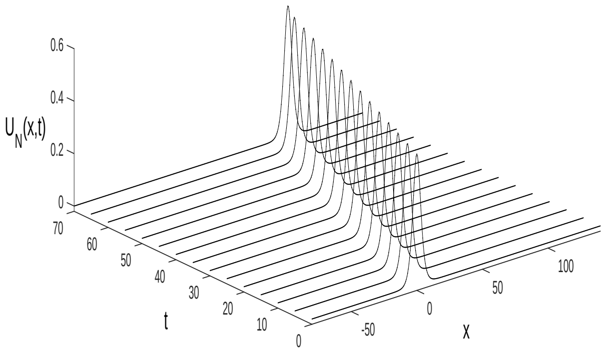


Figure 2: Solitary Wave Movement: $Amp = 0.5$, $\Delta t = 0.001$, $h = 0.5$ and $x_0 = 0$

Table 2: Numerical comparison of L_∞ with those in Ref. [8] for values of $x_0 = 0$ and $h = 0.2$, $\Delta t = 0.001$ at $t = 72$

	present	[8]			
Amp	$L_\infty \times 10^3$	$M1 \times 10^3$	$M2 \times 10^3$	$M3 \times 10^3$	$M4 \times 10^3$
0.25	0.000639	0.27403	0.00164	0.27290	0.00011
0.5	0.008204	1.02472	0.02404	1.004074	0.00080
0.75	0.003077	2.93470	0.13309	2.807873	0.00376
0.9	0.002708	5.00921	0.28976	4.73146	0.00785

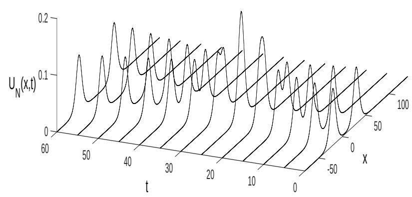
$$u(x,0) = Amp_1 \operatorname{sech}^2\left(\frac{1}{c_1}\sqrt{\frac{Amp_1}{6}}\left(x - x_0^1\right)\right) + Amp_2 \operatorname{sech}^2\left(\frac{1}{c_2}\sqrt{\frac{Amp_2}{6}}\left(x - x_0^2\right)\right)$$
$$u_t(x,0) = v(x,0) = 2Amp_1 \operatorname{sech}^2\left(\frac{1}{c_1}\sqrt{\frac{Amp_1}{6}}\left(x - x_0^1\right)\right)\tanh\left(\frac{1}{c_1}\sqrt{\frac{Amp_1}{6}}\left(x - x_0^1\right)\right)\sqrt{\frac{Amp_1}{6}}$$
$$+2Amp_2 \operatorname{sech}^2\left(\frac{1}{c_2}\sqrt{\frac{Amp_2}{6}}\left(x - x_0^2\right)\right)\tanh\left(\frac{1}{c_2}\sqrt{\frac{Amp_2}{6}}\left(x - x_0^2\right)\right)\sqrt{\frac{Amp_2}{6}}$$

in which $c_1 = \sqrt{1 + \frac{2Amp_1}{3}}$ and $c_2 = -\sqrt{1 + \frac{2Amp_2}{3}}$ represent two solitary waves, respectively. The first one initially located at $x = x_0^1$ and moves to the right side with a velocity of c_1 and amplitude Amp_1, while the second one is initially located $x = x_0^2$ and moves to the left side with a velocity of c_2 and amplitude Amp_2.

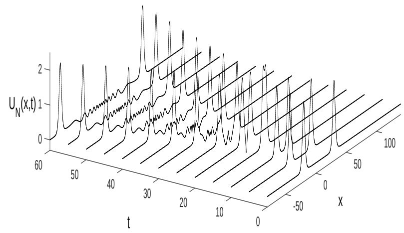
In Figure 3, we have investigated interaction for two situation, the first is for equal amplitudes and the second one is for unequal amplitudes. Then, we have illustrated the interaction for values $x \in [-80, 140]$, $t \in [0, 72]$, $\Delta t = 0.001$ and $h = 0.1$.

For equal amplitudes, we have taken the amplitudes as $Amp_1 = Amp_2 = 0.1$ and $Amp_1 = Amp_2 = 2$ then we have drawn the results in Figs. 3(a)-(b). As, it is seen from these figures that initially waves do not detect each others presence at the beginning. In time, they overlap gradually. After interaction, two waves are seem again then continue traveling. Also, there exists negligible secondary waves behind the solitary waves when amplitude is taken less than or equal to 0.5. However, when the amplitude is taken greater than 0.5, we have also investigated visible the secondary waves(see in Fig. 3(b)).

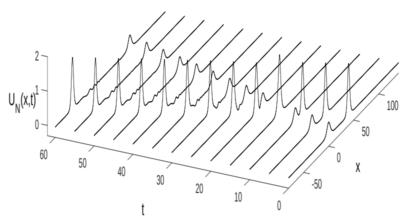
For unequal amplitudes, $Amp_1 = 0.2$, $Amp_2 = 0.4$ and $Amp_1 = 0.4$, $Amp_2 = 1.5$ have chosen and drawn the results in Figs. 3(c)-(d). Waves exhibit similar interaction behavior and similar elasticity results as equal



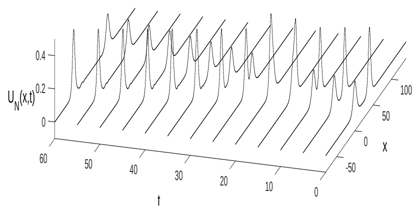
(a) Interaction of two solitary waves: $Amp_1 = 0.1$, $Amp_2 = 0.1$, $\Delta t = 0.001$, $h = 0.1$, $x_0^1 = -20$ and $x_0^2 = 30$.



(b) Interaction of two solitary waves: $Amp_1 = 2$, $Amp_2 = 2, \Delta t = 0.001$, $h = 0.1$, $x_0^1 = -20$ and $x_0^2 = 30$.



(c) Interaction of two solitary waves: $Amp_1 = 0.4$, $Amp_2 = 1.5, \Delta t = 0.001$, $h = 0.1$, $x_0^1 = 0$ and $x_0^2 = 40$.



(d) Interaction of two solitary waves:$Amp_1 = 0.2$, $Amp_2 = 0.4, \Delta t = 0.001$, $h = 0.1$, $x_0^1 = 0$ and $x_0^2 = 40$.

Figure 3: The numerical solutions of two solitary waves during interaction

amplitudes,i.e, if values of amplitudes less than 0.5, the waves act as solitary waves so we can refer that the interaction is clear. For values $Amp > 0.5$, due to the secondary waves which spreads between the interaction area, interaction is not clear. For two situation, we can conclude the nature of wave interaction depends on the amplitudes of interacting waves.

In the literature, there are many article consist of solitary wave interaction for improved Boussinesq equation various choosing of different and equal amplitudes. In Refs. [6, 9] the interactions have discussed for $Amp \ll 1$ and $Amp > 1$. In contrary of the references, we can conclude that for $Amp \leq 0.5$ is clear and for $Amp > 0.5$ is not clear. The newly obtained results are same agreement with those studies in [4, 11, 15, 17].

Besides, in all the conducted experiments, it is also found out that the value of maximum joint amplitude at interaction time is less than the twice of the maximum amplitude of the individual soliton waves in the interaction, as clearly stated in the [4, 6, 9, 11, 15, 17] .

4.3. Wave Break-up

As a third example, we are going to consider the numerical solutions of IBq equation with $f(x)$ the same as defined in Section 4.1 for the first experiment and $g(x) = 0$. The problem is examined on interval $x\epsilon[-80, 140]$ and $t\epsilon[0, 72]$. The initial location of the wave having an amplitude of $Amp = 0.5$ is $x_0 = 30$. Figure 4 illustrates the break-up of the initial stationary wave into two smaller, symmetric solitary waves moving in the opposite directions by leaving oscillating tail behind them. The break-up happens completely at nearly $t = 10$ and the amplitudes of the waves are nearly equal to $Amp = 0.26$.

4.4. Solution Blow-up

In this example, we consider the blow-up solution of Eq.(2) also discussed in [11, 18, 19]. The problem is taken into account on the interval $x \in [0, 1]$ and with the initial conditions

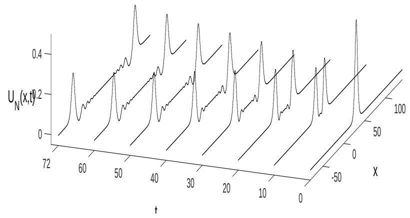


Figure 4: Wave Break up: The numerical solutions when $t \in [0, 70]$.

$$f(x) = -3\sin(\pi x)$$
$$g(x) = -\sin(\pi x).$$

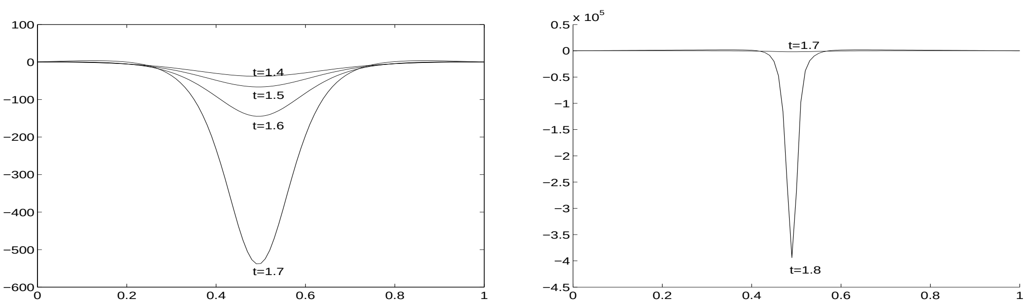
When these conditions are assumed, it is known from [18] that there exists a $T^0 > 0$ such that a unique local solution $u \epsilon C^2([0, T^0) : H^2(0, 1) \cap H_0^1(0, 1))$ exists with the following conditions

$$\|u(., t)\|_{L^2(0,1)} \to +\infty \;\; as \;\; t \to T^0$$

and

$$I(t) = \int_0^1 u(x, t) sin(\pi x) dx \to -\infty \;\; as \;\; t \to T^0.$$

First of all, since the solution changes in large quantities, we have divided the solution interval $[0, 1]$ of the problem into length $h = 0.005$ and have set the time step $\Delta t = 0.00001$, just to generate sensible results until the time $t = 1.8$. The numerical solutions at different times have been demonstrated in Fig. 5. As it is seen from the figure that although there is a small reduction between the times 1.4 and 1.6, this reduction becomes more visible between the times 1.6 and 1.7. In fact, after the time 1.7 there is a huge reduction in the amplitude of wave known also as Blow-up.



(a) (b)

Figure 5: Blow-up solutions: The numerical solutions when $t \in (1.4, 1.6)$ and $t \in (1.7, 1.8)$, respectively.

5. Conclusion

In the present paper, the authors have used Galerkin quadratic B-spline finite element method in order to obtain numerical solutions of Improved Boussinesq equation effectively and efficiently. To show the accuracy of the method, four test problems have been solved using the proposed method. The results also are compared with those available in the literature obtained by different methods and techniques. The comparison shows that the presented results show fairly good agreement with those avaliable in the literature. Thus, the proposed method can also be used to solve a wide range of similar partial differential equations.

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