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On Generalization of Fejér Type Inequalities via Fractional Integral Operators

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Abstract. In this paper we first prove a new lemma for differentiable mapping via a fractional integral operator. Then, using lemma, we establish some new Hermite-Hadamard-Fejer type results for convex functions via fractional integral operators. The results presented here would provide extensions of those given in earlier works.

1. Introduction and Preliminaries

If $f : [a, b] \to \mathbb{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

is known as the Hermite-Hadamard inequality.

Fejér [12] gave a generalization of the inequalities (1) as the following: If $f : [a, b] \to \mathbb{R}$ is a convex function, and $g : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric about $\frac{a+b}{2}$, then

$$f\left(\frac{a+b}{2}\right)\frac{1}{b-a}\int_a^b g(x)dx \leq \frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx. \tag{2}$$

For some results which generalize, improve, and extend the inequalities (1.1) and (1.2) see [21, 25].

In [29], Sarıkaya et al. generalized the Hermite-Hadamard type inequalities via Riemann-Liouville fractional integrals. Then in [17], İşcan extended Sarıkaya's results to Hermite-Hadamard-Fejér type inequalities for fractional integrals. Some other results related to those inequalities involving fractional integrals can be found in the literature, for example, in [23, 24, 26, 29] and the references therein.

The Gronwall inequality and Lyapunov inequalities have been an important role in the field of differential equations. Typical applications involve bounds for eigenvalues, stability criteria for periodic differential

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equations, and estimates for intervals of disconjugacy. Recently, the Gronwall inequality has been generalized for the study of fractional differential equations, with dependence on the Riemann–Liouville fractional derivative and for the Hadamard fractional derivative. Several papers have been presented to give general versions of the Gronwall inequality that is proper for the Katugampola fractional derivative. Several new results, generalizations and improvements can be found in the papers [6, 7, 13–16, 31].

Nowadays fractional calculus is dynamic area of research in mathematics. Various types of fractional integrals were introduced: Riemann-Liouville, Hadamard, conformable and Katugampola are just a few to name [3, 8, 18, 19, 28].

In the following, we will give some necessary definitions and preliminary results which are used and referred to throughout this paper.

Definition 1.1. Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively where $\Gamma(\alpha) = \int_0^{\infty} e^{-t}u^{\alpha-1}du$. Here is $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.

This function $E_{\alpha}(z)$ defined by

$$E_{\rho}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + 1)} \quad (z \in \mathbb{C}; \Re(\rho) > 0),$$

was introduced by Mittag-Leffler [20] and is, therefore, known as th Mittag-Leffler function. More detailed information about this function may be found in the book by Erdélyi et al. ([11], Vol. 3, Section 18.1) and Dzhrbashyan ([9], Chapter III and in [10]).

In [27], Raina introduced a generalized class of Mittag-Leffler functions defined formally by

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x) = \mathcal{F}^{\sigma(0),\sigma(1),\ldots}_{\rho,\lambda}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k \quad (\rho, \lambda > 0; |x| < \mathbf{R}), \tag{3}$$

where the coefficients $\sigma(k)$ $(k \in \mathbb{N} = \mathbb{N} \cup \{0\})$ is a bounded sequence of positive real numbers and $\mathbf{R}$ is the set of real numbers. With the help of (3), Raina [27] and Agarwal et al. [5] defined the following left-sided and right-sided fractional integral operators respectively, as follows:

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi\right)(x) = \int_a^x (x-t)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}[w(x-t)^{\rho}]\varphi(t)dt \qquad (x > a), \tag{4}$$

$$\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi\right)(x) = \int_x^b (t-x)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-x)^{\rho}]\varphi(t)dt \qquad (x < b), \tag{5}$$

where $\lambda, \rho > 0$, $w \in \mathbb{R}$ and $\varphi(t)$ is such that the integral on the right side exits. In recently some new integral inequalities this operator involving have appeared in the literature (see, e.g., [5, 22, 30]).

It is easy to verify that $\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi(x)$ and $\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi(x)$ are bounded integral operators on $L(a,b)$, if

$$\mathfrak{M} := \mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-a)^{\rho}] < \infty. \tag{6}$$

In fact, for $\varphi \in L(a,b)$, we have

$$\|\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi(x)\|_1 \leq \mathfrak{M}(b-a)^{\lambda}\|\varphi\|_1 \tag{7}$$

and

$$\|\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}\varphi(x)\|_1 \leq \mathfrak{M}(b-a)^{\lambda}\|\varphi\|_1 \tag{8}$$

where

$$\|\varphi\|_p := \left(\int_a^b |\varphi(t)|^p dt\right)^{\frac{1}{p}}.$$

Here, many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. For instance the classical Riemann-Liouville fractiona integrals J^{α}_{a+} and J^{α}_{b-} of order α follow easily by setting $\lambda = \alpha$, $\sigma(0) = 1$ and $w = 0$ in (4) and (5).

If f is defined on an interval $[a,b]$, then the action of the $Q-$ operator is defined as $(Qf)(t) = f(a+b-t)$ (*see, e.g.,* [4]). Being symmetric about $\frac{a+b}{2}$ of a function is related to the well-known $Q-$ operator used extensively in fractional calculus for example it has been used recently in [1, 2, 4]. Namely, the meaning of symmetric about $\frac{a+b}{2}$ of f function is $(Qf)(t) = f(t)$ or $f : [a,b] \to \mathbb{R}$ is symmetric to $\frac{a+b}{2}$ if and only if $(Qf)(\frac{a+b}{2}) = f(\frac{a+b}{2})$. Hence if $f(t)$ is symmetric about $\frac{a+b}{2}$, then $QI^{\alpha}_{a^+}f(t) = I^{\alpha}_{b^-}f(t)$.

The following identity is proved by İşcan:

Lemma 1.2. ([17]) *Let $f : [a,b] \to \mathbb{R}$ be differentiable mapping on (a,b) with $a < b$ and $f' \in L[a,b]$. If $g : [a,b] \to \mathbb{R}$ is integrable and symmetric to $\frac{a+b}{2}$, then the following equality for fractional integrals holds*

$$\begin{aligned}&\left(\frac{f(a)+f(b)}{2}\right)[J^{\alpha}_{a+}g(b) + J^{\alpha}_{b-}g(a)] - [J^{\alpha}_{a+}(fg)(b) + J^{\alpha}_{b-}(fg)(a)]\\ =\ & \frac{1}{\Gamma(\alpha)}\int_a^b \left[\int_a^t (b-s)^{\alpha-1}g(s)ds - \int_t^b (s-a)^{\alpha-1}g(s)ds\right] f'(t)dt\end{aligned} \tag{9}$$

with $\alpha > 0$.

İşcan established Fejér type inequalities for convex functions via Lemma 1.2 as follows:

Theorem 1.3. ([17]) *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a < b$. If $|f'|$ is convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$, then the following inequality for fractional integrals holds*

$$\begin{aligned}&\left|\left(\frac{f(a)+f(b)}{2}\right)[J^{\alpha}_{a+}g(b) + J^{\alpha}_{b-}g(a)] - [J^{\alpha}_{a+}(fg)(b) + J^{\alpha}_{b-}(fg)(a)]\right|\\ \leq\ & \frac{(b-a)^{\alpha+1}\|g\|_{\infty}}{(\alpha+1)\Gamma(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)[|f'(a)| + |f'b|]\end{aligned} \tag{10}$$

with $\alpha > 0$.

Theorem 1.4. ([17]) *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a < b$. If $|f'|^q$, $q > 1$, is convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$, then the following inequality for fractional integrals holds*

$$\begin{aligned}&\left|\left(\frac{f(a)+f(b)}{2}\right)[J^{\alpha}_{a+}g(b) + J^{\alpha}_{b-}g(a)] - [J^{\alpha}_{a+}(fg)(b) + J^{\alpha}_{b-}(fg)(a)]\right|\\ \leq\ & \frac{2(b-a)^{\alpha+1}\|g\|_{\infty}}{(b-a)^{\frac{1}{q}}(\alpha+1)\Gamma(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left[\frac{|f'(a)|^q + |f'b|^q}{2}\right]^{\frac{1}{q}}\end{aligned} \tag{11}$$

where $\alpha > 0$ and $\frac{1}{p} + \frac{1}{q}$.

Theorem 1.5. ([17]) *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a < b$. If $|f'|^q$, $q > 1$, is convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$, then the following inequality for fractional integrals holds*

$$\begin{aligned} &\left|\left(\frac{f(a)+f(b)}{2}\right)[J^\alpha_{a+}g(b)+J^\alpha_{b-}g(a)]-[J^\alpha_{a+}(fg)(b)+J^\alpha_{b-}(fg)(a)]\right| \\ \le\ & \frac{2^{\frac{1}{p}}\|g\|_\infty (b-a)^{\alpha+1}}{(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\left(1-\frac{1}{2^{\alpha p}}\right)^{\frac{1}{p}}\left[\frac{|f'(a)|^q+|f'b|^q}{2}\right]^{\frac{1}{q}} \end{aligned} \tag{12}$$

with $\alpha > 0$, where $\frac{1}{p}+\frac{1}{q}=1$.

The purpose of this paper is to derive generalizations of Hermite-Hadamard-Fejér type inequalities obtained by İşcan using the fractional integral operators.

2. Main Results

Throughout this section, let $\|g\|_\infty = \sup_{t\in[a,b]}|g(x)|$, for the continuous function $g : [a,b] \to \mathbb{R}$.

Lemma 2.1. *If $g : [a,b] \to \mathbb{R}$ is integrable and symmetric to $\frac{a+b}{2}$ with $a < b$, then*

$$\mathcal{J}^\sigma_{\rho,\alpha,a+;w}g(b)=\mathcal{J}^\sigma_{\rho,\alpha,b-;w}g(a)=\frac{1}{2}\left[\mathcal{J}^\sigma_{\rho,\alpha,a+;w}g(b)+\mathcal{J}^\sigma_{\rho,\alpha,b-;w}g(a)\right]$$

with $\alpha > 0$.

Proof. Since g is symmetric to $\frac{a+b}{2}$, we have $g(a+b-x)=g(x)$, for all $x \in [a,b]$. Hence, in the following integral setting $x = a+b-t$ and $dx = -dt$ gives

$$\begin{aligned} \mathcal{J}^\sigma_{\rho,\alpha,a+;w}g(b) &= \int_a^b (b-x)^{\alpha-1}\mathcal{F}^\sigma_{\rho,\alpha}[w(b-x)^\rho]g(x)dx \\ &= \int_a^b (t-a)^{\alpha-1}\mathcal{F}^\sigma_{\rho,\alpha}[w(t-a)^\rho]g(a+b-t)dt \\ &= \int_a^b (t-a)^{\alpha-1}\mathcal{F}^\sigma_{\rho,\alpha}[w(t-a)^\rho]g(t)dt \\ &= \mathcal{J}^\sigma_{\rho,\alpha,b-;w}g(a). \end{aligned}$$

This completes the proof. □

Lemma 2.2. *Let $f : [a,b] \to \mathbb{R}$ be differentiable mapping on (a,b) with $a < b$ and $f' \in L[a,b]$. If $g : [a,b] \to \mathbb{R}$ is integrable and symmetric to $\frac{a+b}{2}$, then the following equality for fractional integrals holds:*

$$\begin{aligned} &\left(\frac{f(a)+f(b)}{2}\right)\left[\mathcal{J}^\sigma_{\rho,\alpha,a+;w}g(b)+\mathcal{J}^\sigma_{\rho,\alpha,b-;w}g(a)\right]-\left[\mathcal{J}^\sigma_{\rho,\alpha,a+;w}(fg)(b)+\mathcal{J}^\sigma_{\rho,\alpha,b-;w}(fg)(a)\right] \\ =\ & \int_a^b\left[\int_a^t (b-s)^{\alpha-1}\mathcal{F}^\sigma_{\rho,\alpha}[w(b-s)^\rho]g(s)ds-\int_t^b (s-a)^{\alpha-1}\mathcal{F}^\sigma_{\rho,\alpha}[w(s-a)^\rho]g(s)ds\right]f'(t)dt \end{aligned}$$

with $\alpha > 0$.

Proof. It suffices notice that

$$\begin{aligned} K &= \int_a^b \left[\int_a^t (b-s)^{\alpha-1} \mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}] g(s) ds - \int_t^b (s-a)^{\alpha-1} \mathcal{F}^{\sigma}_{\rho,\alpha}[w(s-a)^{\rho}] g(s) ds \right] f'(t) dt \\ &= \int_a^b \left(\int_a^t (b-s)^{\alpha-1} \mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}] g(s) ds \right) f'(t) dt \\ &\quad + \int_a^b \left(- \int_t^b (s-a)^{\alpha-1} \mathcal{F}^{\sigma}_{\rho,\alpha}[w(s-a)^{\rho}] g(s) ds \right) f'(t) dt \\ &= K_1 + K_2. \end{aligned}$$

By integration by parts and Lemma 2.1, we get

$$\begin{aligned} & K_1 \\ &= \left(\int_a^t (b-s)^{\alpha-1} \mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}] g(s) ds \right) f(t) \Big|_a^b - \int_a^b (b-t)^{\alpha-1} \mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-t)^{\rho}] g(t) f(t) dt \\ &= \left(\int_a^b (b-s)^{\alpha-1} \mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}] g(s) ds \right) f(b) - \int_a^b (b-t)^{\alpha-1} \mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-t)^{\rho}] (fg)(t) dt \\ &= f(b) \mathcal{J}^{\sigma}_{\rho,\alpha,a+;w} g(b) - \mathcal{J}^{\sigma}_{\rho,\alpha,a+;w}(fg)(b) \\ &= \frac{f(b)}{2} \left[\mathcal{J}^{\sigma}_{\rho,\alpha,a+;w} g(b) + \mathcal{J}^{\sigma}_{\rho,\alpha,b-;w} g(a) \right] - \mathcal{J}^{\sigma}_{\rho,\alpha,a+;w}(fg)(b), \end{aligned}$$

and similarly

$$\begin{aligned} & K_2 \\ &= \left(- \int_t^b (s-a)^{\alpha-1} \mathcal{F}^{\sigma}_{\rho,\alpha}[w(s-a)^{\rho}] g(s) ds \right) f(t) \Big|_a^b - \int_a^b (t-a)^{\alpha-1} \mathcal{F}^{\sigma}_{\rho,\alpha}[w(t-a)^{\rho}] g(t) f(t) dt \\ &= \left(\int_a^b (s-a)^{\alpha-1} \mathcal{F}^{\sigma}_{\rho,\alpha}[w(s-a)^{\rho}] g(s) ds \right) f(a) - \int_a^b (t-a)^{\alpha-1} \mathcal{F}^{\sigma}_{\rho,\alpha}[w(t-a)^{\rho}] (fg)(t) dt \\ &= f(a) \mathcal{J}^{\sigma}_{\rho,\alpha,b-;w} g(a) - \mathcal{J}^{\sigma}_{\rho,\alpha,b-;w}(fg)(a) \\ &= \frac{f(a)}{2} \left[\mathcal{J}^{\sigma}_{\rho,\alpha,a+;w} g(b) + \mathcal{J}^{\sigma}_{\rho,\alpha,b-;w} g(a) \right] - \mathcal{J}^{\sigma}_{\rho,\alpha,b-;w}(fg)(a). \end{aligned}$$

Thus, we can write

$$= \left(\frac{f(a)+f(b)}{2} \right) \left[\mathcal{J}^{\sigma}_{\rho,\alpha,a+;w} g(b) + \mathcal{J}^{\sigma}_{\rho,\alpha,b-;w} g(a) \right] - \left[\mathcal{J}^{\sigma}_{\rho,\alpha,a+;w}(fg)(b) + \mathcal{J}^{\sigma}_{\rho,\alpha,b-;w}(fg)(a) \right].$$

This proof is completed. □

Theorem 2.3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a < b$. If $|f'|$ is convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} & \left| \left(\frac{f(a)+f(b)}{2} \right) \left[\mathcal{J}^{\sigma}_{\rho,\alpha,a+;w} g(b) + \mathcal{J}^{\sigma}_{\rho,\alpha,b-;w} g(a) \right] - \left[\mathcal{J}^{\sigma}_{\rho,\alpha,a+;w}(fg)(b) + \mathcal{J}^{\sigma}_{\rho,\alpha,b-;w}(fg)(a) \right] \right| \\ \leq \; & \|g\|_{\infty} (b-a)^{\alpha+1} \mathcal{F}^{\sigma_1}_{\rho,\alpha}[|w|(b-a)^{\rho}] \left[|f'(a)| + |f'(b)| \right] \end{aligned} \tag{13}$$

with $\alpha > 0$

$$\sigma_1(k) := \sigma(k) \frac{1}{(\alpha+\rho k)(\alpha+\rho k+1)} \left(1 - \frac{1}{2^{\alpha+\rho k}} \right).$$

Proof. From Lemma 2.2 we have

$$
\begin{aligned}
&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[\mathcal{J}^{\sigma}_{\rho,\alpha,a+;w}g(b)+\mathcal{J}^{\sigma}_{\rho,\alpha,b-;w}g(a)\right]-\left[\mathcal{J}^{\sigma}_{\rho,\alpha,a+;w}(fg)(b)+\mathcal{J}^{\sigma}_{\rho,\alpha,b-;w}(fg)(a)\right]\right| \\
\leq\ & \int_a^b \left|\int_a^t (b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}]g(s)ds-\int_t^b (s-a)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(s-a)^{\rho}]g(s)ds\right| |f'(t)|dt.
\end{aligned}
\tag{14}
$$

Since $|f'|$ is convex on $[a,b]$, we know that for $t\in[a,b]$

$$
|f'(t)|=\left|f'\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|\leq\frac{b-t}{b-a}|f'(a)|+\frac{t-a}{b-a}|f'(b)| \tag{15}
$$

and since $g:[a,b]\to\mathbb{R}$ is symmetric to $\frac{a+b}{2}$ we can write

$$
\begin{aligned}
\int_t^b (s-a)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(s-a)^{\rho}]g(s)ds &= \int_a^{a+b-t}(b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}]g(a+b-s)ds \\
&= \int_a^{a+b-t}(b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}]g(s)ds,
\end{aligned}
$$

then we have

$$
\begin{aligned}
&\left|\int_a^t (b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}]g(s)ds-\int_t^b (s-a)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(s-a)^{\rho}]g(s)ds\right| \\
=\ & \left|\int_t^{a+b-t}(b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}]g(s)ds\right| \\
\leq\ & \begin{cases} \int_t^{a+b-t}\left|(b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}]g(s)\right|ds, & t\in\left[a,\frac{a+b}{2}\right] \\ \int_{a+b-t}^{t}\left|(b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}]g(s)\right|ds, & t\in\left[\frac{a+b}{2},b\right] \end{cases}
\end{aligned}
\tag{16}
$$

By a combination of (14), (15) and (16), we get

$$
\begin{aligned}
&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[\mathcal{J}^{\sigma}_{\rho,\alpha,a+;w}g(b)+\mathcal{J}^{\sigma}_{\rho,\alpha,b-;w}g(a)\right]-\left[\mathcal{J}^{\sigma}_{\rho,\alpha,a+;w}(fg)(b)+\mathcal{J}^{\sigma}_{\rho,\alpha,b-;w}(fg)(a)\right]\right| \\
\leq\ & \int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|(b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}]g(s)\right|ds\right)\left(\frac{b-t}{b-a}|f'(a)|+\frac{t-a}{b-a}|f'(b)|\right)dt \\
&+\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}\left|(b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}]g(s)\right|ds\right)\left(\frac{b-t}{b-a}|f'(a)|+\frac{t-a}{b-a}|f'(b)|\right)dt
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \frac{\|g\|_\infty}{b-a} \\
& \times\left\{ \int_a^{\frac{a+b}{2}} \left(\int_t^{a+b-t} (b-s)^{\alpha-1} \left(\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k}{\Gamma(\rho k+\alpha)} (b-s)^{\rho k} ds \right) \right) [(b-t)|f'(a)| + (t-a)|f'(b)|]\, dt \right. \\
& \left. + \int_{\frac{a+b}{2}}^{b} \left(\int_{a+b-t}^{t} (b-s)^{\alpha-1} \left(\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k}{\Gamma(\rho k+\alpha)} (b-s)^{\rho k} ds \right) \right) [(b-t)|f'(a)| + (t-a)|f'(b)|]\, dt \right\} \\
=\ & \frac{\|g\|_\infty}{b-a} \left\{ \int_a^{\frac{a+b}{2}} \sum_{k=0}^{\infty} \left(\int_t^{a+b-t} (b-s)^{\alpha+\rho k-1} ds \right) \frac{\sigma(k)|w|^k}{\Gamma(\rho k+\alpha)} [(b-t)|f'(a)| + (t-a)|f'(b)|]\, dt \right. \\
& \left. + \int_{\frac{a+b}{2}}^{b} \sum_{k=0}^{\infty} \left(\int_{a+b-t}^{t} (b-s)^{\alpha+\rho k-1} ds \right) \frac{\sigma(k)|w|^k}{\Gamma(\rho k+\alpha)} [(b-t)|f'(a)| + (t-a)|f'(b)|]\, dt \right\} \\
=\ & \frac{\|g\|_\infty}{b-a} \\
& \times\left\{ \int_a^{\frac{a+b}{2}} \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k}{\Gamma(\rho k+\alpha+1)} \left[(b-t)^{\alpha+\rho k} - (t-a)^{\alpha+\rho k} \right] [(b-t)|f'(a)| + (t-a)|f'(b)|]\, dt \right. \\
& \left. + \int_{\frac{a+b}{2}}^{b} \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k}{\Gamma(\rho k+\alpha+1)} \left[(t-a)^{\alpha+\rho k} - (b-t)^{\alpha+\rho k} \right] [(b-t)|f'(a)| + (t-a)|f'(b)|]\, dt \right\} \\
=\ & \frac{\|g\|_\infty}{b-a} \qquad (17) \\
& \times\left\{ \sum_{k=0}^{\infty} \left(\int_a^{\frac{a+b}{2}} \left[(b-t)^{\alpha+\rho k} - (t-a)^{\alpha+\rho k} \right] [(b-t)|f'(a)| + (t-a)|f'(b)|]\, dt \right) \frac{\sigma(k)|w|^k}{\Gamma(\rho k+\alpha+1)} \right. \\
& \left. + \sum_{k=0}^{\infty} \left(\int_{\frac{a+b}{2}}^{b} \left[(t-a)^{\alpha+\rho k} - (b-t)^{\alpha+\rho k} \right] [(b-t)|f'(a)| + (t-a)|f'(b)|]\, dt \right) \frac{\sigma(k)|w|^k}{\Gamma(\rho k+\alpha+1)} \right\}.
\end{aligned}
$$

By integration by parts, we get

$$
\begin{aligned}
& \int_a^{\frac{a+b}{2}} \left[(b-t)^{\alpha+\rho k} - (t-a)^{\alpha+\rho k} \right] (b-t) dt \qquad (18) \\
=\ & \int_{\frac{a+b}{2}}^{b} \left[(t-a)^{\alpha+\rho k} - (b-t)^{\alpha+\rho k} \right] (t-a) dt \\
=\ & \left(\frac{b-a}{2} \right)^{\alpha+\rho k+2} \frac{1}{(\alpha+\rho k+1)(\alpha+\rho k+2)} \left[(2^{\alpha+\rho k+2} - 1)(\alpha+\rho k+1) - (\alpha+\rho k+3) \right]
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_a^{\frac{a+b}{2}} \left[(b-t)^{\alpha+\rho k} - (t-a)^{\alpha+\rho k} \right] (t-a) dt \qquad (19) \\
=\ & \int_{\frac{a+b}{2}}^{b} \left[(t-a)^{\alpha+\rho k} - (b-t)^{\alpha+\rho k} \right] (b-t) dt \\
=\ & \left(\frac{b-a}{2} \right)^{\alpha+\rho k+2} \frac{1}{(\alpha+\rho k+1)(\alpha+\rho k+2)} \left[(2^{\alpha+\rho k+2}) - 2(\alpha+\rho k+2) \right].
\end{aligned}
$$

Hence, if we use (18) and (19) in (17), we obtain the desired result. This completes the proof. □

Remark 2.4. In Theorem 2.3, if we take $\sigma(0) = 1$ and $w = 0$, then the inequality (13), becomes the inequality (10) of Theorem 1.3.

Theorem 2.5. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a < b$. If $|f'|^q$, $q > 1$, is convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[\mathcal{J}^{\sigma}_{\rho,\alpha,a+;w}g(b)+\mathcal{J}^{\sigma}_{\rho,\alpha,b-;w}g(a)\right]-\left[\mathcal{J}^{\sigma}_{\rho,\alpha,a+;w}(fg)(b)+\mathcal{J}^{\sigma}_{\rho,\alpha,b-;w}(fg)(a)\right]\right| \\
\leq\ &\frac{2\|g\|_\infty(b-a)^{\alpha+1}}{(b-a)^{\frac{1}{q}}}\mathcal{F}^{\sigma_1}_{\rho,\alpha}[|w|(b-a)^\rho]\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}} \qquad (20)
\end{aligned}$$

where $\alpha > 0$, $\frac{1}{p} + \frac{1}{q}$ and

$$\sigma_1(k) := \sigma(k)\frac{1}{(\alpha+\rho k)(\alpha+\rho k+1)}\left(1-\frac{1}{2^{\alpha+\rho k}}\right).$$

Proof. Using Lemma 2.2, Hölder's inequality, (16) and the convexity of $|f'|^q$, it follows that

$$\begin{aligned}
&\left|\left(\frac{f(a)+f(b)}{2}\right)\left[\mathcal{J}^{\sigma}_{\rho,\alpha,a+;w}g(b)+\mathcal{J}^{\sigma}_{\rho,\alpha,b-;w}g(a)\right]-\left[\mathcal{J}^{\sigma}_{\rho,\alpha,a+;w}(fg)(b)+\mathcal{J}^{\sigma}_{\rho,\alpha,b-;w}(fg)(a)\right]\right| \\
\leq\ &\left(\int_a^b\left|\int_t^{a+b-t}(b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^\rho]g(s)ds\right|dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_a^b\left|\int_t^{a+b-t}(b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^\rho]g(s)ds\right||f'(t)|^q\right)^{\frac{1}{q}} \\
\leq\ &\left[\left(\int_a^{\frac{a+b}{2}}\left|\int_t^{a+b-t}(b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^\rho]g(s)\right|ds\right)dt\right. \\
&\left.+\int_{\frac{a+b}{2}}^b\left(\int_{a+b-t}^t\left|(b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^\rho]g(s)\right|ds\right)dt\right]^{1-\frac{1}{q}} \\
&\times\left[\left(\int_a^{\frac{a+b}{2}}\int_t^{a+b-t}\left|(b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^\rho]g(s)\right|ds\right)|f'(t)|^q dt\right. \\
&\left.+\int_{\frac{a+b}{2}}^b\left(\int_{a+b-t}^t\left|(b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^\rho]g(s)\right|ds\right)|f'(t)|^q dt\right]^{\frac{1}{q}} \\
\leq\ &\frac{\|g\|_\infty}{(b-a)^{\frac{1}{q}}}\left\{\int_a^{\frac{a+b}{2}}\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\rho k+\alpha)}\left(\int_t^{a+b-t}(b-s)^{\rho k}(b-s)^{\alpha-1}ds\right)dt\right. \\
&\left.+\int_{\frac{a+b}{2}}^b\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\rho k+\alpha)}\left(\int_{a+b-t}^t(b-s)^{\rho k}(b-s)^{\alpha-1}ds\right)dt\right\}^{1-\frac{1}{q}} \\
&\times\left[\int_a^{\frac{a+b}{2}}\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\rho k+\alpha)}\left(\int_t^{a+b-t}(b-s)^{\rho k}(b-s)^{\alpha-1}ds\right)[(b-t)|f'(a)|^q+(t-a)|f'(b)|^q]\,dt\right. \\
&\left.+\int_{\frac{a+b}{2}}^b\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\rho k+\alpha)}\left(\int_{a+b-t}^t(b-s)^{\rho k}(b-s)^{\alpha-1}ds\right)[(b-t)|f'(a)|^q+(t-a)|f'(b)|^q]\,dt\right]^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
= \quad & \frac{\|g\|_\infty}{(b-a)^{\frac{1}{q}}}\left\{\int_a^{\frac{a+b}{2}}\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\rho k+\alpha)}\left(\int_t^{a+b-t}(b-s)^{\alpha+\rho k-1}ds\right)dt\right. \qquad (21)\\
& \left.+\int_{\frac{a+b}{2}}^{b}\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\rho k+\alpha)}\left(\int_{a+b-t}^{t}(b-s)^{\alpha+\rho k-1}ds\right)dt\right\}^{1-\frac{1}{q}}\\
& \times\left[\int_a^{\frac{a+b}{2}}\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\rho k+\alpha)}\left(\int_t^{a+b-t}(b-s)^{\alpha+\rho k-1}ds\right)\left[(b-t)|f'(a)|^q+(t-a)|f'(b)|^q\right]dt\right.\\
& \left.+\int_{\frac{a+b}{2}}^{b}\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k}{\Gamma(\rho k+\alpha)}\left(\int_{a+b-t}^{t}(b-s)^{\alpha+\rho k-1}ds\right)\left[(b-t)|f'(a)|^q+(t-a)|f'(b)|^q\right]dt\right]^{\frac{1}{q}}
\end{aligned}
$$

where it is easily seen that

$$
\begin{aligned}
& \int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}(b-s)^{\alpha+\rho k-1}ds\right)dt+\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}(b-s)^{\alpha+\rho k-1}ds\right)dt\\
= \quad & \frac{2(b-a)^{\alpha+\rho k+1}}{(\alpha+\rho k)(\alpha+\rho k+1)}\left[1-\frac{1}{2^{\alpha+\rho k}}\right].
\end{aligned}
$$

Hence, if we use (18) and (19) in (21), we obtain the desired result. This completes the proof. □

Remark 2.6. In Theorem 2.5, if we take $\sigma(0)=1$ and $w=0$, then the inequality (20), becomes the inequality (11) of Theorem 1.4.

Theorem 2.7. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I° and $f'\in L[a,b]$ with $a<b$. If $|f'|^q$, $q>1$, is convex on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ is continuous and symmetric to $\frac{a+b}{2}$, then the following inequality for fractional integrals holds:*

$$
\begin{aligned}
& \left|\left(\frac{f(a)+f(b)}{2}\right)\left[\mathcal{J}^{\sigma}_{\rho,\alpha,a+;w}g(b)+\mathcal{J}^{\sigma}_{\rho,\alpha,b-;w}g(a)\right]-\left[\mathcal{J}^{\sigma}_{\rho,\alpha,a+;w}(fg)(b)+\mathcal{J}^{\sigma}_{\rho,\alpha,b-;w}(fg)(a)\right]\right|\\
\leq \quad & \|g\|_\infty(b-a)^{\alpha+1}\left(\mathcal{F}^{\sigma_1}_{\rho,\alpha}[|w|(b-a)^\rho]\right)\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}} \qquad (22)
\end{aligned}
$$

where $\alpha>0$, $\frac{1}{p}+\frac{1}{q}$ and

$$\sigma_1(k):=\sigma(k)\frac{1}{\alpha+\rho k}\left[\frac{2}{(\alpha+\rho k)p+1}\left(1-\frac{1}{2^{(\alpha+\rho k)p}}\right)\right]^{\frac{1}{p}}.$$

Proof. Using Lemma 2.2, Hölder's inequality, (16) and the convexity of $|f'|^q$, it follows that

$$
\begin{aligned}
& \left|\left(\frac{f(a)+f(b)}{2}\right)\left[\mathcal{J}^{\sigma}_{\rho,\alpha,a+;w}g(b)+\mathcal{J}^{\sigma}_{\rho,\alpha,b-;w}g(a)\right]-\left[\mathcal{J}^{\sigma}_{\rho,\alpha,a+;w}(fg)(b)+\mathcal{J}^{\sigma}_{\rho,\alpha,b-;w}(fg)(a)\right]\right|\\
= \quad & \left|\int_a^b\left(\int_a^t(b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^\rho]g(s)ds-\int_t^b(s-a)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(s-a)^\rho]g(s)ds\right)f'(t)dt\right|
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_a^b \left| \int_a^t (b-s)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(b-s)^{\rho}]g(s)ds - \int_t^b (s-a)^{\alpha-1}\mathcal{F}^{\sigma}_{\rho,\alpha}[w(s-a)^{\rho}]g(s)ds \right| |f'(t)dt| \\
&= \int_a^b \left| \sum_{k=0}^{\infty} \frac{\sigma(k)w^k}{\Gamma(\alpha+\rho k)} \int_a^t (b-s)^{\alpha+\rho k-1}g(s)ds - \sum_{k=0}^{\infty} \frac{\sigma(k)w^k}{\Gamma(\alpha+\rho k)} \int_t^b (s-a)^{\alpha+\rho k-1}g(s)ds \right| |f'(t)|dt \\
&= \int_a^b \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)} \left| \int_a^t (b-s)^{\alpha+\rho k-1}g(s)ds - \int_t^b (s-a)^{\alpha+\rho k-1}g(s)ds \right| |f'(t)|dt \\
&= \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)} \int_a^b \left| \int_t^{a+b-t} (b-s)^{\alpha+\rho k-1}g(s)ds \right| |f'(t)|dt \\
&\le \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k)} \left(\int_a^b \left| \int_t^{a+b-t} (b-s)^{\alpha+\rho k-1}g(s)ds \right|^p dt \right)^{\frac{1}{p}} \left(\int_a^b |f'(t)|^q dt \right)^{\frac{1}{q}} \\
&\le \|g\|_{\infty} \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k}{\Gamma(\alpha+\rho k+1)} \\
&\quad \times \Bigg\{ \left(\int_a^{\frac{a+b}{2}} \left[(b-t)^{\alpha+\rho k} - (t-a)^{\alpha+\rho k} \right]^p dt + \int_{\frac{a+b}{2}}^b \left[(t-a)^{\alpha+\rho k} - (b-t)^{\alpha+\rho k} \right]^p dt \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_a^b \left(\frac{b-t}{b-a}|f'(a)|^q + \frac{t-a}{b-a}|f'(b)|^q \right) dt \right)^{\frac{1}{q}} \Bigg\} \\
&= \|g\|_{\infty}(b-a)^{\alpha+1} \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k(b-a)^{\rho k}}{\Gamma(\alpha+\rho k+1)} \\
&\quad \times \Bigg\{ \left(\int_0^{\frac{1}{2}} \left[(1-t)^{\alpha+\rho k} - t^{\alpha+\rho k} \right]^p dt + \int_{\frac{1}{2}}^1 \left[t^{\alpha+\rho k} - (1-t)^{\alpha+\rho k} \right]^p dt \right)^{\frac{1}{p}} \\
&\quad \times \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}} \Bigg\} \\
&\le \|g\|_{\infty}(b-a)^{\alpha+1} \left(\mathcal{F}^{\sigma_1}_{\rho,\alpha}[|w|(b-a)^{\rho}] \right)^{\frac{1}{p}} \left(1 - \frac{1}{2^{(\alpha+\rho k)p}} \right)^{\frac{1}{p}} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}},
\end{aligned}
$$

where it is easily seen that

$$
\begin{aligned}
\int_0^{\frac{1}{2}} \left[(1-t)^{(\alpha+\rho k)p} - t^{(\alpha+\rho k)p} \right] dt &= \int_{\frac{1}{2}}^1 \left[t^{(\alpha+\rho k)p} - (1-t)^{(\alpha+\rho k)p} \right] dt \\
&= \frac{1 - \left(\frac{1}{2}\right)^{(\alpha+\rho k)p+1} - \left(\frac{1}{2}\right)^{(\alpha+\rho k)p+1}}{(\alpha+\rho k)p+1}.
\end{aligned}
$$

Here we use

$$[(1-t)^{\alpha+\rho k} - t^{\alpha+\rho k}]^p \le (1-t)^{(\alpha+\rho)p} - t^{(\alpha+\rho)p}$$

for $t \in \left[0, \frac{1}{2}\right]$ and

$$[t^{\alpha+\rho k} - (1-t)^{\alpha+\rho k}]^p \le t^{(\alpha+\rho)p} - (1-t)^{(\alpha+\rho)p}$$

for $t \in \left[\frac{1}{2}, 1\right]$, which follows from

$$(A-B)^q \le A^q - B^q,$$

for any $A \ge B \ge 0$ and $q \ge 1$. Hence the inequality (22) is proved. □

Remark 2.8. In Theorem 2.7, if we take $\sigma(0) = 1$ and $w = 0$, then the inequality (22), becomes the inequality (12) of Theorem 1.5.

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