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Some Results on Generalized Derivations and $(\sigma,\tau)-$ Lie Ideals

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Abstract. Let R be a prime ring with characteristic not 2 and $\sigma,\tau,\alpha,\beta,\lambda,\mu,\gamma$ automorphisms of R. Let $h : R \longrightarrow R$ be a nonzero left(resp.right)-generalized $(\alpha,\beta)-$derivation, $b \in R$ and $V \neq 0$ a left $(\sigma,\tau)-$Lie ideal of R. The main object in this article is to study the situations. (1) $h(I) \subset C_{\lambda,\mu}(V)$, (2) $bh(I) \subset C_{\lambda,\mu}(V)$ or $h(I)b \subset C_{\lambda,\mu}(V)$, (3) $h\lambda(V) = 0$, (4) $h\lambda(V)b = 0$ or $bh\lambda(V) = 0$.

1. Introduction

Let R be a ring and σ,τ two mappings of R. For each $r,s \in R$ we set $[r,s]_{\sigma,\tau} = r\sigma(s) - \tau(s)r$ and $(r,s)_{\sigma,\tau} = r\sigma(s) + \tau(s)r$. Let U be an additive subgroup of R. If $[U,R] \subset U$ then U is called a Lie ideal of R. The definition of $(\sigma,\tau)-$Lie ideal of R is introduced in [7] as follows: (i) U is called a right $(\sigma,\tau)-$Lie ideal of R if $[U,R]_{\sigma,\tau} \subset U$, (ii) U is called a left $(\sigma,\tau)-$Lie ideal if $[R,U]_{\sigma,\tau} \subset U$. (iii) U is called a $(\sigma,\tau)-$Lie ideal if U is both right and left $(\sigma,\tau)-$Lie ideal of R. Every Lie ideal of R is a $(1,1)-$Lie ideal of R, where $1 : R \to R$ is identity map. If $R = \{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \mid x$ and y are integers$\}$, $U = \{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \mid x$ is integer$\}$, $\sigma\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$ and $\tau\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & -y \\ 0 & 0 \end{pmatrix}$ then U is right $(\sigma,\tau)-$Lie ideal but not a Lie ideal of R.

A derivation d is an additive mapping on R which satisfies $d(rs) = d(r)s + rd(s), \forall r,s \in R$. The notion of generalized derivation was introduced by Brešar [2] as follows. An additive mapping $h : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that $h(xy) = h(x)y + xd(y)$,for all $x,y \in R$.

An additive mapping $d : R \to R$ is said to be a $(\sigma,\tau)-$derivation if $d(rs) = d(r)\sigma(s) + \tau(r)d(s)$ for all $r,s \in R$. Every derivation $d : R \to R$ is a $(1,1)-$derivation. Chang [3] gave the following definition. Let R be a ring, σ and τ automorphisms of R and d a $(\sigma,\tau)-$derivation of R. An additive mapping $h : R \to R$ is said to be a right generalized $(\sigma,\tau)-$derivation of R associated with d if $h(xy) = h(x)\sigma(y) + \tau(x)d(y)$, for all $x,y \in R$ and h is said to be a left generalized $(\sigma,\tau)-$derivation of R associated with d if $h(xy) = d(x)\sigma(y) + \tau(x)h(y)$, for all $x,y \in R$. h is said to be a generalized $(\sigma,\tau)-$derivation of R associated with d if it is both a left and right generalized $(\sigma,\tau)-$derivation of R associated with d.

According to Chang's definition, every $(\sigma,\tau)-$derivation $d : R \to R$ is a generalized $(\sigma,\tau)-$derivation associated with d and every derivation $d : R \to R$ is a generalized $(1,1)-$derivation associated with d. A generalized $(1,1)-$derivation is simply called a generalized derivation. The definition of generalized derivation given in Bresar [2] is a right generalized derivation associated with derivation d according to Chang's definition.

The mapping defined by $h(r) = [r,a]_{\sigma,\tau}, \forall r \in R$ is a right-generalized derivation associated with derivation $d(r) = [r,\sigma(a)], \forall r \in R$ and left-generalized derivation associated with derivation $d_1(r) = [r,\tau(a)], \forall r \in R$.

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The mapping $h(r) = (a, r)_{\sigma,\tau}, \forall r \in R$ is a left-generalized $(\sigma, \tau)-$derivation associated with $(\sigma, \tau)-$derivation $d_1(r) = [a, r]_{\sigma,\tau}, \forall r \in R$ and right-generalized $(\sigma, \tau)-$derivation associated with $(\sigma, \tau)-$derivation $d(r) = -[a, r]_{\sigma,\tau}, \forall r \in R$.

The following result is given in [5]. Let U be a nonzero left $(\sigma, \tau)-$Lie ideal of R and $d : R \longrightarrow R$ a nonzero $(\alpha, \beta)-$derivation. If $d(U) = 0$ then $\sigma(v) + \tau(v) \in Z$ for all $v \in U$. We generalized this result as follows. Let $h : R \longrightarrow R$ be a nonzero left-generalized $(\alpha, \beta)-$derivation associated with a nonzero $(\alpha, \beta)-$derivation $d : R \longrightarrow R$ and V a nonzero left $(\sigma, \tau)-$Lie ideal of R. If $h\lambda(V) = 0$ then $\sigma(v) + \tau(v) \in Z, \forall v \in V$. Kyoo-Hong Park and Yong-Soo Jung [9] proved the following. Let R be a prime ring with characteristic different from two and d be a nonzero $(\alpha, \beta)-$derivation of R. Let U be a left $(\sigma, \tau)-$Lie ideal. If $d[R, U]_{\sigma,\tau} = 0$ then $\sigma(u) + \tau(u) \in Z$ for all $u \in U$. Replacing d with a nonzero left-generalized $(\alpha, \beta)-$derivation $h : R \longrightarrow R$ and a nonzero ideal I with R we generalized this result .

In this paper, we give some other results about left (resp. right)-generalized $(\alpha, \beta)-$derivation on left $(\sigma, \tau)-$Lie ideals of R.

Throughout the paper, R will be a prime ring with characteristic not 2 and $\sigma, \tau, \alpha, \beta, \lambda, \mu, \gamma$ automorphisms of R. We set $C_{\sigma,\tau} = \{c \in R \mid c\sigma(r) = \tau(r)c, \forall\, r \in R\}$, and shall use the following relations frequently:

$[rs, t]_{\sigma,\tau} = r[s, t]_{\sigma,\tau} + [r, \tau(t)]s = r[s, \sigma(t)] + [r, t]_{\sigma,\tau}s$

$[r, st]_{\sigma,\tau} = \tau(s)[r, t]_{\sigma,\tau} + [r, s]_{\sigma,\tau}\sigma(t)$

$(rs, t)_{\sigma,\tau} = r(s, t)_{\sigma,\tau} - [r, \tau(t)]s = r[s, \sigma(t)] + (r, t)_{\sigma,\tau}s$

$(r, st)_{\sigma,\tau} = \tau(s)(r, t)_{\sigma,\tau} + [r, s]_{\sigma,\tau}\sigma(t) = -\tau(s)[r, t]_{\sigma,\tau} + (r, s)_{\sigma,\tau}\sigma(t)$

2. Results

Lemma 2.1. ([1, Lemma 1]) *Let R be a prime ring and $d : R \longrightarrow R$ a $(\sigma, \tau)-$derivation. If U is a right ideal of R and $d(U) = 0$ then $d = 0$.*

Lemma 2.2. ([8, Lemma 4]) *If a prime ring contains a nonzero commutative right ideal then R is commutative.*

Lemma 2.3. ([6, Theorem 2]) *Let V be a noncentral left $(\sigma, \tau)-$Lie ideal of R. Then there exist a nonzero ideal M of R such that $([R, M]_{\sigma,\tau} \subset U$ and $[R, M]_{\sigma,\tau} \not\subseteq C_{\sigma,\tau})$ or $\sigma(v) + \tau(v) \in Z$ for all $v \in V$.*

Lemma 2.4. ([4, Lemma 7]) *Let $h : R \longrightarrow R$ be a nonzero right-generalized $(\sigma, \tau)-$derivation associated with a nonzero $(\sigma, \tau)-$derivation d and let I be a nonzero ideal of R. If $a, b \in R$ such that $[ah(I), b]_{\lambda,\mu} = 0$ then $[a, \mu(b)]a = 0$ or $d\sigma^{-1}\lambda(b) = 0$.*

Lemma 2.5. *Let I be a nonzero ideal of R and $a, b, c \in R$.*
(i) If $b\gamma[I, a]_{\lambda,\mu} = 0$ then $a \in Z$ or $b = 0$.
(ii) If $c\gamma[I, a]_{\lambda,\mu}b = 0$ then $c = 0$ or $[b, \gamma\lambda(a)]b = 0$.

Proof. (i) If $b\gamma[I, a]_{\lambda,\mu} = 0$ then we have for all $r \in R, x \in I$

$$0 = b\gamma[xr, a]_{\lambda,\mu} = b\gamma(x)\gamma[r, \lambda(a)] + b\gamma[x, a]_{\lambda,\mu}\gamma(r) = b\gamma(x)\gamma[r, \lambda(a)].$$

That is

$$b\gamma(I)\gamma[R, \lambda(a)] = 0.$$

Since $\gamma(I)$ is a nonzero ideal of R then we obtain $b = 0$ or $[R, \lambda(a)] = 0$. That is $b = 0$ or $a \in Z$.

(ii) If $c\gamma[I, a]_{\lambda,\mu}b = 0$ then we get for all $x \in I$

$$0 = c\gamma[x\gamma^{-1}(b), a]_{\lambda,\mu}b = c\gamma(x)\gamma[\gamma^{-1}(b), \lambda(a)]b + c\gamma[x, a]_{\lambda,\mu}bb$$
$$= c\gamma(x)\gamma[\gamma^{-1}(b), \lambda(a)]b.$$

That is

$$c\gamma(I)\gamma[\gamma^{-1}(b), \lambda(a)]b = 0.$$

This gives that $c = 0$ or $[b, \gamma\lambda(a)]b = 0$. □

Lemma 2.6. *Let $h : R \longrightarrow R$ be a nonzero left-generalized $(\alpha,\beta)-$derivation associated with a nonzero $(\alpha,\beta)-$derivation $d : R \longrightarrow R$ and I be a nonzero ideal of R. If $b \in R$ such that $h\gamma[I,b]_{\lambda,\mu} = 0$ then $b \in Z$ or $d\gamma\mu(b) = 0$.*

Proof. If $h\gamma[I,b]_{\lambda,\mu} = 0$ then we get for all $x \in I$

$$\begin{aligned} 0 &= h\gamma[\mu(b)x,b]_{\lambda,\mu} = h\gamma(\mu(b)[x,b]_{\lambda,\mu}) = h(\gamma\mu(b)\gamma[x,b]_{\lambda,\mu}) \\ &= d\gamma\mu(b)\alpha\gamma[x,b]_{\lambda,\mu} + \beta\gamma\mu(b)h\gamma[x,b]_{\lambda,\mu} = d\gamma\mu(b)\alpha\gamma[x,b]_{\lambda,\mu}. \end{aligned}$$

That is

$$d\gamma\mu(b)\alpha\gamma[I,b]_{\lambda,\mu} = 0.$$

Using Lemma 2.5 (i) and the last relation we obtain that $d\gamma\mu(b) = 0$ or $b \in Z$. □

Theorem 2.7. *Let $h : R \longrightarrow R$ be a nonzero left-generalized $(\alpha,\beta)-$derivation associated with a nonzero $(\alpha,\beta)-$derivation $d : R \longrightarrow R$ and I, M be nonzero ideals of R. Let V be a left $(\sigma,\tau)-$Lie ideal of R.*

(i) If $h\gamma[I,M]_{\lambda,\mu} = 0$ then R is commutative.

(ii) If $h\lambda(V) = 0$ then $\sigma(v) + \tau(v) \in Z, \forall v \in V$.

(iii) If $h\gamma[I,V]_{\lambda,\mu} = 0$ then $\sigma(v) + \tau(v) \in Z$ for all $v \in V$.

Proof. (i) If $h\gamma[I,m]_{\lambda,\mu} = 0, \forall m \in M$ then using Lemma 2.6, for any $m \in M$ we get

$$m \in Z \text{ or } d\gamma\mu(m) = 0.$$

Let $K = \{m \in M \mid m \in Z\}$ and $L = \{m \in M \mid d\gamma\mu(m) = 0\}$. Then K and L are additive proper subgroups of M moreover $M = K \cup L$. Then it must be $M = K$ or $M = L$. We have $M \subset Z$ or $d\gamma\mu(M) = 0$. Since $\gamma\mu(M)$ is a nonzero ideal of R and d is a nonzero derivation then using Lemma 2.1, $d\gamma\mu(M) \neq 0$. Hence we obtain $M \subset Z$. This gives that R is commutative from Lemma 2.2.

(ii) If $V \subset Z$ then $\sigma(v) + \tau(v) \in Z$ for all $v \in V$. If $V \not\subseteq Z$ then using Lemma 2.3, there exist a nonzero ideal M of R such that

$$([R,M]_{\sigma,\tau} \subset V \text{ and } [R,M]_{\sigma,\tau} \not\subseteq C_{\sigma,\tau}) \text{ or } \sigma(v) + \tau(v) \in Z, \forall v \in V.$$

If $[R,M]_{\sigma,\tau} \subset V$ then $h\lambda(V) = 0$ implies that $h\lambda[R,M]_{\sigma,\tau} = 0$. If we use (i) then we have R is commutative and so $V \subset Z$. This contradicts with $V \not\subseteq Z$. Hence finally we obtain that $\sigma(v) + \tau(v) \in Z, \forall v \in V$ for any cases.

(iii) If $h\gamma[I,V]_{\lambda,\mu} = 0$ then for any $v \in V$, we have $v \in Z$ or $d\gamma\mu(v) = 0$ from Lemma 2.6. Considering as in the proof of (i) we get $V \subset Z$ or $d\gamma\mu(V) = 0$. It is clear that $V \subset Z$ gives that $\sigma(v) + \tau(v) \in Z, \forall v \in V$. On the other hand, since d is a nonzero $(\alpha,\beta)-$derivation then d is a nonzero left (and right)-generalized $(\alpha,\beta)-$derivation associated with d. Hence, if $d\gamma\mu(V) = 0$ then we obtain that $\sigma(v) + \tau(v) \in Z, \forall v \in V$ by (ii). □

Corollary 2.8. ([9, Corollary 5]) *Let R be a prime ring with characteristic different from two, let d be a nonzero $(\theta,\varphi)-$derivation of R and let U be a left $(\sigma,\tau)-$ Lie ideal. If $d[R,U]_{\sigma,\tau} = 0$ then $\sigma(u) + \tau(u) \in Z$ for all $u \in U$.*

Corollary 2.9. *Let V be a left $(\sigma,\tau)-$Lie ideal of R. If $a \in R$ such that $(a,\lambda(V))_{\alpha,\beta} = 0$ then $a \in C_{\alpha,\beta}$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$.*

Proof. The mapping defined by $g(r) = (a,r)_{\alpha,\beta}, \forall r \in R$ is a left-generalized $(\alpha,\beta)-$derivation associated with $(\alpha,\beta)-$derivation $d(r) = [a,r]_{\alpha,\beta}, \forall r \in R$. If $g = 0$ then we have $d = 0$ and so $a \in C_{\alpha,\beta}$. Let $g \neq 0$ and $d \neq 0$. If $(a,\lambda(V))_{\alpha,\beta} = 0$ then we can write $g\lambda(V) = 0$. Using Theorem 2.7 (ii) we obtain that $\sigma(v) + \tau(v) \in Z$ for all $v \in V$. □

Lemma 2.10. *Let $h : R \longrightarrow R$ be a nonzero left-generalized $(\alpha,\beta)-$derivation associated with a nonzero $(\alpha,\beta)-$derivation d and I be a nonzero ideal of R. If $a, b \in R$ such that $h\lambda[I,a]_{\sigma,\tau}b = 0$ then $d\lambda\tau(a) = 0$ or $[b,\alpha\lambda\sigma(a)]b = 0$.*

Proof. If $h\lambda[I,a]_{\sigma,\tau}b=0$ then we have for all $x\in I$

$$0 = h\lambda[\tau(a)x,a]_{\sigma,\tau}b = h(\lambda\tau(a)\lambda[x,a]_{\sigma,\tau})b$$
$$= d\lambda\tau(a)\alpha\lambda[x,a]_{\sigma,\tau}b + \beta\lambda\tau(a)h\lambda[x,a]_{\sigma,\tau}b = d\lambda\tau(a)\alpha\lambda[x,a]_{\sigma,\tau}b.$$

That is $d\lambda\tau(a)\alpha\lambda[I,a]_{\sigma,\tau}b=0$.

Using Lemma 2.5 (ii) we obtain that $d\lambda\tau(a)=0$ or $[b,\alpha\lambda\sigma(a)]b=0$. □

Lemma 2.11. *Let V be a left (σ,τ)−Lie ideal of R and $a,b\in R$. If $[a,\lambda(V)]_{\alpha,\beta}b=0$ then $a\in C_{\alpha,\beta}$ or $b\in Z$ or $\sigma(v)+\tau(v)\in Z$ for all $v\in V$.*

Proof. If $V\subset Z$ then $\sigma(v)+\tau(v)\in Z$ for all $v\in V$. If $V\not\subseteq Z$ then using Lemma 2.3, there exist a nonzero ideal M of R such that

$$([R,M]_{\sigma,\tau}\subset V \text{ and } [R,M]_{\sigma,\tau}\not\subseteq C_{\sigma,\tau}) \text{ or } \sigma(v)+\tau(v)\in Z, \forall v\in V.$$

Let $[R,M]_{\sigma,\tau}\subset V$. The mapping $d(r)=[a,r]_{\alpha,\beta}, \forall r\in R$ is a (α,β)−derivation and so left (and right)-generalized (α,β)−derivation with d. If $d=0$ then $a\in C_{\alpha,\beta}$ is obtained. Let $d\neq 0$.

If $[a,\lambda(V)]_{\alpha,\beta}b=0$ then we have $d\lambda(V)b=0$ and so $d\lambda[R,M]_{\sigma,\tau}b=0$. Using Lemma 2.10, this gives that for any $m\in M$

$$d\lambda\tau(m)=0 \text{ or } [b,\alpha\lambda\sigma(m)]b=0.$$

Considering as in the proof of Theorem 2.7 (i) we obtain that

$$d\lambda\tau(M)=0 \text{ or } [b,\alpha\lambda\sigma(M)]b=0.$$

Since $\lambda\tau(M)$ is a nonzero ideal of R and d is a nonzero derivation then using by Lemma 2.1, $d\lambda\tau(M)\neq 0$. On the other hand, if $[b,\alpha\lambda\sigma(M)]b=0$ then we get for all $m,m_1\in M$

$$0=[b,\alpha\lambda\sigma(mm_1)]b=\alpha\lambda\sigma(m)[b,\alpha\lambda\sigma(m_1)]b+[b,\alpha\lambda\sigma(m)]\alpha\lambda\sigma(m_1)b$$
$$=[b,\alpha\lambda\sigma(m)]\alpha\lambda\sigma(m_1)b.$$

That is

$$[b,\alpha\lambda\sigma(M)]\alpha\lambda\sigma(M)b=0.$$

Since $\alpha\lambda\sigma(M)$ is a nonzero ideal of R then we have $b\in Z$. Finally we obtain that $a\in C_{\alpha,\beta}$ or $b\in Z$ or for all $v\in V$, $\sigma(v)+\tau(v)\in Z$ for all case. □

Corollary 2.12. *Let V be a left (σ,τ)−Lie ideal of R and $a,b\in R$. If $[a,\lambda(V)]b=0$ then $a\in Z$ or $b\in Z$ or $\sigma(v)+\tau(v)\in Z$ for all $v\in V$.*

Proof. Taking $\alpha=\beta=1$ an identity map of R in Lemma 2.11 we get the required result. □

Theorem 2.13. *Let $h: R\longrightarrow R$ be a nonzero left-generalized (α,β)−derivation associated with (α,β)−derivation d and let V be a left (σ,τ)−Lie ideal of R. If $b\in R$ such that $h\lambda(V)b=0$ then $b\in Z$ or $\sigma(v)+\tau(v)\in Z, \forall v\in V$.*

Proof. If $h\lambda(V)b=0$ then we have $h\lambda[R,V]_{\sigma,\tau}b=0$. Using Lemma 2.10, for any $v\in V$, we obtain that $d\lambda\tau(v)=0$ or $[b,\alpha\lambda\sigma(v)]b=0$. Let $K=\{v\in V \mid d\lambda\tau(v)=0\}$ and $L=\{v\in V \mid [b,\alpha\lambda\sigma(v)]b=0\}$. Considering as in the proof of Theorem 2.7 (i), we get

$$d\lambda\tau(V)=0 \text{ or } [b,\alpha\lambda\sigma(V)]b=0.$$

Since d is a (α,β)−derivation then left-generalized (α,β)−derivation with d. If $d\lambda\tau(V)=0$ then using Theorem 2.7 (ii), we have $\sigma(v)+\tau(v)\in Z, \forall v\in V$. On the other hand if $[b,\alpha\lambda\sigma(V)]b=0$ then using Corollary 2.12, we obtain that $b\in Z$ or $\sigma(v)+\tau(v)\in Z$ for all $v\in V$. □

Corollary 2.14. *Let $h : R \longrightarrow R$ be a nonzero left-generalized derivation associated with derivation d and let V be a left $(\sigma,\tau)-$Lie ideal of R. If $b \in R$ such that $h(V)b = 0$ then $b \in Z$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$.*

Proof. Taking $\alpha = \beta = \lambda = 1$ an identity map of R in Theorem 2.13, we get the required result. □

Remark 2.15. Let M be a nonzero ideal of R. If $b[b,\lambda(M)] = 0$ then $b \in Z$.

Proof. If $b[b,\lambda(M)] = 0$ then for all $m \in M, r \in R$

$$0 = b[b,\lambda(mr)] = b\lambda(m)[b,\lambda(r)] + b[b,\lambda(m)]\lambda(r) = b\lambda(m)[b,\lambda(r)].$$

That is $b\lambda(M)[b,R] = 0$. This gives that $b \in Z$. □

Lemma 2.16. *Let $h : R \longrightarrow R$ be a nonzero right-generalized $(\alpha,\beta)-$derivation associated with a nonzero $(\alpha,\beta)-$derivation $d : R \longrightarrow R$. Let I be a nonzero ideal of R and $a, b \in R$. If $bh\lambda[I,a]_{\sigma,\tau} = 0$ then $b[b,\beta\lambda\tau(a)] = 0$ or $d\lambda\sigma(a) = 0$.*

Proof. If $bh\lambda[I,a]_{\sigma,\tau} = 0$ then we have for all $x \in I$

$$0 = bh\lambda[x\sigma(a),a]_{\sigma,\tau} = b\left\{h(\lambda[x,a]_{\sigma,\tau}\lambda\sigma(a))\right\} = bh\lambda[x,a]_{\sigma,\tau}\alpha\lambda\sigma(a) + b\beta\lambda[x,a]_{\sigma,\tau}d\lambda\sigma(a).$$

That is

$$b\beta\lambda[x,a]_{\sigma,\tau}d\lambda\sigma(a) = 0, \forall x \in I. \tag{1}$$

Replacing x by $\lambda^{-1}\beta^{-1}(b)x$ in (1) we get for all $x \in I$

$$\begin{aligned}0 &= b\beta\lambda[\lambda^{-1}\beta^{-1}(b)x,a]_{\sigma,\tau}d\lambda\sigma(a)\\ &= bb\beta\lambda[x,a]_{\sigma,\tau}d\lambda\sigma(a) + b\beta\lambda[\lambda^{-1}\beta^{-1}(b),\tau(a)]\beta\lambda(x)d\lambda\sigma(a)\\ &= b\beta\lambda[\lambda^{-1}\beta^{-1}(b),\tau(a)]\beta\lambda(x)d\lambda\sigma(a).\end{aligned}$$

That is

$$b\beta\lambda[\lambda^{-1}\beta^{-1}(b),\tau(a)]\beta\lambda(I)d\lambda\sigma(a) = 0.$$

Since $\beta\lambda(I)$ an ideal of R then we have $b\beta\lambda[\lambda^{-1}\beta^{-1}(b),\tau(a)] = 0$ or $d\lambda\sigma(a) = 0$. That is $b[b,\beta\lambda\tau(a)] = 0$ or $d\lambda\sigma(a) = 0$. □

Corollary 2.17. *Let $h : R \longrightarrow R$ be a nonzero right-generalized $(\alpha,\beta)-$derivation associated with a nonzero $(\alpha,\beta)-$derivation d and M be a nonzero ideal of R. If $b \in R$ such that $bh\lambda[R,M]_{\sigma,\tau} = 0$ then $b \in Z$.*

Proof. If $bh\lambda[R,m]_{\sigma,\tau} = 0, \forall m \in M$ then using Lemma 2.16 we have for any $m \in M$

$$b[b,\beta\lambda\tau(m)] = 0 \text{ or } d\lambda\sigma(m) = 0.$$

Let $K = \{m \in M \mid b[b,\beta\lambda\tau(m)] = 0\}$ and $L = \{m \in M \mid d\lambda\sigma(m) = 0\}$. Considering as in the proof of Theorem 2.7 (i) we obtain $b[b,\beta\lambda\tau(M)] = 0$ or $d\lambda\sigma(M) = 0$. Since d is a nonzero derivation then $d\lambda\sigma(M) \neq 0$ from Lemma 2.1. If $b[b,\beta\lambda\tau(M)] = 0$ then using Remark 2.15, $b \in Z$ is obtained. □

Lemma 2.18. *Let V be a left $(\sigma,\tau)-$Lie ideal of R and $a, b \in R$. If $b[a,\lambda(V)]_{\alpha,\beta} = 0$ then $a \in C_{\alpha,\beta}$ or $b \in Z$ or $\sigma(v) + \tau(v) \in Z$ for all $v \in V$.*

Proof. If $V \subset Z$ then $\sigma(v) + \tau(v) \in Z$ for all $v \in V$. If $V \not\subseteq Z$ then using Lemma 2.3, there exist a nonzero ideal M of R such that

$$([R,M]_{\sigma,\tau} \subset V \text{ and } [R,M]_{\sigma,\tau} \not\subseteq C_{\sigma,\tau}) \text{ or } \sigma(v) + \tau(v) \in Z, \forall v \in V.$$

Let $[R,M]_{\sigma,\tau} \subset V$. The mapping $d(r) = [a,r]_{\alpha,\beta}, \forall r \in R$ is a $(\alpha,\beta)-$derivation and so right (and left)-generalized $(\alpha,\beta)-$derivation associated with d. If $d = 0$ then $a \in C_{\alpha,\beta}$ is obtained. Assume that d is a nonzero derivation.

If $b[a,\lambda(V)]_{\alpha,\beta} = 0$ then we have $bd\lambda(V) = 0$ and so $bd\lambda[R,M]_{\sigma,\tau} = 0$. This gives that by Lemma 2.16, for any $m \in M$

$$d\lambda\sigma(m) = 0 \text{ or } b[b,\beta\lambda\tau(m)] = 0.$$

Considering as in the proof of Theorem 2.7 (i), we obtain that

$$d\lambda\sigma(M) = 0 \text{ or } b[b,\beta\lambda\tau(M)] = 0.$$

Since $\lambda\sigma(M)$ is a nonzero ideal of R and d is a nonzero derivation then $d\lambda\sigma(M) \neq 0$ by Lemma 2.1. On the other hand, if $b[b,\beta\lambda\tau(M)] = 0$ then using Remark 2.15, we have $b \in Z$.

Finally we obtain that $a \in C_{\alpha,\beta}$ or $b \in Z$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$ for all case. □

Corollary 2.19. *Let V be a left $(\sigma,\tau)-$Lie ideal of R and $a, b \in R$. If $b[a,\lambda(V)] = 0$ then $a \in Z$ or $b \in Z$ or $\sigma(v) + \tau(v) \in Z$ for all $v \in V$.*

Proof. Taking $\alpha = \beta = 1$ an identity map of R in Lemma 2.18, we get the required result. □

Theorem 2.20. *Let $h : R \longrightarrow R$ be a nonzero right-generalized $(\alpha,\beta)-$derivation associated with nonzero $(\alpha,\beta)-$derivation d and $a, b \in R$. Let V be a left $(\sigma,\tau)-$Lie ideal and $I \neq 0$ an ideal of R.*

(i) If $bh\lambda(V) = 0$ then $b \in Z$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$.

(ii) If $bh\lambda[I,V]_{\sigma,\tau} = 0$ then $b \in Z$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$.

Proof. (i) If $V \subset Z$ then it is clear that $\sigma(v) + \tau(v) \in Z, \forall v \in V$. If $V \not\subseteq Z$ then using Lemma 2.3, there exist a nonzero ideal M of R such that ($[R,M]_{\sigma,\tau} \subset V$ and $[R,M]_{\sigma,\tau} \not\subseteq C_{\sigma,\tau}$) or $\sigma(v) + \tau(v) \in Z, \forall v \in V$. If $[R,M]_{\sigma,\tau} \subset V$ then $bh\lambda(V) = 0$ means that $bh\lambda[R,M]_{\sigma,\tau} = 0$. Using Corollary 2.17, we get $b \in Z$.

Finally we obtain that $b \in Z$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$ for any case.

(ii) If $bh\lambda[I,v]_{\sigma,\tau} = 0, \forall v \in V$ then using Lemma 2.16, we have for any $v \in V$

$$b[b,\beta\lambda\tau(v)] = 0 \text{ or } d\lambda\sigma(v) = 0.$$

Let $K = \{v \in V \mid b[b,\beta\lambda\tau(v)] = 0\}$ and $L = \{v \in V \mid d\lambda\sigma(v) = 0\}$. Considering as the proof of Theorem 2.7 (i), we obtain

$$b[b,\beta\lambda\tau(V)] = 0 \text{ or } d\lambda\sigma(V) = 0.$$

Since d is a $(\alpha,\beta)-$derivation then d is a left-generalized $(\alpha,\beta)-$derivation with d. Hence $d\lambda\sigma(V) = 0$ implies that $\sigma(v) + \tau(v) \in Z, \forall v \in V$ from Theorem 2.7(ii). On the other hand,

if $b[b,\beta\lambda\tau(V)] = 0$ then using Corollary 2.19, we get $b \in Z$ or $\sigma(v) + \tau(v) \in Z$ for all $v \in V$. □

Corollary 2.21. *Let $h : R \longrightarrow R$ be a nonzero right-generalized derivation associated with nonzero derivation d and $a, b \in R$. Let V be a left $(\sigma,\tau)-$Lie ideal and $I \neq 0$ an ideal of R. If $bh(V) = 0$ then $b \in Z$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$.*

Proof. Taking $\alpha = \beta = \lambda = 1$ an identity map of R in Theorem 2.20 (i), we get the required result. □

Theorem 2.22. *Let V be a nonzero left $(\sigma,\tau)-$Lie ideal of R. and $a, b \in R$.*

(i) If $b\,[V,a]_{\alpha,\beta} = 0$ or $[V,a]_{\alpha,\beta}\,b = 0$ then $a \in Z$ or $b \in Z$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$.

(ii) If $(a,V)_{\alpha,\beta}b = 0$ then $a \in C_{\alpha,\beta}$ or $b \in Z$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$

(iii) If $(V,a)_{\alpha,\beta}b = 0$ then $a \in Z$ or $b \in Z$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$.

Proof. (i) Let us consider the mapping defined by $h(r) = [r, a]_{\alpha,\beta}, \forall r \in R$. Then h is a right(left resp.)-generalized derivation associated with derivation $d_1(r) = [r, \alpha(a)], \forall r \in R$ and derivation $d_2 = [r, \beta(a)], \forall r \in R$ respectively. Because

$$h(rs) = [rs, a]_{\alpha,\beta} = r[s, \alpha(a)] + [r, a]_{\alpha,\beta} s = h(r)s + rd_1(s), \forall r, s \in R,$$

$$h(rs) = [rs, a]_{\alpha,\beta} = r[s, a]_{\alpha,\beta} + [r, \beta(a)] s = d_2(r)s + rh(s), \forall r, s \in R.$$

If $h = 0$ then $d_1 = 0$ and $d_2 = 0$ and so $a \in Z$ is obtained. Let $h \neq 0$, $d_1 \neq 0$ and $d_2 \neq 0$.

If $b[V, a]_{\alpha,\beta} = 0$ then we have $bh(V) = 0$. Since h is a right-generalized derivation associated with derivation d_1 then using Corollary 2.21, we get $b \in Z$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$.

Similarly if $[V, a]_{\alpha,\beta} b = 0$ then we write $h(V)b = 0$. Since h is a left-generalized derivation associated with derivation d_2 then using Corollary 2.14 we have $b \in Z$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$. Finally we obtain that $a \in Z$ or $b \in Z$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$ for two case.

(ii) Let $g(r) = (a, r)_{\alpha,\beta}, \forall r \in R$ and $d_3(r) = [a, r]_{\alpha,\beta}, \forall r \in R$. Then g is a left-generalized $(\alpha, \beta)-$derivation associated with $(\alpha, \beta)-$derivation d_3. Because for all $r, s \in R$

$$g(rs) = (a, rs)_{\alpha,\beta} = \beta(r)(a, s)_{\alpha,\beta} + [a, r]_{\alpha,\beta}\alpha(s) = d_3(r)\alpha(s) + \beta(r)g(s).$$

If $g = 0$ then we have $d_3 = 0$ and so $a \in C_{\alpha,\beta}$ is obtained. Let $g \neq 0$ and $d_3 \neq 0$.

If $(a, V)_{\alpha,\beta} b = 0$ then we have $g(V)b = 0$. Since g is a nonzero left-generalized $(\alpha, \beta)-$derivation associated with $(\alpha, \beta)-$derivation d_3 then considering Theorem 2.13, we get $b \in Z$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$. Finally we obtain that $a \in C_{\alpha,\beta}$ or $b \in Z$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$ for two case.

(iii) The mapping $f(r) = (r, a)_{\alpha,\beta}, \forall r \in R$ is a left-generalized derivation associated with derivation $d_4(r) = -[r, \beta(a)], \forall r \in R$. Because for all $r, s \in R$

$$f(rs) = (rs, a)_{\alpha,\beta} = r(s, a)_{\alpha,\beta} - [r, \beta(a)]s = d_4(r)s + rf(s).$$

If $f = 0$ then $d_4 = 0$ and so $a \in Z$ is obtained.

Let $f \neq 0$ and $d_4 \neq 0$. If $(V, a)_{\alpha,\beta} b = 0$ then we have $f(V)b = 0$. This gives that $b \in Z$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$ by Corollary 2.14. □

Lemma 2.23. *Let $h : R \longrightarrow R$ be a nonzero left-generalized $(\alpha, \beta)-$derivation associated with a nonzero $(\alpha, \beta)-$derivation $d : R \longrightarrow R$. Let V be a nonzero left $(\sigma, \tau)-$Lie ideal of R and I be a nonzero ideal of R. If $a, b \in R$ such that $[h(I)b, a]_{\lambda,\mu} = 0$ then $d\beta^{-1}\mu(a) = 0$ or $b[b, \lambda(a)] = 0$.*

Proof. If $[h(x)b, a]_{\lambda,\mu} = 0, \forall x \in I$ then we have

$$h(x)[b, \lambda(a)] + [h(x), a]_{\lambda,\mu} b = 0, \forall x \in I. \quad (2)$$

Replacing x by $sx, s \in R$ in (2) and using (2), we get for all $x \in I, s \in R$

$$\begin{aligned} 0 &= h(sx)[b, \lambda(a)] + [h(sx), a]_{\lambda,\mu} b \\ &= d(s)\alpha(x)[b, \lambda(a)] + \beta(s)h(x)[b, \lambda(a)] + [d(s)\alpha(x) + \beta(s)h(x), a]_{\lambda,\mu} b \\ &= d(s)\alpha(x)[b, \lambda(a)] + \beta(s)h(x)[b, \lambda(a)] + d(s)[\alpha(x), \lambda(a)]b \\ &+ [d(s), a]_{\lambda,\mu}\alpha(x)b + \beta(s)[h(x), a]_{\lambda,\mu} b + [\beta(s), \mu(a)]h(x)b \\ &= d(s)\alpha(x)[b, \lambda(a)] + d(s)[\alpha(x), \lambda(a)]b + [d(s), a]_{\lambda,\mu}\alpha(x)b + [\beta(s), \mu(a)]h(x)b. \end{aligned}$$

If we take $\beta^{-1}\mu(a)$ instead of s and say that $k = d\beta^{-1}\mu(a)$ then the last relation gives that

$$k\alpha(x)[b, \lambda(a)] + k[\alpha(x), \lambda(a)]b + [k, a]_{\lambda,\mu}\alpha(x)b = 0, \forall x \in I. \quad (3)$$

Replacing x by $x\alpha^{-1}(b)$ in (3) we have for all $x \in I$

$$0 = k\alpha(x)b[b, \lambda(a)] + k\alpha(x)[b, \lambda(a)]b + k[\alpha(x), \lambda(a)]bb + [k, a]_{\lambda,\mu}\alpha(x)bb = k\alpha(x)b[b, \lambda(a)]$$

and so $k\alpha(I)b[b, \lambda(a)] = 0$.
Since $\alpha(I)$ is a nonzero ideal of R then we obtain $d\beta^{-1}\mu(a) = 0$ or $b[b, \lambda(a)] = 0$. □

Theorem 2.24. *Let $h : R \longrightarrow R$ be a nonzero left-generalized $(\alpha, \beta)-$derivation associated with a nonzero $(\alpha, \beta)-$derivation $d : R \longrightarrow R$. Let V be a nonzero left $(\sigma, \tau)-$Lie ideal of R and I be a nonzero ideal of R. If $b \in R$ such that $h(I)b \subset C_{\lambda,\mu}(V)$ then $b \in Z$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$.*

Proof. If $h(I)b \subset C_{\lambda,\mu}(V)$ then we have $[h(x)b, v]_{\lambda,\mu} = 0, \forall v \in V, x \in I$. Using Lemma 2.23, we get $d\beta^{-1}\mu(v) = 0$ or $b[b, \lambda(v)] = 0$.

Let $K = \{v \in V \mid b[b, \lambda(v)] = 0\}$ and $L = \left\{v \in V \mid d\beta^{-1}\mu(v) = 0\right\}$. Considering as in the proof of Theorem 2.7 (i) we obtain

$$d\beta^{-1}\mu(V) = 0 \text{ or } b[b, \lambda(V)] = 0.$$

If $d\beta^{-1}\mu(V) = 0$ then using Theorem 2.7 (ii), we have $\sigma(v) + \tau(v) \in Z, \forall v \in V$. On the other hand using Corollary 2.19, $b[b, \lambda(V)] = 0$ means that $b \in Z$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$. □

Corollary 2.25. *Let V be a nonzero left $(\sigma, \tau)-$Lie ideal of R and $a, b \in R$. If $(a, I)_{\sigma,\tau}b \subset C_{\lambda,\mu}(V)$ then $a \in C_{\sigma,\tau}$ or $b \in Z$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$.*

Proof. The mapping defined by $h(r) = (a, r)_{\sigma,\tau}, \forall r \in R$ is a left-generalized $(\sigma, \tau)-$derivation associated with $(\sigma, \tau)-$derivation $d(r) = [a, r]_{\sigma,\tau}, \forall r \in R$. If $h = 0$ then $d = 0$ and so we have $a \in C_{\sigma,\tau}$. Let $h \neq 0$ and $d \neq 0$.

If $(a, I)_{\sigma,\tau}b \subset C_{\lambda,\mu}(V)$ then we can write $h(I)b \subset C_{\lambda,\mu}(V)$. Using Theorem 2.24, we get $b \in Z$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$. □

Theorem 2.26. *Let $h : R \longrightarrow R$ be a nonzero left-generalized $(\alpha, \beta)-$derivation associated with nonzero $(\alpha, \beta)-$derivation $d : R \longrightarrow R$ and I be a nonzero ideal of R. Let V be a nonzero left $(\sigma, \tau)-$Lie ideal of R . If $h(I) \subset C_{\lambda,\mu}(V)$ then $\sigma(v) + \tau(v) \in Z, \forall v \in V$.*

Proof. If $h(I) \subset C_{\lambda,\mu}(V)$ then we have $[h(x), v]_{\lambda,\mu} = 0, \forall x \in I, v \in V$. Let us consider the following relations

$$[rs, t]_{\lambda,\mu} = r[s, \lambda(t)] + [r, t]_{\lambda,\mu}s \text{ , } \forall r, s, t \in R \text{ and}$$
$$[rs, t]_{\lambda,\mu} = r[s, t]_{\lambda,\mu} + [r, \mu(t)]s, \forall r, s, t \in R.$$

Using the hypothesis and the above relations we get for all $v \in V, x \in I, s \in R$

$$\begin{aligned} 0 = [h(sx), v]_{\lambda,\mu} &= [d(s)\alpha(x) + \beta(s)h(x), v]_{\lambda,\mu} \\ &= [d(s)\alpha(x), v]_{\lambda,\mu} + [\beta(s)h(x), v]_{\lambda,\mu} \\ &= d(s)[\alpha(x), \lambda(v)] + [d(s), v]_{\lambda,\mu}\alpha(x) + \beta(s)[h(x), v]_{\lambda,\mu} + [\beta(s), \mu(v)]h(x) \\ &= d(s)[\alpha(x), \lambda(v)] + [d(s), v]_{\lambda,\mu}\alpha(x) + [\beta(s), \mu(v)]h(x) \end{aligned}$$

and so

$$d(s)[\alpha(x), \lambda(v)] + [d(s), v]_{\lambda,\mu}\alpha(x) + [\beta(s), \mu(v)]h(x) = 0, \forall v \in V, x \in I, s \in R. \tag{4}$$

If we take $\beta^{-1}\mu(v)$ instead of s in (4) and say that $k(v) = d\beta^{-1}\mu(v)$ then we obtain

$$k(v)[\alpha(x), \lambda(v)] + [k(v), v]_{\lambda,\mu}\alpha(x) = 0, \forall v \in V, x \in I. \tag{5}$$

Replacing x by $xr, r \in R$ in (5) and using (5) we have for all $v \in V, x \in I, r \in R$

$$\begin{aligned} 0 &= k(v)\alpha(x)[\alpha(r), \lambda(v)] + k(v)[\alpha(x), \lambda(v)]\alpha(r) + [k(v), v]_{\lambda,\mu}\alpha(x)\alpha(r) \\ &= k(v)\alpha(x)[\alpha(r), \lambda(v)] \end{aligned}$$

and so

$$k(v)\alpha(I)[R, \lambda(v)] = 0, \forall v \in V. \tag{6}$$

For any $v \in V$, the relation (6) gives that $d\beta^{-1}\mu(v) = 0$ or $v \in Z$.

Let $K = \left\{v \in V \mid d\beta^{-1}\mu(v) = 0\right\}$ and $L = \{v \in V \mid v \in Z\}$. Considering as in the proof of Theorem 2.7 (i), we get $d\beta^{-1}\mu(V) = 0$ or $V \subset Z$.

If $V \subset Z$ then $\sigma(v) + \tau(v) \in Z, \forall v \in V$. On the other hand, since d is a $(\alpha, \beta)-$derivation and so a left-generalized $(\alpha, \beta)-$derivation associated with d then $d\beta^{-1}\mu(V) = 0$ means that $\sigma(v) + \tau(v) \in Z, \forall v \in V$ by Theorem 2.7 (ii). □

Corollary 2.27. *Let V be a nonzero left $(\sigma, \tau)-$Lie ideal of R and $b \in R$. If $(b, I)_{\alpha,\beta} \subset C_{\lambda,\mu}(V)$ then $b \in C_{\alpha,\beta}$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$.*

Proof. Let us consider the mappings defined by $h(r) = (b, r)_{\alpha,\beta}, \forall \in R$ and $d(r) = [b, r]_{\alpha,\beta}, \forall r \in R$. Since

$$h(rs) = (b, rs)_{\alpha,\beta} = \tau(r)(b, s)_{\alpha,\beta} + [b, r]_{\alpha,\beta}\sigma(s) = d(r)\sigma(s) + \tau(r)h(s), \forall r, s \in R.$$

Then h is a left-generalized $(\alpha, \beta)-$ derivation associated with $(\alpha, \beta)-$ derivation d. If $h = 0$ then $d = 0$ and so $b \in C_{\alpha,\beta}$ is obtained.

If $(b, I)_{\alpha,\beta} \subset C_{\lambda,\mu}(V)$ then we have $h(I) \subset C_{\lambda,\mu}(V)$. Let $h \neq 0$ and $d \neq 0$. Using Theorem 2.26, we obtain $\sigma(v) + \tau(v) \in Z, \forall v \in V$. □

Theorem 2.28. *Let $h : R \longrightarrow R$ be a nonzero right-generalized $(\alpha, \beta)-$derivation associated with $(\alpha, \beta)-$derivation $0 \neq d : R \longrightarrow R$. Let V be a nonzero left $(\sigma, \tau)-$Lie ideal of R and I be a nonzero ideal of R. If $b \in R$ such that $bh(I) \subset C_{\lambda,\mu}(V)$ then $b \in Z$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$.*

Proof. If $bh(I) \subset C_{\lambda,\mu}(V)$ then $[bh(I), v]_{\lambda,\mu} = 0, \forall v \in V$. Using Lemma 2.4 , for any $v \in V$, we obtain that $[b, \mu(v)]b = 0$ or $d\alpha^{-1}\lambda(v) = 0$.

Let $K = \{v \in V \mid [b, \mu(v)]b = 0\}$ and $L = \left\{v \in V \mid d\alpha^{-1}\lambda(v) = 0\right\}$. Considering as in the proof of Theorem 2.7 (i), we have

$$[b, \mu(V)]b = 0 \text{ or } d\alpha^{-1}\lambda(V) = 0.$$

If $d\alpha^{-1}\lambda(V) = 0$ then $\sigma(v) + \tau(v) \in Z, \forall v \in V$ is obtained by Theorem 2.7 (ii). On the other hand $[b, \mu(V)]b = 0$ means that $b \in Z$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$ by Corollary 2.12. □

Corollary 2.29. *Let I be a nonzero ideal of R and $a, b \in R$. If $b(a, I)_{\sigma,\tau} \subset C_{\lambda,\mu}(V)$ then $a \in C_{\sigma,\tau}$ or $b \in Z$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$.*

Proof. The mapping defined by $h(r) = (a, r)_{\sigma,\tau}, \forall r \in R$ is a right-generalized $(\sigma, \tau)-$derivation associated with $(\sigma, \tau)-$derivation $d(r) = [a, r]_{\sigma,\tau}, \forall r \in R$. If $h = 0$ then $d = 0$ and so we have $a \in C_{\sigma,\tau}$. Assume that h and d are nonzero.

If $b(a, I)_{\sigma,\tau} \subset C_{\lambda,\mu}(V)$ then we have $bh(I) \subset C_{\lambda,\mu}(V)$. Since h is right-generalized $(\sigma, \tau)-$derivation then using Theorem 2.2, we get $b \in Z$ or $\sigma(v) + \tau(v) \in Z, \forall v \in V$. □

References

[1] N. Aydın, K. Kaya, Some generalizations in prime rings with $(\sigma, \tau)-$derivation, Doğa-Tr. J. Math. 16, (1992) 169–176.
[2] M. Bresar, On the distance of the composition of two derivations to the generalized derivations, Glasgow Math. J. 33 (1991) 89–93.
[3] J.C. Chang., On the identity h(x) = af(x) + g(x)b, Taiwanese J. Math. 7 (2003) 103–113.
[4] E. Güven, One sided $(\sigma, \tau)-$Lie ideals and generalized derivations in prime rings, Palestine J. Math. 7 (2018) 479–486.
[5] E. Güven, K. Kaya, M. Soytürk, Some results on $(\sigma, \tau)-$Lie ideals, Math. J. Okayama Univ. 49 (2007) 59–64.
[6] K. Kaya, N. Aydın, Some results on generalized Lie ideals, Albasapr Sci. J. (Jordan Univ. for Woman) 3:1 (1999) 53–61.
[7] K. Kaya, $(\sigma, \tau)-$Lie ideals in prime rings, Anal. Univ Timisoara Ser. Stiinte Math. (1992) 251–255.
[8] J.H. Mayne, Centralizing mappings of prime rings, Canad. Math. Bull. 27 (1984) 122–126.
[9] K.H. Park, Y.S. Jung, Some result concerning $(\theta, \varphi)-$derivations on prime rings, J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. 10:4 (2003) 207–215.