Filomat 32:16 (2018), 5519–5526
https://doi.org/10.2298/FIL1816519U

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Generalization Conformable Fractional Integral Inequalities

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Abstract. The main issues addressed in this paper are making generalization of Gronwall, Volterra and Pachpatte type inequalities for conformable differential equations. By using the Katugampola definition for conformable calculus we found some upper and lower bound for integral inequalities. The established results are extensions of some existing Gronwall, Volterra and Pachpatte type inequalities in the previous published studies.

1. Introduction

Until quite recently, the question of how to take non-integer order of derivative or integration was phenomenon among the mathematicians. However together with the development of mathematics knowledge, this question was answered via Fractional Calculus which is a generalization of ordinary differentiation and integration to arbitrary (non-integer) order. During three centuries, the theory of fractional calculus developed as a pure theoretical field, useful only for mathematicians, we refer to [11], see also [13]. Then In conjunction with the development of theoretical progress of fractional calculus, a number of mathematicians have started to applied the obtained results to real world problems consist of fractional derivatives and integrals [2, 3].

An important point is that the fractional derivative at a point x is a local property only when a is an integer; in non-integer cases we cannot say that the fractional derivative at x of a function f depends only on values of f very near x, in the way that integer-power derivatives certainly do. Therefore it is expected that the theory involves some sort of boundary conditions, involving information on the function further out. To use a metaphor, the fractional derivative requires some peripheral vision. As far as the existence of such a theory is concerned, the foundations of the subject were laid by Liouville in a paper from 1832. The fractional derivative of a function to order a is often now defined by means of the Fourier or Mellin integral transforms. Various types of fractional derivatives were introduced: Riemann- Liouville, Caputo, Hadamard, Erdelyi-Kober, Grunwald-Letnikov, Marchaud and Riesz are just a few to name [11, 13].

Now, all these definitions satisfy the property that the fractional derivative is linear. This is the only property inherited from the first derivative by all of the definitions. However, all definitions do not provide some properties such as Product Rule (Leibniz Rule), Quotient Rule, Chain Rule, Rolle's Theorem and Mean Value Theorem. In addition most of the fractional derivatives except Caputo-type derivatives, do not satisfy $D^{\alpha}(f)(1) = 0$ if α is not a natural number.

2010 *Mathematics Subject Classification*. Primary 26D15; Secondary 26A51, 26A33, 26A42
Keywords. Gronwall's inequality, confromable fractional integrals
Received: 29 May 2017; Revised: 27 September 2017; Accepted: 30 September 2017
Communicated by Ljubiša D.R. Kočinac
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Recently, a new local, limit-based definition of a so-called conformable derivative has been formulated in [1, 10], with several follow-up papers [4, 5, 7, 18]. This new idea was quickly generalized by Katugampola [8], whose definition forms the basis for this work and is referred to here as the Katugampola derivative (D_α will henceforth be referring to the Katugampola derivative). This definition has several practical properties which are summarized below.

Note that if f is fully differentiable at t; then the derivative is $D_\alpha(f)(t) = t^{1-\alpha} f'(t)$. (Here, operators of a very similar form, $t^\alpha D_1$, have been applied in combinatorial theory [9]). Of course, for $t = 0$ this is not valid and it would be useful to deal with equations and solutions with singularities. Additionally it must be noted that conformable derivative is conformable at $\alpha = 1$, as

$$\lim_{\alpha\to 1} D^\alpha(f) = f', \tag{1}$$

but

$$\lim_{\alpha\to 0^+} D^\alpha(f) \neq f'. \tag{2}$$

Remark 1.1. Some authors [12, 15] have argued that conformable fractional derivative is not a truly fractional operator since fractional derivatives does not satisfy Leibniz rule. This question seems today to still be open and perhaps it is a philosophical issue. Moreover, the incompleteness of this argument was pointed out in [6, 16, 17] reassess that for some versions of the fractional derivative the Leibniz formula can hold. However, in any case, the study of Gronwall, Volterra and Pachpatte type integral inequalities involving this new derivative has, in our opinion, a point of interest and deserves to be researched in more detail.

The remainder of this work is organized as follows: In Section 2, the related definitions and theorems are summarized. In Section 3, the general versions of Gronwall, Volterra and Pachpatte type integral inequalities are obtained while some conclusions and remarks are discussed in Section 4.

2. Preliminaries

In more recent times, in [8] Katugampola introduced the idea of fractional derivative which obeys the Product rule, Quotient rule and has results similar to the Rolle's Theorem and the Mean Value Theorem in classical calculus.

In this study, we use the Katugampola derivative formulation of conformable derivative of order for $\alpha \in (0,1]$ and $t \in [0,\infty)$ given by

$$D^\alpha(f)(t) = \lim_{\varepsilon\to 0} \frac{f\left(te^{\varepsilon t^{-\alpha}}\right) - f(t)}{\varepsilon}, \; D^\alpha(f)(0) = \lim_{t\to 0} D^\alpha(f)(t), \tag{3}$$

provided the limits exist (for detail see, [8]). If f is fully differentiable at t, then

$$D^\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t). \tag{4}$$

A function f is $\alpha-$differentiable at a point $t \geq 0$ if the limit in (3) exists and is finite. This definition yields the following results;

Theorem 2.1. *Let $\alpha \in (0,1]$ and f, g be $\alpha-$differentiable at a point $t > 0$. Then*
i. $D^\alpha(af + bg) = aD^\alpha(f) + bD^\alpha(g)$, *for all* $a, b \in \mathbb{R}$,
ii. $D^\alpha(\lambda) = 0$, *for all constant functions* $f(t) = \lambda$,
iii. $D^\alpha(fg) = fD^\alpha(g) + gD^\alpha(f)$,
iv. $D^\alpha\left(\frac{f}{g}\right) = \frac{fD^\alpha(g) - gD^\alpha(f)}{g^2}$
v. $D^\alpha(t^n) = nt^{n-\alpha}$ *for all* $n \in \mathbb{R}$
vi. $D^\alpha(f \circ g)(t) = f'(g(t))D^\alpha(g)(t)$ *for* f *is differentiable at* $g(t)$.

Definition 2.2. [Conformable fractional integral] Let $\alpha \in (0,1]$ and $0 \le a < b$. A function $f : [a,b] \to \mathbb{R}$ is α-fractional integrable on $[a,b]$ if the integral

$$\int_a^b f(x)\, d_\alpha x := \int_a^b f(x)\, x^{\alpha-1} dx$$

exists and is finite. All α-fractional integrable on $[a,b]$ is indicated by $L^1_\alpha([a,b])$

Remark 2.3.

$$I^a_\alpha(f)(t) = I^a_1\left(t^{\alpha-1} f\right) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1]$.

We will also use the following important results, which can be derived from the results above.

Lemma 2.4. *Let the conformable differential operator D^α be given as in (3), where $\alpha \in (0,1]$ and $t \ge 0$, and assume the functions f and g are α-differentiable as needed. Then*
i. $D^\alpha(\ln t) = t^{-\alpha}$ *for* $t > 0$
ii. $D^\alpha\left[\int_a^t f(t,s)\, d_\alpha s\right] = f(t,t) + \int_a^t D^\alpha[f(t,s)]\, d_\alpha s$
iii. $\int_a^b f(x) D^\alpha(g)(x)\, d_\alpha x = fg\big|_a^b - \int_a^b g(x) D^\alpha(f)(x)\, d_\alpha x.$

The definition given in below is a generalization of the limit definition of the derivative for the case of a function with many variables.

Definition 2.5. Let f be a function with n variables $t_1, ..., t_n$ and the conformable partial derivative of f of order $\alpha \in (0,1]$ in x_i is defined as follows

$$\frac{\partial^\alpha}{\partial t_i^\alpha} f(t_1, ..., t_n) = \lim_{\varepsilon \to 0} \frac{f(t_1, ..., t_{i-1}, t_i e^{\varepsilon t_i^{-\alpha}}, ..., t_n) - f(t_1, ..., t_n)}{\varepsilon}. \tag{5}$$

The below theorem is the generalization of Theorem 2.10 of [5] which the detailed proof can be found in [14].

Theorem 2.6. *Assume that $f(t,s)$ is function for which $\partial_t^\alpha\left[\partial_s^\beta f(t,s)\right]$ and $\partial_s^\beta\left[\partial_t^\alpha f(t,s)\right]$ exist and are continuous over the domain $D \subset \mathbb{R}^2$, then*

$$\partial_t^\alpha\left[\partial_s^\beta f(t,s)\right] = \partial_s^\beta\left[\partial_t^\alpha f(t,s)\right]. \tag{6}$$

This prospective study was designed to investigate the new generalization of Gronwall, Volterra and Pachpatte type inequalities for conformable differential equations. The established results are extensions of some existing Gronwall, Volterra and Pachpatte type inequalities in the literature.

3. Main Findings and Cumulative Results

Throughout this paper, all the functions which appear in the inequalities are assumed to be real-valued and all the integrals involved exist on the respective domains of their definitions, and $C(M,S)$ and $C^1(M,S)$ denote the class of all continuous functions and the first order conformable derivative, respectively, defined on set M with range in the set S.

Theorem 3.1. *Let $k, y, x, g \in C(\mathbb{R}^+, \mathbb{R}^+)$, $r \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ and assume that r is non-decreasing with $r(t) \leq t$ for $t \geq 0$. If $u \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfies*

$$u(t) \leq k(t) + y(t) \int_0^{r(t)} [x(s)u(s) + g(s)] d_\alpha s, \quad t \geq 0, \tag{7}$$

then

$$u(t) \leq k(t) + y(t) \int_0^t e^{\int_{r(\tau)}^{r(t)} x(s)y(s)d_\alpha s} [x(r(\tau))k(r(\tau)) + g(r(\tau))] D^\alpha r(\tau) d_\alpha \tau, \quad t \geq 0. \tag{8}$$

Proof. If we set

$$z(t) = \int_0^{r(t)} [x(s)u(s) + g(s)] d_\alpha s$$

then, by using conformable rules we see that

$$\begin{aligned} D^\alpha z(t) &= [x(r(t))u(r(t)) + g(r(t))] D^\alpha r(t) \\ &\leq \{x(r(t))[k(r(t)) + y(r(t))z(r(t))] + g(r(t))\} D^\alpha r(t) \\ &\leq \{x(r(t))[k(r(t)) + y(r(t))z(t)] + g(r(t))\} D^\alpha r(t). \end{aligned}$$

Thus, we have

$$D^\alpha z(t) - x(r(t))y(r(t))z(t) D^\alpha r(t) \leq [x(r(t))k(r(t)) + g(r(t))] D^\alpha r(t).$$

Multiplying the above inequality by $e^{-\int_0^{r(t)} x(s)y(s)d_\alpha s}$, we obtain that

$$\frac{\partial^\alpha}{\partial t^\alpha}\left(z(t)e^{-\int_0^{r(t)} x(s)y(s)d_\alpha s}\right) \leq e^{-\int_0^{r(t)} x(s)y(s)d_\alpha s} [x(r(t))k(r(t)) + g(r(t))] D^\alpha r(t).$$

Integrating this from 0 to t yields

$$\begin{aligned} z(t) &\leq e^{\int_0^{r(t)} x(s)y(s)d_\alpha s} \int_0^t e^{-\int_0^{r(\tau)} x(s)y(s)d_\alpha s} [x(r(\tau))k(r(\tau)) + g(r(\tau))] D^\alpha r(\tau) d_\alpha \tau \\ &= \int_0^t e^{\int_{r(\tau)}^{r(t)} x(s)y(s)d_\alpha s} [x(r(\tau))k(r(\tau)) + g(r(\tau))] D^\alpha r(\tau) d_\alpha \tau \end{aligned}$$

and hence the claim follows because of $u(t) \leq k(t) + y(t)z(t)$. The proof is complete. □

Remark 3.2. If we take $g(t) = 0$ in Theorem 3.1, then Theorem 3.1 reduces to Theorem 4 which has been proved by Sarikaya in [14].

Corollary 3.3. *Assume y, x, k are as in Theorem 3.1 and $r(t) = \frac{t^\alpha}{\alpha}$. If $u \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfies (7), then*

$$u(t) \leq k(t) + y(t) \int_0^t e^{\int_{\frac{\tau^\alpha}{\alpha}}^{\frac{t^\alpha}{\alpha}} x(s)y(s)d_\alpha s} [x(\tau)k(\tau) + g(\tau)] d_\alpha \tau, \quad t \geq 0.$$

Theorem 3.4. *Let $k, y, x, g \in C(\mathbb{R}^+, \mathbb{R}^+)$, $r \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ and assume that r is non-decreasing with $r(t) \le t$ for $t \ge 0$. If $u \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfies*

$$u(t) \le k(t) + \sum_{i=1}^{n} y_i(t) \int_0^{r(t)} [x_i(s)u(s) + g_i(s)] d_\alpha s, \quad t \ge 0, \tag{9}$$

then

$$u(t) \le k(t) + Y(t) \int_0^t e^{\int_{r(\tau)}^{r(t)} \sum_{i=1}^n x_i(s)y(s)d_\alpha s} \sum_{i=1}^{n} [x_i(r(\tau))k(r(\tau)) + g_i(r(\tau))] D^\alpha r(\tau) d_\alpha \tau, \quad t \ge 0. \tag{10}$$

where $Y(t) = \sup_{i=1,\dots,n} y_i(t)$.

Proof. The inequality (9) implies that

$$u(t) \le k(t) + Y(t) \int_0^{r(t)} \sum_{i=1}^{n} [x_i(s)u(s) + g_i(s)] d_\alpha s.$$

Now an application of Theorem 3.1 provides the desired inequality (10). □

Theorem 3.5. *Let $v, y, h \in C(\mathbb{R}^+, \mathbb{R}^+)$, $r, p \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ and assume that p is non-decreasing with $p(x) \le x$ for $x \ge 0$. If $u \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfies*

$$u(t) \ge v(x) + y(t) \int_{p(x)}^{r(t)} h(s)v(s) d_\alpha s, \quad 0 \le x \le t, \tag{11}$$

then

$$u(t) \ge v(x) e^{y(t) \int_{p(x)}^{r(\tau)} h(\chi) d_\alpha \chi}, \quad 0 \le x \le t, \tag{12}$$

Proof. Denote

$$z(x) = u(t) - y(t) \int_{p(x)}^{r(t)} h(s)v(s) d_\alpha s$$

hence, by using conformable rules we have

$$\begin{aligned} D^\alpha z(x) &= -y(t)h(p(x))v(p(x))D^\alpha p(x) \\ &\ge -y(t)h(p(x))z(p(x))D^\alpha p(x) \\ &\ge -y(t)h(p(x))z(x)D^\alpha p(x). \end{aligned}$$

Thus, we have

$$D^\alpha z(x) + y(t)h(p(x))z(x)D^\alpha p(x) \ge 0.$$

Multiplying the above inequality by $e^{y(t) \int_{p(x)}^{r(t)} h(s) d_\alpha s}$, we obtain that

$$\frac{\partial^\alpha}{\partial x^\alpha} \left(z(x) e^{y(t) \int_{p(x)}^{r(t)} h(s) d_\alpha s} \right) \ge 0.$$

Then if $q(x) = e^{y(t) \int_{p(x)}^{r(t)} h(s) d_\alpha s}$, we have $\frac{\partial^\alpha}{\partial x^\alpha}(zq)(x) \ge 0$ and so $(zq)(t) \ge (zq)(x)$ on $[0, t]$. Now $z(x) \ge v(x)$ and $z(t) = u(t)$ and we have the result given in (12). This result is the best possible in the sense that if equation (11) holds on $[0, t]$, then equation (11) holds on $[0, t]$. □

Theorem 3.6. *Let* $k, m, f, g \in C(\mathbb{R}^+, \mathbb{R}^+)$, $y \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, $r \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ *with* $(t,s) \to \partial_t^\alpha y(t,s) \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$. *Assume in additional that r is non-decreasing and* $r(t) \le t$ *for* $t \ge 0$. *If* $u \in C(\mathbb{R}^+, \mathbb{R}^+)$ *satisfies*

$$u(t) \le k(t) + m(t)\int_0^{r(t)} y(t,s)\,[f(s)u(s) + g(s)]d_\alpha s, \tag{13}$$

then

$$\begin{aligned} u(t) \quad &\le \quad k(t) + m(t)e^{\int_0^{r(t)} y(t,s)m(s)f(s)d_\alpha s} \int_0^t e^{-\int_0^{r(\tau)} y(\tau,s)m(s)f(s)d_\alpha s} \\ &\times \frac{\partial^\alpha}{\partial \tau^\alpha}\left(\int_0^{r(\tau)} y(\tau,s)\,[f(s)k(s) + g(s)]d_\alpha s\right) d_\alpha \tau \end{aligned} \tag{14}$$

for $t \ge 0$.

Proof. Let describe

$$z(t) = \int_0^{r(t)} y(t,s)\,[f(s)u(s) + g(s)]d_\alpha s$$

then our assumptions on y, f, g and r imply that z is non-decreasing on $\mathbb{R}^+$. Thus, for $t \ge 0$, by using Lemma 2.4 (ii), we get

$$\begin{aligned} D^\alpha z(t) \quad &= \quad y(t, r(t))\,[f(r(t))u(r(t)) + g(r(t))]D^\alpha r(t) + \int_0^{r(t)} \left[\frac{\partial^\alpha}{\partial t^\alpha} y(t,s)\right][f(s)u(s) + g(s)]d_\alpha s \\ &\le \quad y(t, r(t))\,[f(r(t))\{k(r(t)) + m(r(t))z(r(t))\} + g(r(t))]\,D^\alpha r(t) + \\ &+ \quad \int_0^{r(t)} \left[\frac{\partial^\alpha}{\partial t^\alpha} y(t,s)\right][f(s)\{k(s) + m(s)z(s)\} + g(s)]\,d_\alpha s \\ &\le \quad y(t, r(t))\,[f(r(t))\{k(r(t)) + m(r(t))z(t)\} + g(r(t))]\,D^\alpha r(t) + \\ &+ \quad \int_0^{r(t)} \left[\frac{\partial^\alpha}{\partial t^\alpha} y(t,s)\right][f(s)k(s) + g(s)]d_\alpha s + z(t)\int_0^{r(t)} \frac{\partial^\alpha}{\partial t^\alpha} y(t,s)\,[m(s)y(s)]d_\alpha s \end{aligned}$$

or, equivalently

$$D^\alpha z(t) - z(t)\frac{\partial^\alpha}{\partial t^\alpha}\left(\int_0^{r(t)} y(t,s)\,m(s)f(s)d_\alpha s\right) \le \frac{\partial^\alpha}{\partial t^\alpha}\left(\int_0^{r(t)} y(t,s)\,[f(s)k(s) + g(s)]d_\alpha s\right).$$

Multiplying the above inequality by $e^{-\int_0^{r(t)} y(t,s)m(s)f(s)d_\alpha s}$, we obtain that

$$\frac{\partial^\alpha}{\partial t^\alpha}\left(z(t)e^{-\int_0^{r(t)} y(t,s)m(s)f(s)d_\alpha s}\right) \le e^{-\int_0^{r(t)} y(t,s)m(s)f(s)d_\alpha s}\frac{\partial^\alpha}{\partial t^\alpha}\left(\int_0^{r(t)} y(t,s)\,[f(s)k(s) + g(s)]d_\alpha s\right).$$

Integrating this from 0 to t yields

$$z(t) \le e^{\int_0^{r(t)} y(t,s)m(s)f(s)d_\alpha s}\int_0^t e^{-\int_0^{r(\tau)} y(\tau,s)m(s)f(s)d_\alpha s}\frac{\partial^\alpha}{\partial \tau^\alpha}\left(\int_0^{r(\tau)} y(\tau,s)\,[f(s)k(s) + g(s)]d_\alpha s\right)d_\alpha \tau.$$

Combine the above inequality with $u(t) \le k(t) + m(t)z(t)$ this imply (14). The proof is complete. □

Remark 3.7. If we take $r(t) = t$, $k(t) = k$ (a constant), $m(t) = 1$, $f(s) = 1$ and $g(s) = 0$ in Theorem 3.6, then the inequality given by Theorem 3.6 reduces to Gronwall's inequality for conformable integrals in [1].

Theorem 3.8. *Let $f, g \in C(\mathbb{R}^+, \mathbb{R}^+)$, $r \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ and assume that r is non-decreasing with $r(t) \le t$ for $t \ge 0$. If $u \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfies*

$$u(t) \le u_0 + \int_0^{r(t)} f(s)u(s)d_\alpha s + \int_0^{r(t)} f(s)\left[\int_0^s g(n)u(n)d_\alpha n\right] d_\alpha s, \quad t \ge 0, \tag{15}$$

then

$$u(t) \le u_0 + u_0 \int_0^t f(s)e^{\int_0^s [f(n)+g(n)]d_\alpha n} d_\alpha s, \quad t \ge 0. \tag{16}$$

Proof. Let denote $z(t)$ the right hand side of inequality (15). Then $u(t) \le z(t)$ and $z(0) = u_0$ and

$$\begin{aligned} D^\alpha z(t) &= f(r(t))u(r(t))D^\alpha r(t) + f(r(t))D^\alpha r(t) \int_0^{r(t)} g(n)u(n)d_\alpha n \\ &\le f(r(t))z(t)D^\alpha r(t) + f(r(t))D^\alpha r(t) \int_0^{r(t)} g(n)z(n)d_\alpha n \\ &\le f(r(t))D^\alpha r(t)\left[z(t) + \int_0^{r(t)} g(n)z(n)d_\alpha n\right]. \end{aligned}$$

Define a function $m(t)$ by

$$m(t) = z(t) + \int_0^{r(t)} g(n)z(n)d_\alpha n$$

then $m(0) = z(0) = u_0$, $D^\alpha z(t) \le f(r(t))D^\alpha r(t)m(t)$, $z(t) \le m(t)$ and

$$\begin{aligned} D^\alpha m(t) &= D^\alpha z(t) + g(r(t))z(r(t))D^\alpha r(t) \\ &\le D^\alpha z(t) + g(r(t))z(t)D^\alpha r(t). \end{aligned}$$

So we get

$$D^\alpha m(t) \le [f(r(t)) + g(r(t))]m(t)D^\alpha r(t). \tag{17}$$

The inequality (17) implies the estimation of $m(t)$ such that

$$m(t) \le u_0 e^{\int_0^{r(t)} [f(n)+g(n)]d_\alpha n}.$$

Then

$$D^\alpha z(t) \le u_0 f(r(t))D^\alpha r(t)e^{\int_0^{r(t)} [f(n)+g(n)]d_\alpha n}. \tag{18}$$

Now by setting $r(t) = s$ in (18) and integrating from 0 to t and substituting the bound $z(t)$ in $u(t) \le z(t)$ we get

$$z(t) \le u_0 + u_0 \int_0^t f(s)e^{\int_0^s [f(n)+g(n)]d_\alpha n} d_\alpha s$$

which this proves our claim. □

4. Concluding Remark

The present study was designed to make the generalization of some inequalities for conformable differential equations. For this purpose we use the Katugampola derivative formulation of conformable derivative of order for $\alpha \in (0,1]$. The findings of this investigation complement those of earlier studies. In other words the present study confirms previous findings and contributes additional evidence by making generalization.

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