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Hermite-Hadamard Type Inequalities for F-Convex Function Involving Fractional Integrals

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Abstract. In this study, we firstly give some properties the family F and $F-$convex function which are defined by B. Samet. Then, we establish Hermite-Hadamard type inequalities involving fractional integrals via $F-$convex function. Some previous results are also recaptured as special cases

1. Introduction

Let $f : I \subseteq R \rightarrow R$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. If f is a convex function then the following double inequality, which is well known in the literature as the Hermite–Hadamard inequality, holds [14]

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Note that some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the mapping f . Both inequalities hold in the reversed direction if f is concave (1).

It is well known that the Hermite–Hadamard inequality plays an important role in nonlinear analysis. Over the last decade, this classical inequality has been improved and generalized in a number of ways; there have been a large number of research papers written on this subject, (see, [2, 3, 7, 8, 10, 13, 19, 20]) and the references therein.

Over the years, many type of convexity have been defined, such as quasi-convex [1], pseudo-convex [11], strongly convex [16], $\varepsilon-$convex [6], $s-$convex [5], $h-$convex [22] etc. Recently, Samet [17] have defined a new concept of convexity that depends on a certain function satisfying some axioms, that generalizes different types of convexity, including $\varepsilon-$convex functions, $\alpha-$convex functions, $h-$convex functions, and many others.

Recall the family $\mathcal{F}$ of mappings $F : \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times[0, 1] \rightarrow \mathbb{R}$ satisfying the following axioms:

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(A1) If $u_i \in L^1(0,1)$, $i = 1,2,3$, then for every $\lambda \in [0,1]$, we have

$$\int_0^1 F(u_1(t), u_2(t), u_3(t), \lambda)dt = F\left(\int_0^1 u_1(t)dt, \int_0^1 u_2(t)dt, \int_0^1 u_3(t)dt, \lambda\right).$$

(A2) For every $u \in L^1(0,1)$, $w \in L^\infty(0,1)$ and $(z_1, z_2) \in \mathbb{R}^2$, we have

$$\int_0^1 F(w(t)u(t), w(t)z_1, w(t)z_2, t)dt = T_{F,w}\left(\int_0^1 w(t)u(t)dt, z_1, z_2\right),$$

where $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function that depends on (F, w), and it is nondecreasing with respect to the first variable.

(A3) For any $(w, u_1, u_2, u_3) \in \mathbb{R}^4$, $u_4 \in [0,1]$, we have

$$wF(u_1, u_2, u_3, u_4) = F(wu_1, wu_2, wu_3, u_4) + L_w$$

where $L_w \in \mathbb{R}$ is a constant that depends only on w.

Definition 1.1. Let $f : [a,b] \to \mathbb{R}$, $(a,b) \in \mathbb{R}^2$, $a < b$, be a given function. We say that f is a convex function with respect to some $F \in \mathcal{F}$ (or F−convex function) if

$$F(f(tx + (1-t)y), f(x), f(y), t) \leq 0, \quad (x, y, t) \in [a,b] \times [a,b] \times [0,1].$$

Remark 1.2. 1) Let $\varepsilon \geq 0$, and let $f : [a,b] \to \mathbb{R}$, $(a,b) \in \mathbb{R}^2$, $a < b$, be an ε-convex function, that is (see [6])

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon, \ (x, y, t) \in [a,b] \times [a,b] \times [0,1].$$

Define the functions $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0,1] \to \mathbb{R}$ by

$$F(u_1, u_2, u_3, u_4) = u_1 - u_4 u_2 - (1-u_4)u_3 - \varepsilon \tag{2}$$

and $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$T_{F,w}(u_1, u_2, u_3) = u_1 - \left(\int_0^1 tw(t)dt\right)u_2 - \left(\int_0^1 (1-t)w(t)dt\right)u_3 - \varepsilon. \tag{3}$$

For

$$L_w = (1-w)\varepsilon, \tag{4}$$

it is clear that $F \in \mathcal{F}$ and

$$F(f(tx + (1-t)y), f(x), f(y), t) = f(tx + (1-t)y) - tf(x) - (1-t)f(y) - \varepsilon \leq 0,$$

that is f is an F−convex function. Particularly, taking $\varepsilon = 0$, we show that if f is a convex function then f is an F−convex function with respect to F defined above.

2) Let $f : [a,b] \to \mathbb{R}$, $(a,b) \in \mathbb{R}^2$, $a < b$, be an α-convex function,$\alpha \in (0,1]$, that is

$$f(tx + (1-t)y) \leq t^\alpha f(x) + (1-t^\alpha)f(y), \ (x, y, t) \in [a,b] \times [a,b] \times [0,1].$$

Define the functions $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0,1] \to \mathbb{R}$ by

$$F(u_1, u_2, u_3, u_4) = u_1 - u_4^\alpha u_2 - (1-u_4^\alpha)u_3 \tag{5}$$

and $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$T_{F,w}(u_1, u_2, u_3) = u_1 - \left(\int_0^1 t^{\alpha} w(t) dt \right) u_2 - \left(\int_0^1 (1 - t^{\alpha}) w(t) dt \right) u_3. \tag{6}$$

For $L_w = 0$, it is clear that $F \in \mathcal{F}$ and

$$F(f(tx + (1 - t)y), f(x), f(y), t) = f(tx + (1 - t)y) - t^{\alpha} f(x) - (1 - t^{\alpha}) f(y) \leq 0,$$

that is f is an $F-$convex function.

3) Let $h : J \to [0, \infty)$ be a given function which is not identical to 0, where J is an interval in $\mathbb{R}$ such that $(0, 1) \subseteq J$. Let $f : [a, b] \to [0, \infty)$, $(a, b) \in \mathbb{R}^2$, $a < b$, be an h-convex function, that is (see [22])

$$f(tx + (1 - t)y) \leq h(t) f(x) + h(1 - t) f(y), \ \ (x, y, t) \in [a, b] \times [a, b] \times [0, 1].$$

Define the functions $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0, 1] \to \mathbb{R}$ by

$$F(u_1, u_2, u_3, u_4) = u_1 - h(u_4) u_2 - h(1 - u_4) u_3 \tag{7}$$

and $T_{F,w} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$T_{F,w}(u_1, u_2, u_3) = u_1 - \left(\int_0^1 h(t) w(t) dt \right) u_2 - \left(\int_0^1 h(1 - t) w(t) dt \right) u_3. \tag{8}$$

For $L_w = 0$, it is clear that $F \in \mathcal{F}$ and

$$F(f(tx + (1 - t)y), f(x), f(y), t) = f(tx + (1 - t)y) - h(t) f(x) - h(1 - t) f(y) \leq 0,$$

that is f is an F-convex function.

In [17], the author established the following Hermite-Hadamard type inequalities using the new convexity concept:

Theorem 1.3. *Let $f : [a, b] \to \mathbb{R}$, $(a, b) \in \mathbb{R}^2$, $a < b$, be an F-convex function, for some $F \in \mathcal{F}$. Suppose that $f \in L_1[a, b]$. Then*

$$F\left(f\left(\frac{a + b}{2} \right), \frac{1}{b - a} \int_a^b f(x) dx, \frac{1}{b - a} \int_a^b f(x) dx, \frac{1}{2} \right) \leq 0,$$

$$T_{F,1}\left(\frac{1}{b - a} \int_a^b f(x) dx, f(a), f(b) \right) \leq 0.$$

Theorem 1.4. *Let $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $(a, b) \in I^{\circ} \times I^{\circ}$, $a < b$. Suppose that*

(i) $\left| f' \right|$ is F-convex on $[a, b]$, for some $F \in \mathcal{F}$

(ii) the function $t \in (0, 1) \to L_{w(t)}$ belongs to $L^1(0, 1)$, where $w(t) = |1 - 2t|$. Then,

$$T_{F,w}\left(\frac{2}{b - a} \left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) dx \right|, \left| f'(a) \right|, \left| f'(b) \right| \right) + \int_0^1 L_{w(t)} dt \leq 0.$$

Theorem 1.5. *Let $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $(a,b) \in I^{\circ} \times I^{\circ}$, $a < b$ and let $p > 1$. Suppose that $\left|f'\right|^{p/(p-1)}$ is F-convex on $[a,b]$, for some $F \in \mathcal{F}$ and $\left|f'\right| \in L^{p/(p-1)}(a,b)$. Then*

$$T_{F,1}\left(A(p,f), \left|f'(a)\right|^{p/(p-1)}, \left|f'(b)\right|^{p/(p-1)}\right) \leq 0$$

where

$$A(p,f) = \left(\frac{2}{b-a}\right)^{\frac{p}{p-1}} (p+1)^{\frac{1}{p-1}} \left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right|^{\frac{p}{p-1}}.$$

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult [4, 9, 12, 15].

Definition 1.6. Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $x \geq a$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.

It is remarkable that Sarikaya et al. [21] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1.7. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{9}$$

with $\alpha > 0$.

Meanwhile, Sarikaya et al. [21] presented the following important integral identity including the first-order derivative of f to establish many interesting Hermite-Hadamard type inequalities for convexity functions via Riemann-Liouville fractional integrals of the order $\alpha > 0$.

Lemma 1.8. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L[a,b]$, then the following equality for fractional integrals holds:*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] = \frac{b-a}{2} \int_0^1 \left[(1-t)^{\alpha} - t^{\alpha}\right] f'(ta + (1-t)b)\, dt. \tag{10}$$

2. Hermite-Hadamard Type Inequality Involving Fractional Integrals

In this section, we establish some inequalities of Hermite-Hadamard type including fractional integrals via $F-$convex functions.

Theorem 2.1. *Let $I \subseteq R$ be an interval, $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping on I°, $a, b \in I^\circ$, $a < b$. If f is F-convex on $[a, b]$, for some $F \in \mathcal{F}$, then we have the inequalities*

$$F\left(f\left(\frac{a+b}{2}\right), \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} J_{a^+}^\alpha f(b), \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} J_{b^-}^\alpha f(a), \frac{1}{2}\right) + \int_0^1 L_{w(t)} dt \leq 0 \tag{11}$$

and

$$T_{F,w}\left(\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right], f(a)+f(b), f(a)+f(b)\right) + \int_0^1 L_{w(t)} dt \leq 0 \tag{12}$$

where $w(t) = \alpha t^{\alpha-1}$.

Proof. Since f is $F-$convex, we have

$$F\left(f\left(\frac{x+y}{2}\right), f(x), f(y), \frac{1}{2}\right) \leq 0, \ x, y \in [a, b]$$

For $x = ta + (1-t)b$ and $y = tb + (1-t)a$, we have

$$F\left(f\left(\frac{a+b}{2}\right), f(ta+(1-t)b), f(tb+(1-t)a), \frac{1}{2}\right) \leq 0, \ t \in [0, 1].$$

Multiplying this inequality by $w(t) = \alpha t^{\alpha-1}$ and using axiom (A3), we get

$$F\left(\alpha t^{\alpha-1} f\left(\frac{a+b}{2}\right), \alpha t^{\alpha-1} f(ta+(1-t)b), \alpha t^{\alpha-1} f(ta+(1-t)b), \frac{1}{2}\right) + L_{w(t)} \leq 0,$$

for $t \in [0, 1]$. Integrating over $[0, 1]$ with respect to the variable t and using axiom (A1), we obtain

$$F\left(f\left(\frac{a+b}{2}\right) \alpha \int_0^1 t^{\alpha-1} dt, \alpha \int_0^1 t^{\alpha-1} f(ta+(1-t)b) dt, \alpha \int_0^1 t^{\alpha-1} f(ta+(1-t)b) dt, \frac{1}{2}\right) + \int_0^1 L_{w(t)} dt \leq 0.$$

Using the facts that

$$\int_0^1 t^{\alpha-1} f(ta+(1-t)b) dt = \frac{1}{(b-a)^\alpha} \int_a^b (b-x)^{\alpha-1} f(x) dx = \frac{\Gamma(\alpha)}{(b-a)^\alpha} J_{a^+}^\alpha f(b)$$

and

$$\int_0^1 t^{\alpha-1} f(ta+(1-t)b) dt = \frac{1}{(b-a)^\alpha} \int_a^b (x-a)^{\alpha-1} f(x) dx = \frac{\Gamma(\alpha)}{(b-a)^\alpha} J_{b^-}^\alpha f(a),$$

we obtain

$$F\left(f\left(\frac{a+b}{2}\right), \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} J_{a^+}^\alpha f(b), \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} J_{b^-}^\alpha f(a), \frac{1}{2}\right) + \int_0^1 L_{w(t)} dt \leq 0$$

which gives (11).

On the other hand, since f is F-convex, we have

$$F(f(ta+(1-t)b), f(a), f(b), t) \leq 0, \ t \in [0, 1]$$

and

$$F(f(tb+(1-t)a), f(b), f(a), t) \leq 0, \ t \in [0, 1].$$

Using the linearity of F, we get

$$F\left(f\left(ta+(1-t)b\right)+f\left(tb+(1-t)a\right), f(a)+f(b), f(a)+f(b), t\right)\leq 0,\ t\in[0,1].$$

Applying the axiom (A3) for $w(t)=\alpha t^{\alpha-1}$, we obtain

$$F\left(\alpha t^{\alpha-1}\left[f\left(ta+(1-t)b\right)+f\left(tb+(1-t)a\right)\right], \alpha t^{\alpha-1}\left[f(a)+f(b)\right], \alpha t^{\alpha-1}\left[f(a)+f(b)\right], t\right)+L_{w(t)}\leq 0,$$

for $t\in[0,1]$. Integrating over $[0,1]$ and using axiom (A2), we have

$$T_{F,w}\left(\int_0^1 \alpha t^{\alpha-1}\left[f\left(ta+(1-t)b\right)+f\left(tb+(1-t)a\right)\right]dt, f(a)+f(b), f(a)+f(b)\right)+\int_0^1 L_{w(t)}dt\leq 0,$$

that is

$$T_{F,w}\left(\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{a^+}^{\alpha}f(b)+\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b^-}^{\alpha}f(a), f(a)+f(b), f(a)+f(b)\right)+\int_0^1 L_{w(t)}dt\leq 0.$$

This completes the proof. □

Corollary 2.2. *If we choose $F(u_1,u_2,u_3,u_4)=u_1-u_4u_2-(1-u_4)u_3-\varepsilon$ in Theorem 2.1, then the function f is ε-convex on $[a,b]$, $\varepsilon\geq 0$ and we have the inequality*

$$f\left(\frac{a+b}{2}\right)-\varepsilon\leq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\leq\frac{f(a)+f(b)}{2}+\frac{\varepsilon}{2}$$

Proof. Using (4)with $w(t)=\alpha t^{\alpha-1}$, we have

$$\int_0^1 L_{w(t)}dt=\varepsilon\int_0^1(1-\alpha t^{\alpha-1})dt=0. \tag{13}$$

Using (2), (11) and (13), we get

$$\begin{aligned}0 &\geq F\left(f\left(\frac{a+b}{2}\right), \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{a^+}^{\alpha}f(b), \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b^-}^{\alpha}f(a), \frac{1}{2}\right)+\int_0^1 L_{w(t)}dt\\ &= f\left(\frac{a+b}{2}\right)-\frac{1}{2}\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]-\varepsilon,\end{aligned}$$

that is

$$f\left(\frac{a+b}{2}\right)-\varepsilon\leq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right].$$

On the other hand, using (3) with $w(t)=\alpha t^{\alpha-1}$, we have

$$T_{F,w}(u_1,u_2,u_3) = u_1-\alpha\left(\int_0^1 t^{\alpha}dt\right)u_2-\alpha\left(\int_0^1(1-t)t^{\alpha-1}dt\right)u_3-\varepsilon=u_1-\frac{\alpha u_2+u_3}{\alpha+1}-\varepsilon \tag{14}$$

for $u_1, u_2, u_3 \in \mathbb{R}$. Hence, from (12) and (14), we obtain

$$\begin{aligned} 0 &\geq T_{F,w}\left(\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right], f(a)+f(b), f(a)+f(b)\right)+\int_0^1 L_{w(t)}dt \\ &= \frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]-\frac{1}{\alpha+1}\left[\alpha\left(f(a)+f(b)\right)+\left(f(a)+f(b)\right)\right]-\varepsilon \\ &= \frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]-\left(f(a)+f(b)\right)-\varepsilon. \end{aligned}$$

This implies that

$$\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]\leq f(a)+f(b)+\varepsilon$$

and thus the proof is completed. □

Remark 2.3. If we take $\varepsilon = 0$ in Corollary 2.2, then f is convex and we have the inequality (9).

Corollary 2.4. *If we choose $F(u_1, u_2, u_3, u_4) = u_1 - h(u_4)u_2 - h(1-u_4)u_3$ in Theorem 2.1, then the function f is h-convex on $[a, b]$ and we have the inequality*

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]\leq\alpha\left(\int_0^1\left[h(t)+h(1-t)\right]t^{\alpha-1}dt\right)\frac{f(a)+f(b)}{2}.$$

Proof. Using (4) and (11) with $L_{w(t)} = 0$, we have

$$\begin{aligned} 0 &\geq F\left(f\left(\frac{a+b}{2}\right),\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}J_{a^+}^\alpha f(b),\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}J_{b^-}^\alpha f(a),\frac{1}{2}\right)+\int_0^1 L_{w(t)}dt \\ &= f\left(\frac{a+b}{2}\right)-h\left(\frac{1}{2}\right)\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right], \end{aligned}$$

that is

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right].$$

On the other hand, using (8) and (12) with $w(t) = \alpha t^{\alpha-1}$, we obtain

$$\begin{aligned} 0 &\geq T_{F,w}\left(\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right], f(a)+f(b), f(a)+f(b)\right)+\int_0^1 L_{w(t)}dt \\ &= \frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]-\alpha\left[\int_0^1 h(t)t^{\alpha-1}dt+\int_0^1 h(1-t)t^{\alpha-1}dt\right]\left[f(a)+f(b)\right] \\ &= \frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]-\alpha\left(\int_0^1\left[h(t)+h(1-t)\right]t^{\alpha-1}dt\right)\left[f(a)+f(b)\right], \end{aligned}$$

that is,

$$\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]\leq\alpha\left(\int_0^1\left[h(t)+h(1-t)\right]t^{\alpha-1}dt\right)\left[f(a)+f(b)\right]$$

and thus the proof is completed. □

Theorem 2.5. *Let $I \subseteq R$ be an interval, $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$, $a < b$. Suppose that $\left|f'\right|$ is $F-$convex on $[a,b]$, for some $F \in \mathcal{F}$ and the function $t \in [0,1] \to L_{w(t)}$ belongs to $L_1[0,1]$, where $w(t) = |(1-t)^\alpha - t^\alpha|$. Then, we have the inequality*

$$T_{F,w}\left(\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right|, \left|f'(a)\right|, \left|f'(b)\right|\right) + \int_0^1 L_{w(t)}dt \le 0.$$

Proof. Since $\left|f'\right|$ is F-convex, we have

$$F\left(\left|f'(ta+(1-t)b)\right|, \left|f'(a)\right|, \left|f'(b)\right|, t\right) \le 0,\ t \in [0,1].$$

Using axiom (A3) with $w(t) = |(1-t)^\alpha - t^\alpha|$, we get

$$F\left(w(t)\left|f'(ta+(1-t)b)\right|, w(t)\left|f'(a)\right|, w(t)\left|f'(b)\right|, t\right) + L_{w(t)} \le 0,\ t \in [0,1].$$

Integrating over $[0,1]$ and using axiom (A2), we obtain

$$T_{F,w}\left(\int_0^1 w(t)\left|f'(ta+(1-t)b)\right|dt, \left|f'(a)\right|, \left|f'(b)\right|\right) + \int_0^1 L_{w(t)}dt \le 0,\ t \in [0,1].$$

From Lemma 1.8, we have

$$\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right| \le \int_0^1 w(t)\left|f'(ta+(1-t)b)\right|dt.$$

Since $T_{F,w}$ is nondecreasing with respect to the first variable, we establish

$$T_{F,w}\left(\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right|, \left|f'(a)\right|, \left|f'(b)\right|\right) + \int_0^1 L_{w(t)}dt \le 0.$$

The proof is completed. □

Corollary 2.6. *Under assumptions of Theorem 2.5, if we choose $F(u_1, u_2, u_3, u_4) = u_1 - u_4u_2 - (1-u_4)u_3 - \varepsilon$, then the function $\left|f'\right|$ is ε-convex on $[a,b]$, $\varepsilon \ge 0$ and we have the inequality*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right]\right| \\ \le\ & \frac{b-a}{2(\alpha+1)}\left(1 - \frac{1}{2^\alpha}\right)\left[\left|f'(a)\right| + \left|f'(b)\right| + 2\varepsilon\right]. \end{aligned}$$

Proof. From (4) with $w(t) = |(1-t)^\alpha - t^\alpha|$, we have

$$\begin{aligned} \int_0^1 L_{w(t)}dt &= \varepsilon\int_0^1 (1 - |(1-t)^\alpha - t^\alpha|)dt \\ &= \varepsilon\left[\int_0^{1/2}(1-(1-t)^\alpha + t^\alpha)dt + \int_{1/2}^1 (1+(1-t)^\alpha - t^\alpha)dt\right] \\ &= \varepsilon\left(1 - \frac{2}{\alpha+1}\left(1 - \frac{1}{2^\alpha}\right)\right). \end{aligned}$$

Using (3) with $w(t) = |(1-t)^{\alpha} - t^{\alpha}|$

$$\begin{aligned} T_{F,w}(u_1, u_2, u_3) &= u_1 - \alpha\left(\int_0^1 t\,|(1-t)^{\alpha} - t^{\alpha}|\,dt\right)u_2 - \alpha\left(\int_0^1 (1-t)\,|(1-t)^{\alpha} - t^{\alpha}|\,dt\right)u_3 - \varepsilon \\ &= u_1 - \frac{1}{\alpha+1}\left(1 - \frac{1}{2^{\alpha}}\right)(u_2 + u_3) - \varepsilon \end{aligned}$$

for $u_1, u_2, u_3 \in \mathbb{R}$. Then, by Theorem 2.5, we have

$$\begin{aligned} 0 &\geq T_{F,w}\left(\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]\right|, \left|f'(a)\right|, \left|f'(b)\right|\right) + \int_0^1 L_{w(t)}dt \\ &= \frac{2}{b-a}\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]\right| \\ &\quad - \frac{1}{\alpha+1}\left(1 - \frac{1}{2^{\alpha}}\right)\left[\left|f'(a)\right| + \left|f'(b)\right|\right] - \varepsilon + \varepsilon\left(1 - \frac{2}{\alpha+1}\left(1 - \frac{1}{2^{\alpha}}\right)\right). \end{aligned}$$

This completes the proof. □

Remark 2.7. If we choose $\varepsilon = 0$ in Corollary 2.6, then $\left|f'\right|$ is convex and we have the inequality

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]\right| \\ \leq\ &\frac{b-a}{2(\alpha+1)}\left(1 - \frac{1}{2^{\alpha}}\right)\left[\left|f'(a)\right| + \left|f'(b)\right|\right] \end{aligned}$$

which is given by Sarikaya et. al in [21].

Corollary 2.8. *Under assumption of Theorem 2.5, if we choose $F(u_1, u_2, u_3, u_4) = u_1 - h(u_4)u_2 - h(1-u_4)u_3$, then the function $\left|f'\right|$ is h-convex on $[a, b]$ and we have the inequality*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]\right| \\ \leq\ &\frac{b-a}{2}\left(\int_0^1 h(t)\,|(1-t)^{\alpha} - t^{\alpha}|\,dt\right)\left[\left|f'(a)\right| + \left|f'(b)\right|\right]. \end{aligned}$$

Proof. From (8) with $w(t) = |(1-t)^{\alpha} - t^{\alpha}|$, we have

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]\right| \\ \leq\ &\frac{b-a}{2}\left(\int_0^1 h(t)\,|(1-t)^{\alpha} - t^{\alpha}|\,dt\right)\left[\left|f'(a)\right| + \left|f'(b)\right|\right]. \end{aligned}$$

for $u_1, u_2, u_3 \in \mathbb{R}$. Then, by Theorem 2.5,

$$
\begin{aligned}
& T_{F,w}\left(\frac{2}{b-a}\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\right|, \left|f'(a)\right|, \left|f'(b)\right|\right) \\
= & \frac{2}{b-a}\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\right| \\
& -\left(\int_0^1 h(t)\left|(1-t)^{\alpha}-t^{\alpha}\right| dt\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right] \le 0.
\end{aligned}
$$

This completes the proof. □

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