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On The Circulant Matrices with Ducci Sequences and Fibonacci Numbers

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Abstract. A Ducci sequence generated by $A = (a_1, a_2, ..., a_n) \in \mathbb{Z}^n$ is the sequence $\{A, DA, D^2A, ...\}$ where the Ducci map $D : \mathbb{Z}^n \to \mathbb{Z}^n$ is defined by

$$\begin{aligned} D(A) &= D(a_1, a_2, ..., a_n) \\ &= (|a_2 - a_1|, |a_3 - a_2|, ..., |a_n - a_{n-1}|, |a_n - a_1|). \end{aligned}$$

In this study, we examine some properties of the matrices C_n, DC_n, D^2C_n, where C_n =Circ$(c_0, c_1, ..., c_{n-1})$ is a circulant matrix whose entries consist of Fibonacci numbers.

1. Introduction

1.1. Ducci Sequences

Let $A = (a_1, a_2, ..., a_n)$ be an $n-$tuple of integers and $D : \mathbb{Z}^n \to \mathbb{Z}^n$ be a map defined by

$$D(a_1, a_2, ..., a_n) := (|a_2 - a_1|, |a_3 - a_2|, ..., |a_n - a_{n-1}|, |a_n - a_1|).$$

The map D is called the *Ducci map* and the sequence $\{A, DA, D^2A, ...\}$ is called a *Ducci sequence*. Professor E.Ducci made some observations on the map D in the 1800's [6], so when Ducci sequences were first introduced in 1937 [7], their name was attributed to E. Ducci. Under the Ducci map, the behavior of the starting vector $A = (a_1, a_2, ..., a_n)$ is interesting and has been examined in a number of papers $[2-6, 9, 16, 21, 22]$ .When $n = 2^k$ for some positive integer k, every starting vector ends in a tuple having all components equal [7] and converges to the zero vector [3, 4]. Under the Ducci map, every starting vector converges to a periodic orbit [9] and reaches to $(x_1, x_2, ..., x_n)$ when $n \neq 2^k$, where $x_j \in \{0, m\}$ and m is a positive constant [4, 6]. Also, there are integers s and r with $0 \leq s < r$ such that $D^sA = D^rA$. Thus, we say that the Ducci sequence $\{A, DA, D^2A, ...\}$ has period $r - s$, when r and s are as small as possible [2].

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Ducci matrix sequences are closely related to the set of rational and irrational numbers [11, 17] . Thus, the researches on Ducci matrix sequences have increased in recent years [3, 11, 17, 20]. For example, Solak and Bahşi [20] have established relationships between the spectral norm, Frobenius norm, l_p norm, determinant and eigenvalues of circulant matrix $\mathrm{Circ}(A) = \mathrm{Circ}(a_1, a_2, ..., a_n)$ and its image under the Ducci map.

Let us consider the following starting vector

$$F = (F_1, F_2, ..., F_n),$$

where F_s is the sth Fibonacci number defined by (6) in Section 2. Then, its consecutive images under the Ducci map are

$$\begin{aligned} DF &= (F_2 - F_1, F_3 - F_2, \ldots, F_n - F_{n-1}, F_n - F_1) \\ &= (F_0, F_1, \ldots, F_{n-2}, F_n - 1) \end{aligned}$$

and

$$\begin{aligned} D^2F &= (F_1 - F_0, F_2 - F_1, \ldots, F_{n-2} - F_{n-3}, F_n - 1 - F_{n-2}, F_n - 1 - F_0) \\ &= (F_{-1}, F_0, F_1, \ldots, F_{n-4}, F_{n-1} - 1, F_n - 1) \\ &= (1, F_0, F_1, \ldots, F_{n-4}, F_{n-1} - 1, F_n - 1)\,, \end{aligned}$$

where $F_{-1} := 1$ (see (7) in Section 2 for the Fibonacci numbers with negative indices). Thus, the above starting vector and its consecutive images under the Ducci map, for $n \geq 3$, yield the following $n \times n$ matrices:

$$\mathrm{Circ}(F) := \begin{bmatrix} F_1 & F_2 & \cdots & F_n \\ F_n & F_1 & \cdots & F_{n-1} \\ \vdots & \vdots & & \vdots \\ F_2 & F_3 & \cdots & F_1 \end{bmatrix}, \tag{1}$$

$$\mathrm{Circ}(DF) := \begin{bmatrix} F_0 & F_1 & \cdots & F_{n-2} & F_n - 1 \\ F_n - 1 & F_0 & \cdots & F_{n-3} & F_{n-2} \\ \vdots & \vdots & & \vdots & \vdots \\ F_1 & F_2 & \cdots & F_n - 1 & F_0 \end{bmatrix} \tag{2}$$

and

$$\mathrm{Circ}(D^2F) := \begin{bmatrix} 1 & F_0 & \cdots & F_{n-4} & F_{n-1} - 1 & F_n - 1 \\ F_n - 1 & 1 & \cdots & F_{n-5} & F_{n-4} & F_{n-1} - 1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ F_0 & F_1 & \cdots & F_{n-1} - 1 & F_n - 1 & 1 \end{bmatrix}. \tag{3}$$

Now that we have seen the matrices $\mathrm{Circ}(F)$, $\mathrm{Circ}(DF)$ and $\mathrm{Circ}(D^2F)$, we would like to determine if there is any relationship between the norms, determinants and eigenvalues of them. Because they are all circulant matrices, we will begin our study with some properties of circulant matrices.

1.2. Circulant Matrices

Let $(c_0, c_1, \ldots, c_{n-1})$ be an $n-$tuple of complex numbers. Then, the matrix

$$
C := \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & \cdots & c_{n-3} \\ \vdots & \vdots & & \vdots \\ c_1 & c_2 & \cdots & c_0 \end{bmatrix}
$$

is called a *circulant matrix* associated with $n-$tuple $(c_0, c_1, \ldots, c_{n-1})$. Thus, we denote the circulant matrix C by $C = \mathrm{Circ}(c_0, c_1, \ldots, c_{n-1})$. By means of [8, Chapter 3] the properties of circulant matrices are well known. Some of them are:

1. Let C be an $n \times n$ matrix. Then, C is a circulant matrix if and only if

$$CP = PC,$$

where P is the $n \times n$ matrix $P = \mathrm{Circ}(0, 1, 0, ..., 0)$.

2. $\mathrm{Circ}(c_0, c_1, \ldots, c_{n-1}) = c_0 I + c_1 P + ... + c_{n-1} P^{n-1}$, where I denotes the $n \times n$ identity matrix and P is the $n \times n$ matrix $P = \mathrm{Circ}(0, 1, 0, ..., 0)$.

3. All circulant matrices of the same order commute. If C is a circulant matrix so is C^*, where C^* denotes conjugate transpose of C. Hence C and C^* commute and therefore all circulant matrices are normal matrices.

4. The eigenvalues of matrix $C = \mathrm{Circ}(c_0, c_1, \ldots, c_{n-1})$ are

$$\lambda_j = \sum_{k=0}^{n-1} c_k w^{-jk}, \quad \text{for} \ \ 0 \le j \le n-1,$$

where $w = e^{\frac{2\pi i}{n}}$ and $i = \sqrt{-1}$.

There are many studies on general and special circulant matrices [1, 10, 13, 14, 23]. In recent years, researchers have been especially attracted to the study of norms and determinants of the circulant matrices with Fibonacci and Lucas numbers as entries [13, 18 − 20]. The Ducci map was first applied to circulant matrices by Solak and Bahşi [20]. By applying the Ducci map to each row of the circulant matrix

$$\mathrm{Circ}(A) = \mathrm{Circ}(a_1, a_2, ..., a_n) \tag{4}$$

associated with $A = (a_1, a_2, ..., a_n) \in \mathbb{Z}^n$, they have defined the circulant matrix

$$\mathrm{Circ}(DA) = \mathrm{Circ}(|a_2 - a_1|, |a_3 - a_2|, ..., |a_n - a_{n-1}|, |a_n - a_1|) \tag{5}$$

associated with $D(A) = (|a_2 - a_1|, |a_3 - a_2|, ..., |a_n - a_{n-1}|, |a_n - a_1|)$. Then, they have established some relationships between spectral norm, Frobenius norm, l_p norm, determinant and eigenvalues of the matrices $\mathrm{Circ}(A)$ and $\mathrm{Circ}(DA)$.

This paper is organized as follows: In Section 2 we introduce some definitions and lemmas related to our study. In Section 3 we prove some equalities and inequalities involving norms, determinants and eigenvalues of circulant matrices $\mathrm{Circ}(F)$, $\mathrm{Circ}(DF)$ and $\mathrm{Circ}(D^2F)$.

Throughout this study matrices $\mathrm{Circ}(F)$, $\mathrm{Circ}(DF)$ and $\mathrm{Circ}(D^2F)$ are the $n \times n$ circulant matrices defined by (1), (2) and (3).

2. Preliminaries

The *Fibonacci numbers* are defined by the second order linear recurrence relation:

$$
\begin{cases} F_{n+1} := F_n + F_{n-1} & \text{for} \ \ n \ge 1 \\ F_0 := 0 \\ F_1 := 1. \end{cases} \tag{6}
$$

Also, we have the backwards rule

$$F_{-n} := (-1)^{n+1} F_n, \tag{7}$$

where $n > 0$. The Fibonacci numbers have many interesting identities [15, Chapter 5] such as

$$F_{n+1}^2 + F_n^2 = F_{2n+1}, \tag{8}$$

$$F_{n+1}^2 - F_n^2 = F_{n-1} F_{n+2} \tag{9}$$

and

$$\sum_{s=0}^{n-1} F_s^2 = F_{n-1} F_n. \tag{10}$$

In addition, one can see easily that

$$F_n - F_{n-3} = 2F_{n-2}. \tag{11}$$

Definition 2.1. *[12, pp. 291] Let $A = \left(a_{jk}\right)$ be any $m \times n$ matrix. The l_p $(1 < p < \infty)$ norm of A is*

$$\|A\|_p := \left(\sum_{j=1}^{m} \sum_{k=1}^{n} \left|a_{ij}\right|^p \right)^{\frac{1}{p}}.$$

In the particular case $p = 2$, this is the Frobenius norm of A and we denote it by

$$\|A\|_F := \sqrt{\left(\sum_{j=1}^{m} \sum_{k=1}^{n} \left|a_{ij}\right|^2 \right)}.$$

Definition 2.2. *[12, pp. 295] Let $A = \left(a_{jk}\right)$ be any $m \times n$ matrix. The spectral norm of A is*

$$\|A\|_2 := \sqrt{\max \lambda_s \left(A^* A\right)},$$

where $\lambda_s\left(A^*A\right)$ are eigenvalues of A^*A and A^* is conjugate transpose of A.

In the next three lemmas, matrices Circ(A) and Circ(DA) denote the circulant matrices in (4) and (5), respectively.

Lemma 2.3. *[20, Theorem 5] Let μ_j and λ_j $(j = 0, 1, ..., n-1)$ be eigenvalues of the matrices Circ(DA) and Circ(A), respectively. If $a_1 \le a_2 \le ... \le a_n$, then*

$$\mu_j = \left(\lambda_j + 2a_n - 2a_1\right) w^{-j} - \lambda_j, \quad \text{for } 0 \le j \le n-1$$

where $w = e^{\frac{2\pi i}{n}}$ and $i = \sqrt{-1}$.

Lemma 2.4. *[20, Theorem 6] The spectral norm of the matrix Circ(DA) with $a_1 \le a_2 \le ... \le a_n$ satisfies*

$$\|Circ(DA)\|_2 = 2\left(a_n - a_1\right).$$

Lemma 2.5. *[20, Theorem 4] The determinant of the matrix Circ(DA) satisfies*

$$\left|\det Circ(DA)\right| \le \frac{1}{n^{\frac{n}{2}}} \|Circ(DA)\|_E^n.$$

3. Main Results

Theorem 3.1. *For the Frobenius norms of the $n \times n$ matrices $Circ(D^2F)$ and $Circ(DF)$, for $n \geq 3$, we have*

$$\left\|Circ(D^2F)\right\|_F^2 - \|Circ(DF)\|_F^2 = 2n\,(F_{n-2} - 1)\,(F_{n-3} - 1)\,.$$

Proof. By the definition of the Frobenius norm, we have

$$\|\text{Circ}(DF)\|_F^2 = n\left[\sum_{k=0}^{n-2} F_k^2 + (F_n - 1)^2\right]$$

and

$$\begin{aligned}
\left\|\text{Circ}(D^2F)\right\|_F^2 &= n\left[\sum_{k=-1}^{n-4} F_k^2 + (F_{n-1} - 1)^2 + (F_n - 1)^2\right] \\
&= n\left[1 + \sum_{k=0}^{n-2} F_k^2 + (F_{n-1} - 1)^2 + (F_n - 1)^2 - F_{n-2}^2 - F_{n-3}^2\right] \\
&= n\left[\sum_{k=0}^{n-2} F_k^2 + (F_n - 1)^2\right] + n\left[F_{n-1}^2 - 2F_{n-1} - F_{n-2}^2 - F_{n-3}^2 + 2\right] \\
&= \|\text{Circ}(DF)\|_F^2 + n\left[F_{n-1}^2 - 2F_{n-1} - F_{n-2}^2 - F_{n-3}^2 + 2\right]. \qquad (12)
\end{aligned}$$

By (9) and (11), we have

$$\begin{aligned}
F_{n-1}^2 - 2F_{n-1} - F_{n-2}^2 - F_{n-3}^2 + 2 &= -2F_{n-1} - F_{n-3}^2 + F_{n-3}F_n + 2 \\
&= -2F_{n-2} - 2F_{n-3} + F_{n-3}\,(F_n - F_{n-3}) + 2 \\
&= -2F_{n-2} - 2F_{n-3} + 2F_{n-2}F_{n-3} + 2 \\
&= -2\,(F_{n-2} - 1) + 2F_{n-3}\,(F_{n-2} - 1) \\
&= 2\,(F_{n-2} - 1)\,(F_{n-3} - 1)\,.
\end{aligned}$$

This completes the proof. □

Theorem 3.2. *For the Frobenius norms of the $n \times n$ matrices $Circ(D^2F)$ and $Circ(F)$, for $n \geq 3$, we have*

$$\left\|Circ(D^2F)\right\|_F^2 - \|Circ(F)\|_F^2 = -n\,(2F_{n+1} + F_{2n-5} - 3)\,.$$

Proof. By [20, Example 2], we have:

$$\|\text{Circ}(F)\|_F^2 - \|\text{Circ}(DF)\|_F^2 = n\left(F_{n-1}^2 + 2F_n - 1\right). \qquad (13)$$

By (8), (12) and (13), we have

$$\begin{aligned}
\left\|\text{Circ}(D^2F)\right\|_F^2 &= \|\text{Circ}(F)\|_F^2 - n\left[\ F_{n-1}^2 + 2F_n - 1 - F_{n-1}^2 + 2F_{n-1} + F_{n-2}^2 + F_{n-3}^2 - 2\ \right] \\
&= \|\text{Circ}(F)\|_F^2 - n\,[2F_{n+1} + F_{2n-5} - 3]\,,
\end{aligned}$$

which completes the proof of the theorem. □

Theorem 3.3. *For the l_p norms of the $n \times n$ matrices Circ(F), Circ(DF) and Circ(D^2F), for $n \geq 3$, we have*

$$\left\|Circ(D^2F)\right\|_p^p - \|Circ(DF)\|_p^p = n\left[(F_{n-1}-1)^p - F_{n-2}^p - F_{n-3}^p + 1\right] \tag{14}$$

and

$$\|Circ(F)\|_p^p - \left\|Circ(D^2F)\right\|_p^p = n\left[\; F_n^p + F_{n-1}^p + F_{n-2}^p + F_{n-3}^p - (F_n-1)^p - (F_{n-1}-1)^p - 1 \;\right]. \tag{15}$$

Proof. By the definition of l_p norm, we have

$$\left\|\mathrm{Circ}(D^2F)\right\|_p^p = n\left[\sum_{k=-1}^{n-4} F_k^p + (F_{n-1}-1)^p + (F_n-1)^p\right],$$

$$\|\mathrm{Circ}(DF)\|_p^p = n\left[\sum_{k=0}^{n-2} F_k^p + (F_n-1)^p\right] \tag{16}$$

and

$$\|\mathrm{Circ}(F)\|_p^p = n\sum_{k=1}^{n} F_k^p. \tag{17}$$

Thus

$$\begin{aligned}\left\|\mathrm{Circ}(D^2F)\right\|_p^p &= n\left[1 + \sum_{k=0}^{n-2} F_k^p + (F_{n-1}-1)^p + (F_n-1)^p - F_{n-2}^p - F_{n-3}^p\right] \\ &= n\left[\sum_{k=0}^{n-2} F_k^p + (F_n-1)^p\right] + n\left[(F_{n-1}-1)^p - F_{n-2}^p - F_{n-3}^p + 1\right] \\ &= \|\mathrm{Circ}(DF)\|_p^p + n\left[(F_{n-1}-1)^p - F_{n-2}^p - F_{n-3}^p + 1\right],\end{aligned}$$

which yields formula (14). On the other hand,

$$\begin{aligned}\left\|\mathrm{Circ}(D^2F)\right\|_p^p &= n\left[\; 1 + \sum_{k=0}^{n} F_k^p + (F_{n-1}-1)^p + (F_n-1)^p - F_n^p - F_{n-1}^p - F_{n-2}^p - F_{n-3}^p \;\right] \\ &= n\sum_{k=0}^{n} F_k^p + n\left[\; (F_{n-1}-1)^p + (F_n-1)^p - F_n^p - F_{n-1}^p - F_{n-2}^p - F_{n-3}^p + 1 \;\right] \\ &= \|\mathrm{Circ}(F)\|_p^p + n\left[\; (F_{n-1}-1)^p + (F_n-1)^p - F_n^p - F_{n-1}^p - F_{n-2}^p - F_{n-3}^p + 1 \;\right],\end{aligned}$$

from which formula (15) follows, and the proof is completed. □

We remark that the next equality, which is an immediate consequence of (16) and (17), already appeared in [20, Example 3]:

$$\|\mathrm{Circ}(F)\|_p^p - \|\mathrm{Circ}(DF)\|_p^p = n\left[F_{n-1}^p + F_n^p - (F_n-1)^p\right].$$

Theorem 3.4. *The determinant of the $n \times n$ matrix Circ(D^2F) satisfies*

$$\left|\det Circ(D^2F)\right| \leq (F_{n-4}F_{n-3} + F_{2n-1} - 2F_{n+1} + 3)^{\frac{n}{2}},$$

where $n \geq 4$.

Proof. If we take $DF = A$, then $DA = D(DF) = D^2F$. Thus, for the determinant of the matrix $\mathrm{Circ}(D^2F)$, Lemma (2.5) and the equations (8) and (10) yield

$$\begin{aligned}\left|\det \mathrm{Circ}(D^2F)\right| &\leq \frac{1}{n^{\frac{n}{2}}}\left(\left\|\mathrm{Circ}(D^2F)\right\|_E\right)^n \\ &= \frac{1}{n^{\frac{n}{2}}}\left[\left(n\left[1+\sum_{k=0}^{n-4}F_k^2+(F_{n-1}-1)^2+(F_n-1)^2\right]\right)^{\frac{1}{2}}\right]^n \\ &= \frac{1}{n^{\frac{n}{2}}}\left[\left(n\left[F_{n-4}F_{n-3}+F_{n-1}^2+F_n^2-2F_{n-1}-2F_n+3\right]\right)^{\frac{1}{2}}\right]^n \\ &= \left(F_{n-4}F_{n-3}+F_{2n-1}-2F_{n+1}+3\right)^{\frac{n}{2}}.\end{aligned}$$

Hence, the desired result is obtained. □

For determinants of matrices $\mathrm{Circ}(F)$ and $\mathrm{Circ}(DF)$, the inequalities

$$|\det \mathrm{Circ}(F)| \leq (F_nF_{n+1})^{\frac{n}{2}}$$

and

$$|\det \mathrm{Circ}(DF)| \leq \left(F_{n-2}F_{n-1}+(F_n-1)^2\right)^{\frac{n}{2}}$$

are given in [20, Example 4].

Theorem 3.5. *Let η_j, μ_j and λ_j ($j = 0, 1, ..., n-1$) be eigenvalues of the $n \times n$ ($n \geq 3$), matrices Circ(D^2F), Circ(DF) and Circ(F), respectively. Then*

$$\eta_j = \left(\mu_j + 2F_n - 2\right)w^{-j} - \mu_j$$

and

$$\eta_j = \left[\left(\lambda_j + 2F_n - 2\right)w^{-j} - 2\lambda_j\right]w^{-j} + \lambda_j,$$

where $w = e^{\frac{2\pi i}{n}}$ and $i = \sqrt{-1}$.

Proof. Since $F_0 \leq F_1 \leq ... \leq F_n - 1$ for $n \geq 3$, Lemma (2.3) immediately yields

$$\eta_j = \left(\mu_j + 2F_n - 2\right)w^{-j} - \mu_j$$

and

$$\mu_j = \left(\lambda_j + 2F_n - 2\right)w^{-j} - \lambda_j,$$

for $0 \leq j \leq n-1$. Thus

$$\begin{aligned}\eta_j &= \left[\left(\lambda_j + 2F_n - 2\right)w^{-j} - \lambda_j + 2F_n - 2\right]w^{-j} - \left[\left(\lambda_j + 2F_n - 2\right)w^{-j} - \lambda_j\right] \\ &= \left[\left(\lambda_j + 2F_n - 2\right)w^{-j} - 2\lambda_j\right]w^{-j} + \lambda_j.\end{aligned}$$

□

Corollary 3.6. *For the spectral norms of the $n \times n$ matrices Circ(D^2F), for $n \geq 3$, we have*

$$\left\|Circ(D^2F)\right\|_2 = 2F_n - 2.$$

Proof. Since $F_0 \leq F_1 \leq ... \leq F_n - 1$ for $n \geq 3$, we have from Lemma (2.4)

$$\left\|\mathrm{Circ}(D^2F)\right\|_2 = 2F_n - 2.$$

□

By [13, Theorem 1] and [20, Example 6], we have

$$\|\mathrm{Circ}(F)\|_2 = F_{n+2} - 1$$

and

$$\|\mathrm{Circ}(DF)\|_2 = 2F_n - 2$$

for the spectral norms of matrices Circ(F) and Circ(DF).

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