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Weak Markov Operators

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Abstract. Let A and B be f-algebras with unit elements e_A and e_B respectively. A positive operator T from A to B satisfying $T(e_A) = e_B$ is called a Markov operator. In this definition we replace unit elements with weak order units and, in this case, call T to be a weak Markov operator. In this paper, we characterize extreme points of the weak Markov operators.

1. Introduction

Let A and B be f-algebras with point separating order duals and unit elements e_A and e_B respectively. A positive linear operator T from A to B satisfying $T(e_A) = e_B$ is called a Markov operator. In this definition we replace unit elements with weak order units and, in this case, call T to be a weak Markov operator.The set of all weak Markov operators is convex. In this paper, we characterize extreme points of the weak Markov operators. In this regard, first we give an alternate and quick proof of [6] for f-algebras instead of order complete real vector spaces (Proposition 2.5). Then we give an alternate proof of Theorem 5.7 in [5] (Theorem 2.6). In addition we present another necessary and sufficient condition to this Theorem. Then we show that a weak Markov operator is a lattice homomorphism if and only if it is an extreme point in the collection of all weak Markov operators from A into B provided B is order complete.

2. Preliminaries

For unexplained terminology and the basic results on vector lattices and semiprime f-algebras we refer to [1, 10, 13]. Let us recall some definitions.

Definition 2.1. *The real algebra A is called a Riesz algebra or lattice-ordered algebra if A is a Riesz space such that $ab \in A$ whenever a, b are positive elements in A. The Riesz algebra is called an f-algebra if A satisfies the condition that*

$$a \wedge b = 0 \text{ implies } ac \wedge b = ca \wedge b = 0 \text{ for all } 0 \leq c \in A.$$

In an Archimedean f-algebra A, all nilpotent elements have index 2. Throughout this paper A will show an Archimedean semiprime f-algebra with point separating order dual. By definition, if zero is the unique nilpotent element of A, that is, $a^2 = 0$ implies $a = 0$, A is called semiprime f-algebra. It is well known that every f-algebra with unit element is semiprime.

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Definition 2.2. *The relatively uniform completion of an Archimedean Riesz space A is the closure of A, $\overline{A}$, in its Dedekind completion $\widehat{A}$, relative to the uniform topology of $\widehat{A}$* [9].

If A is an semiprime f-algebra then the multiplication in A can be extended in a unique way into a lattice ordered algebra multiplication on $\overline{A}$ such that A becomes a sub-algebra of $\overline{A}$ and $\overline{A}$ is an relatively uniformly complete semiprime f-algebra. We also recall that $\overline{A}$ satisfies the Stone condition (that is, $x \wedge nI^{*} \in \overline{A}$, for all $x \in \overline{A}$, where I denotes the identity on A of $OrthA$) due to Theorem 2.5 in [4]. For $a \in A$, the mapping $\pi_a : A \to OrthA$, defined by $\pi_a(b) = a.b$ is an orthomorphism on A. Since A is a Archimedean semiprime f-algebra,, the mapping $\pi : A \to OrthA$, defined by $\pi(a) = \pi_a$ is an injective f-algebra homomorphism. Hence we shall identify A with $\pi(A)$.The ideal center $Z(A)$of A is defined as the order ideal in $OrthA$ generated by the identity mapping I which is a unital f-algebra.

Proposition 2.3. *Let A and B be semiprime f-algebras and $T : A \to B$ an order bounded linear operator. If $T(A \cap Z(A)) = \{0\}$ then $T(A) = \{0\}$.*

Proof. T can be regarded as an element of the collection of all order bounded linear operators from A to $\widehat{B}$. Therefore there exist two order bounded positive operators T_1 and T_2 such that $T = T_1 - T_2$. Thus we can assume that T is positive. Since $\widehat{B}$ is relatively uniformly complete, by Theorem 3.3 in [11] there exists a unique positive relatively uniformly continuos extension $\overline{T} : \overline{A} \to \widehat{B}$ of T to the relatively uniformly completion $\overline{A}$ of A, defined by,

$$\overline{T}(x) = \sup\{Ta : 0 \leq a \leq x\}$$

for $x \in \overline{A}$. Let $0 \leq x \in \overline{A} \cap Z\left(\overline{A}\right)$ and $a \in A \cap [0,x]$. Then $a \leq \lambda I_A$ holds for some positive real number λ. By the hypothesis, $Ta = 0$. From here we conclude that $\overline{T}(x) = 0$ for all $x \in \overline{A} \cap Z\left(\overline{A}\right)$. Let $0 \leq a \in A$. Then $\left(a \wedge n\overline{I}\right)_n$ is a sequencein $\overline{A} \cap Z\left(\overline{A}\right)$ and it is converging to a by Proposition 2.1 (i) in [5]. It follows from the relatively uniformly continuity of $\overline{T}$ that $0 = \overline{T}(a \wedge nI_A) \to \overline{T}(a) = T(a)$ (relatively uniformly). Thus $T(a) = 0$ for all $a \in A$. □

Proposition 2.4. *Let A and B be f-algebras with unit elements e_A and e_B respectively and $T : A \to B$ an order bounded linear operator satisfying $T(e_A) = e_B$. If $| Ta | \leq e_B$ whenever $| a | \leq e_A$, then T is positive.*

Proof. T can be regarded as an element of the collection of all order bounded linear operators from A to $\widehat{B}$. As $\widehat{B}$ is Dedekind complete, $| Ta | \leq | T | (a) = \sup\{| Tb | : | b | \leq a, b \in A\}$ holds for all $0 \leq a \in A$. Let $a \in A$ be an element such that $| a | \leq e_A$. Then by the hypothesis, $| Ta | \leq e_B$. This shows that $| Te_A | = e_B \leq | T | (e_A) \leq e_B$. Thus $| T | (e_A) = e_B$. Let $a \in A \cap Z(A)$. Then there exists a positive real number λ such that $| a | \leq \lambda e_A$. From here we derive that

$$(| T | - T)(| a |) \leq \lambda (| T | - T)(e_A) = 0$$

Thus the order bounded operator $| T | - T$ vanishes on $A \cap Z(A)$. By Proposition 2.3, we conclude that $| T | = T$ on A, so T is positive. □

The following proposition was proved in [6] for an order complete real vector space. In the following proposition we will give an alternate and quick proof for Archimedean semiprime f-algebras.

Proposition 2.5. *Let A be a semiprime f-algebra. A positive element a_0 in A is a weak order unit if and only if $Inf\{| a - \lambda a_0 | : \lambda \in \mathbb{R}\} = 0$ for all $a \in A$.*

Proof. Let $0 < a_0 \in A$ be a weak order unit. Assume that, on the contrary, there exist $a \in A$ and $b \in \widehat{A}$ satisfying $0 < b \leq | a - \lambda a_0 |$for all $\lambda \in \mathbb{R}$. Then

$$\lambda^2 a_0^2 - 2\lambda a a_0 + a^2 - b^2 \geq 0 \text{ for all } \lambda \in \mathbb{R}$$

Using Proposition 3.3 in [3], we get $(a_0 b)^2 = 0$. Since $\widehat{A}$ is semiprime, the last equality implies that

$$0 = a_0 b = \sup \{a a_0 \in A : a \leq b,\ a \in A\}.$$

As a_0 is weak order unit, $a a_0 = 0$ implies that $a = 0$ for all $a \in A$ satisfying $a \leq b$. Thus $b = 0$. Conversely, suppose that there exists $0 \neq b \in A$ such that $b a_0 = 0$. Then since $0 < \frac{|b|}{2} \leq | b - \lambda a_0 |$ for all $\lambda \in \mathbb{R}$, we get $Inf \{| b - \lambda a_0 | : \lambda \in \mathbb{R}\} \neq 0$. □

A positive linear operator T between two unital f-algebras A and B is said to be a Markov operator for which $T(e_A) = e_B$ where e_A, e_B are the unit elements of A and B respectively. In both [5, Theorem 5.7] and [12], it is proved that an operator is an extreme point of Markov operators if and only if it is a lattice homomorphism. In the following theorem we shall give an alternate proof of this theorem and another sufficient and necessary condition for the extreme point of a Markov operator.

Theorem 2.6. *Let A and B be f-algebras with unit elements e_A and e_B respectively and $T : A \to B$ an order bounded linear operator satisfying $T(e_A) = e_B$ and $| Ta | \leq e_B$ whenever $| a | \leq e_A$. Then the following are equivalent;*
(i) T is an extreme point of Markov operators.
(ii) $Inf \{T | a - \lambda e_A | : \lambda \in \mathbb{R}\} = 0$ for all $a \in A$.
(iii) T is a lattice homomorphism.

Proof. $(i) \Rightarrow (ii)$: By Proposition 2.4, T is positive. Suppose, on the contrary, that $Inf \{T | a_0 - \lambda e_A | : \lambda \in \mathbb{R}\} \neq 0$ for some $0 \leq a_0 \in A$. Then there exists $b \in \widehat{B}$ such that $0 < b \leq T | a_0 - \lambda e_A |$ for all $\lambda \in \mathbb{R}$. Taking square from both side and applying the Schwarz inequality [3, Corollary 3.5], one can get easily

$$\lambda^2 e_B - 2\lambda T a_0 + T\left(a_0^2\right) - b^2 \geq 0 \text{ for all } \lambda \in \mathbb{R}.$$

Taking into account Proposition 3.3 in [3], we derive that

$$(T a_0)^2 \leq T\left(a_0^2\right) - b^2. \tag{1}$$

Let, $a_n = \frac{a_0}{n} \wedge e_A$ $(n = 1, 2, ...)$. Then $0 < a_n$, as $Inf \{T | a_0 - \lambda e_A | : \lambda \in \mathbb{R}\} \neq 0$, for each n. Define,

$$T_{a_n}(b) = T(a_n b) - T(a_n) T(b) \tag{2}$$

for $b \in A$. Clearly the operators $T \mp T_{a_n}$ are positive and $T_{a_n}(e_A) = 0$. Hence $T \mp T_{a_n}$ are Markov operators. From here we conclude that $T_{a_n} = 0$, as T is the extreme point of Markov operators. Hence the equality (2) implies that

$$nT(a_n b) = nT(a_n) T(b)$$

and by setting a_n, we have

$$T((a_0 \wedge n e_A) a_0) = T(a_0 \wedge n e_A) T(a_0).$$

Applying Proposition 2.1 (i) in [5] and using the positivity of T, one may get,

$$T\left(a_0^2\right) \leq (T(a_0))^2.$$

Thus the inequality (1) implies that both $T\left(a_0^2\right) = (T(a_0))^2$ and $b^2 = 0$. From here we conclude that $b = 0$, as $\hat{B}$ is semiprime whenever B is semiprime.
$(ii) \Rightarrow (iii)$: Let $a \in A$ and $\lambda \in \mathbb{R}$. Then

$$T \mid a \mid \leq T \mid a - \lambda e_A \mid + T \mid \lambda e_A \mid = T \mid a - \lambda e_A \mid + \mid T(\lambda e_A) \mid \leq 2T \mid a - \lambda e_A \mid + \mid Ta \mid$$

it follows from (ii) that $T \mid a \mid \leq \mid Ta \mid$ and since T is positive, $T \mid a \mid = \mid Ta \mid$ which implies that T is a lattice homomorphism.
$(iii) \Rightarrow (i)$: By Proposition 2.5, we get that $Inf\{\mid Ta - \lambda e_B \mid : \lambda \in \mathbb{R}\} = 0$ for all $a \in A$, as e_B is a weak order unit in B. To prove that T is an extreme point of Markov operators it is enough to show that for any Markov operator S and $0 < \alpha \in \mathbb{R}$ satisfying $\alpha T - S \geq 0$ implies that $T = S$. Let $a \in A$ and $\lambda \in \mathbb{R}$. It follows from

$$\mid Sa - S(\lambda e_A) \mid \leq S \mid a - \lambda e_A \mid \leq \alpha T \mid a - \lambda e_A \mid = \alpha \mid Ta - \lambda e_B \mid$$

that

$$\mid Ta - Sa \mid \leq \mid Ta - \lambda e_B \mid + \mid Sa - \lambda e_B \mid \leq (1 + \alpha) \mid Ta - \lambda e_B \mid$$

Hence $Ta = Sa$ for $a \in A$. □

Definition 2.7. *Let A and B be f-algebras with point separating order duals and weak order units e_A and e_B respectively. In this case, we call a positive linear operator T from A to B satisfying $T(e_A) = e_B$ to be a weak Markov operator (briefly WMO).*

Now we remark that in the last step $(iii) \Rightarrow (i)$ of the above proof we proved the following Corollary as well;

Corollary 2.8. *Let A and B be a semiprime f-algebras with weak order units e_A and e_B respectively and let $T : A \to B$ be a weak Markov operator. If T is a lattice homomorphism, then it is an extreme point in the set of all weak Markov operators.*

In [2, Lemma 4.2], it was proved that if A and B are semiprime f-algebras with Stone condition and if $T : A \to B$ is a linear positive operator satisfying both $T(a \wedge I_A) = T(a) \wedge I_B$, for all $a \in A$ and $T([0, I_A]) \subseteq [0, I_B]$ then T is a lattice homomorphism. At this point we remark that this result is true for all positive linear operator between any semiprime f-algebras. For the completeness, we repeat this proof in the following Proposition.

Proposition 2.9. *Let A and B be semiprime f- algebras and $T : A \to B$ a positive linear operator. If there exists $0 \leq a_0 \in A$ and $0 \leq b_0 \in B$ such that $T(a \wedge a_0) = T(a) \wedge b_0$, for all $a \in A$, then T is a lattice homomorphism.*

Proof. We remark that, for all elements a in A or B, the following equations hold;

$$\begin{aligned} (a \wedge a_0)^+ &= a^+ \wedge a_0 \\ (a \wedge a_0)^- &= a^- \end{aligned}$$

Since T is positive, in order to prove that T is a lattice homomorphism it is enough to show that $T(a^+) \leq (Ta)^+$ for all $a \in A$. By above remark,

$$\begin{aligned} Ta &= T\left(a^+ - a^- - (a \wedge \frac{1}{n}a_0)^+ + (a \wedge \frac{1}{n}a_0)^-\right) + T(a \wedge \frac{1}{n}a_0) \\ &= T\left(a^+ - (a \wedge \frac{1}{n}a_0)^+\right) + (T(a \wedge \frac{1}{n}a_0)) \end{aligned}$$

and

$$\begin{aligned} Ta &= (Ta)^+ - (Ta)^- - (T(a \wedge \frac{1}{n}a_0))^+ + (T(a \wedge \frac{1}{n}a_0))^- + T(a \wedge \frac{1}{n}a_0) \\ &= (Ta)^+ - (T(a \wedge \frac{1}{n}a_0))^+ + (T(a \wedge \frac{1}{n}a_0)) \end{aligned}$$

combining these results, we get

$$T(a^{+}) - (Ta)^{+} =^{-} T\left(a^{+} \wedge \frac{1}{n}a_{0}\right) - (Ta)^{+} \wedge \frac{1}{n}b_{0} \leq \frac{1}{n}T(a_{0})$$

Passing limit as $n \to \infty$, we have the desired result. □

Corollary 2.10. *Let A and B be semiprime f- algebras with weak order units e_A and e_B respectively and $T : A \to B$ a weak Markov operator. If there exists $0 \leq a_0 \in A$ and $0 \leq b_0 \in B$ such that $T(a \wedge a_0) = T(a) \wedge b_0$, for all $a \in A$, then T is an extreme point of weak Markov operators.*

Proof. By Proposition 2.9, T is a lattice homomorphism and by Corollary 2.8, T is an extreme point of weak Markov operators. □

Theorem 2.11. *Let A and B be a semiprime f-algebras with weak order units e_A and e_B respectively and $T : A \to B$ a weak Markov operator. If B is order complete, then T is an extreme point of the set of all weak Markov operators if and only if T is a lattice homomorphism.*

Proof. First we remark that any weak Markov operator is an extension of the operator $f : M \to B$, defined by $f(\lambda e_A) = \lambda e_B$, where $M = \{\lambda e_A : \lambda \in \mathbb{R}\}$. It is not difficult to see that the extreme point of the collection of all weak Markov operators is the same of the extreme point of all positive extensions of f to the whole A. Let T be an extreme point in the set of all weak Markov operators. Then T is an extreme point of the set of all positive extensions of f to A. Since f is a lattice homomorphism, by Theorem $2(a)$ in [7], we derive that T is a lattice homomorphism. Conversely assume that T is a lattice homomorphism. Taking into account Proposition 2.5 and Theorem $2(b)$ in [7], we conclude that T is an extreme point of the weak Markov operators. □

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