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Nonexistence of Nontrivial Weak Solutions for Anisotropic Elliptic Problems

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Abstract. In this paper, we consider anisotropic elliptic problems on a bounded smooth domain, nonexistence of nontrivial weak solutions are obtained by two different simplified methods. We point out the results of nonexistence of nontrivial weak solutions not only depend on the size and shape of domain, but also the constants p_1 and p_N in anisotropic elliptic problems.

1. Introduction and Main Results

We consider the following anisotropic elliptic problem

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) = \lambda |u|^{p_1-2}u + |u|^{p_N-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $\lambda > 0$, $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ is a bounded smooth domain, $1 < p_1 \leqslant p_2 \leqslant \cdots \leqslant p_N$, $\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right)$ is the anisotropic Laplace operator, which appears in physics (Lions [11], Tang [13]), biology (Bendahmane-Langlais-Saad [5], Bendahmane-Karlsen [6]), fluid mechanics (Antontsev-Diaz-Shmarev [2], Bear [3]), and image processing (Weickert [15]). If $p_i = p$, $i = 1, \cdots, N$, then the anisotropic Laplace operator is the pseudo-p-Laplace (Belloni-Kawohl [4]). If $p_i = 2$, $i = 1, \cdots, N$, then the anisotropic Laplace operator is the classical Laplace operator. We say that $u \in W_0^{1,(p_i)}(\Omega)$ is a weak solution of the problem (1) if for any $\phi \in W_0^{1,(p_i)}(\Omega)$, there holds

$$\sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i} dx = \lambda \int_\Omega |u|^{p_1-2}u\phi dx + \int_\Omega |u|^{p_N-2}u\phi dx, \tag{2}$$

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where $W_0^{1,(p_i)}(\Omega)$ is an anisotropic Sobolev space and defined by

$$W_0^{1,(p_i)}(\Omega) = \left\{ u \in W_0^{1,1}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega), i = 1, \cdots, N \right\}$$

with the corresponding norm

$$\|u\|_{W_0^{1,(p_i)}(\Omega)} = \sum_{i=1}^{N} \left(\int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx \right)^{\frac{1}{p_i}}.$$

In recent years, there have been a large number of papers and an increasing interest in the results of existence for anisotropic elliptic problems, with no hope of being complete, let us mention Alves-El Hamidi [1], Di Castro-Montefusco [7], El Hamidi-Rakotoson [8], Figueiredo-Santos Júnior-Suárez [9], Fragalà-Gazzola-Kawohl [10] and Vétoiss [14]. However, to the best of our knowledge, very little is known about nonexistence of nontrivial weak solutions of the problem (1), we only refer to Di Castro-Montefusco [7] and Fragalà-Gazzola-Kawohl [10], so our present paper is a new contribution in this direction.

In this paper, we aim to establish nonexistence of nontrivial weak solutions for the problem (1) by two different simplified methods. Firstly we use anisotropic Poincaré inequality and combine an argument of contradiction to obtain nonexistence of nontrivial weak solutions for the problem (1). Secondly we also obtain nonexistence of nontrivial weak solutions for the problem (1) by anisotropic Pohožaev identity on a strictly star-shaped bounded smooth domain with respect to the origin. Hence we generalize the results of nonexistence of nontrivial weak solutions in [12], and provide some effective methods to study more general anisotropic elliptic problems. Furthermore, we point out the results of nonexistence of nontrivial weak solutions for the problem (1) not only depend on the size and shape of domain, but also the constants p_1 and p_N.

Below we introduce anisotropic Poincaré inequality needed in this paper:

Lemma 1.1. *(anisotropic Poincaré inequality, [10]) Let $\{e_1, e_2, \cdots, e_N\}$ be the canonical basis of $\mathbb{R}^N$. Then for any $u \in W_0^{1,(p_i)}(\Omega)$ and every $q \geqslant 1$, we have*

$$\left(\int_\Omega |u|^q dx \right)^{\frac{1}{q}} \leqslant \frac{aq}{2} \left(\int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^q dx \right)^{\frac{1}{q}} \tag{3}$$

in the direction of e_i, where $a > 0$ denotes the width of Ω in the direction of e_i, namely $a = \sup_{x,y\in\Omega} (x - y, e_i)$.

Our main results of this paper are as follows:

Theorem 1.2. *If there exists a constant $\lambda^* = \frac{1}{(\frac{ap_1}{2})^{p_1}(\frac{ap_N}{2})^{p_N}} > 0$, then the problem (1) has no nontrivial weak solutions for all $\lambda \in (0, \lambda^*)$.*

Theorem 1.3. *Let $u \in W_0^{1,(p_i)}(\Omega)$ be weak solutions for the problem (1). Then there holds anisotropic Pohožaev identity*

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\partial\Omega} \frac{p_i - 1}{p_i} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} (x \cdot \nu)\, ds \\ &= \sum_{i=1}^{N} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx + N \int_\Omega \left(\frac{\lambda}{p_1} |u|^{p_1} + \frac{1}{p_N} |u|^{p_N} \right) dx - \sum_{i=1}^{N} \int_\Omega \frac{N}{p_i} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx, \end{aligned} \tag{4}$$

where $\nu = \nu(x) = (\nu_1(x), \nu_2(x), \cdots, \nu_N(x))$ denotes the exterior unit normal at $x \in \partial\Omega$, $x \cdot \nu = \sum\limits_{j=1}^{N} x_j \nu_j$, $\sum\limits_{i=1}^{N} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx = \int_\Omega (\lambda |u|^{p_1} + |u|^{p_N})\, dx$. If Ω is a strictly star-shaped bounded smooth domain with respect to the origin, then the problem (1) has no nontrivial weak solutions if

$$\lambda\left(\frac{p_N - N}{p_N} + \frac{N}{p_1}\right)|u|^{p_1} + |u|^{p_N} < 0, u \neq 0. \tag{5}$$

The following nonexistence result can be obtained from Theorem 1.3 directly.

Corollary 1.4. *Assume Ω is a strictly star-shaped bounded smooth domain with respect to the origin, $f : \mathbb{R} \to \mathbb{R}$ is continuous and $F(u) = \int_0^u f(t)dt$ satisfies $F(0) = 0$. Then there is no nontrivial weak solutions for anisotropic elliptic problem*

$$\begin{cases} -\sum\limits_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\right) = f(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{6}$$

if

$$\left(\frac{p_N - N}{p_N}\right) u f(u) + NF(u) < 0, u \neq 0.$$

Remark 1.5. *In the special case, $p_i = 2 (i = 1, \cdots, N)$ in Corollary 1.4, which is just the result of nonexistence of nontrivial weak solutions for semilinear elliptic problem $-\Delta u = f(u)$ obtained by Pohožaev [12].*

2. Proof of Theorem 1.2

Proof of Theorem 1.2. Taking $\phi = u \in W_0^{1,(p_i)}(\Omega)$ in (2), we get

$$\sum_{i=2}^{N-1} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx + \int_\Omega \left|\frac{\partial u}{\partial x_1}\right|^{p_1} dx + \int_\Omega \left|\frac{\partial u}{\partial x_N}\right|^{p_N} dx = \lambda \int_\Omega |u|^{p_1} dx + \int_\Omega |u|^{p_N} dx. \tag{7}$$

Applying anisotropic Poincaré inequality (3) with $q = p_1$ and p_N respectively, we obtain

$$\int_\Omega |u|^{p_1} dx \leqslant \left(\frac{ap_1}{2}\right)^{p_1} \int_\Omega \left|\frac{\partial u}{\partial x_1}\right|^{p_1} dx \tag{8}$$

and

$$\int_\Omega |u|^{p_N} dx \leqslant \left(\frac{ap_N}{2}\right)^{p_N} \int_\Omega \left|\frac{\partial u}{\partial x_N}\right|^{p_N} dx. \tag{9}$$

Substituting (8) and (9) into (7) yields that

$$\sum_{i=2}^{N-1} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx \leqslant \left(\lambda\left(\frac{ap_1}{2}\right)^{p_1} - 1\right) \int_\Omega \left|\frac{\partial u}{\partial x_1}\right|^{p_1} dx + \left(\left(\frac{ap_N}{2}\right)^{p_N} - 1\right) \int_\Omega \left|\frac{\partial u}{\partial x_N}\right|^{p_N} dx. \tag{10}$$

Noting that if

$$\begin{cases} \lambda\left(\frac{ap_1}{2}\right)^{p_1} - 1 < 0, \\ \left(\frac{ap_N}{2}\right)^{p_N} - 1 < 0, \end{cases}$$

namely if $\lambda\left(\frac{ap_1}{2}\right)^{p_1}\left(\frac{ap_N}{2}\right)^{p_N} < 1$, then the right hand side of (10) becomes

$$\left(\lambda\left(\frac{ap_1}{2}\right)^{p_1} - 1\right)\int_\Omega \left|\frac{\partial u}{\partial x_1}\right|^{p_1} dx + \left(\left(\frac{ap_N}{2}\right)^{p_N} - 1\right)\int_\Omega \left|\frac{\partial u}{\partial x_N}\right|^{p_N} dx < 0,$$

which contradicts with $\sum_{i=2}^{N-1} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx > 0$ of (10). Hence there exists a constant

$$\lambda^* = \frac{1}{\left(\frac{ap_1}{2}\right)^{p_1}\left(\frac{ap_N}{2}\right)^{p_N}} > 0$$

such that the problem (1) has no nontrivial weak solutions for all $\lambda \in (0, \lambda^*)$.

3. Proof of Theorem 1.3

Proof of Theorem 1.3. Firstly multiplying the equation in the problem (1) by u and integrating it over Ω, there holds

$$\sum_{i=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx = \int_\Omega (\lambda|u|^{p_1} + |u|^{p_N})\, dx. \tag{11}$$

Next multiplying the equation in the problem (1) by $\sum_{j=1}^N x_j \frac{\partial u}{\partial x_j}$ and integrating it over Ω, there holds

$$-\sum_{i,j=1}^N \int_\Omega x_j \frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right)dx = \sum_{j=1}^N \int_\Omega x_j \frac{\partial u}{\partial x_j}\left(\lambda|u|^{p_1-2}u + |u|^{p_N-2}u\right)dx. \tag{12}$$

In virtue of $u = 0$ for $x \in \partial\Omega$ and an application of the divergence theorem yields that

$$\begin{aligned}
0 &= \int_{\partial\Omega}\left(\frac{\lambda}{p_1}|u|^{p_1} + \frac{1}{p_N}|u|^{p_N}\right)(x\cdot\nu)\, ds \\
&= \sum_{j=1}^N \int_{\partial\Omega}\left(\frac{\lambda}{p_1}|u|^{p_1} + \frac{1}{p_N}|u|^{p_N}\right)x_j\nu_j ds \\
&= \sum_{j=1}^N \int_\Omega \frac{\partial}{\partial x_j}\left[x_j\left(\frac{\lambda}{p_1}|u|^{p_1} + \frac{1}{p_N}|u|^{p_N}\right)\right]dx \\
&= N\int_\Omega\left(\frac{\lambda}{p_1}|u|^{p_1} + \frac{1}{p_N}|u|^{p_N}\right)dx + \sum_{j=1}^N \int_\Omega x_j\frac{\partial u}{\partial x_j}\left(\lambda|u|^{p_1-2}u + |u|^{p_N-2}u\right)dx,
\end{aligned}$$

which implies

$$\sum_{j=1}^N \int_\Omega x_j\frac{\partial u}{\partial x_j}\left(\lambda|u|^{p_1-2}u + |u|^{p_N-2}u\right)dx = -N\int_\Omega\left(\frac{\lambda}{p_1}|u|^{p_1} + \frac{1}{p_N}|u|^{p_N}\right)dx. \tag{13}$$

In virtue of

$$\begin{aligned}
&\sum_{i,j=1}^N \int_\Omega x_j \frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right)dx \\
&= \sum_{i,j=1}^N \int_\Omega \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}x_j\frac{\partial u}{\partial x_j}\right)dx - \sum_{i,j=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_i}\left(x_j\frac{\partial u}{\partial x_j}\right)dx,
\end{aligned} \tag{14}$$

where

$$\sum_{i,j=1}^{N}\int_{\Omega}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}x_j\frac{\partial u}{\partial x_j}\right)dx=\sum_{i,j=1}^{N}\int_{\partial\Omega}\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}x_j\frac{\partial u}{\partial x_j}\nu_i ds$$
$$=\sum_{i,j=1}^{N}\int_{\partial\Omega}\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}x_j\frac{\partial u}{\partial x_i}\nu_j ds$$
$$=\sum_{i=1}^{N}\int_{\partial\Omega}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}(x\cdot\nu)ds \tag{15}$$

and

$$\sum_{i,j=1}^{N}\int_{\Omega}\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_i}\left(x_j\frac{\partial u}{\partial x_j}\right)dx$$
$$=\sum_{i,j=1}^{N}\int_{\Omega}\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\left(\frac{\partial x_j}{\partial x_i}\frac{\partial u}{\partial x_j}+x_j\frac{\partial^2 u}{\partial x_i\partial x_j}\right)dx$$
$$=\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx+\sum_{i,j=1}^{N}\int_{\Omega}\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}x_j\frac{\partial^2 u}{\partial x_i\partial x_j}dx$$
$$=\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx+\sum_{i,j=1}^{N}\int_{\Omega}\frac{1}{p_i}x_j\frac{\partial}{\partial x_j}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\right)dx$$
$$=\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx+\sum_{i,j=1}^{N}\int_{\Omega}\frac{1}{p_i}\left[\frac{\partial}{\partial x_j}\left(x_j\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\right)-\frac{\partial x_j}{\partial x_j}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\right]dx$$
$$=\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx+\sum_{i,j=1}^{N}\int_{\Omega}\frac{1}{p_i}\frac{\partial}{\partial x_j}\left(x_j\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\right)dx-\sum_{i=1}^{n}\int_{\Omega}\frac{N}{p_i}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx$$
$$=\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx+\sum_{i,j=1}^{N}\int_{\partial\Omega}\frac{1}{p_i}x_j\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\nu_j ds-\sum_{i=1}^{N}\int_{\Omega}\frac{N}{p_i}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx$$
$$=\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx+\sum_{i=1}^{N}\int_{\partial\Omega}\frac{1}{p_i}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}(x\cdot\nu)ds-\sum_{i=1}^{N}\int_{\Omega}\frac{N}{p_i}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx, \tag{16}$$

where we used $\frac{\partial u}{\partial x_i}\nu_j=\frac{\partial u}{\partial x_j}\nu_i$ on $\partial\Omega$. Substituting (15) and (16) into (14) yields that

$$\sum_{i,j=1}^{N}\int_{\Omega}x_j\frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right)dx$$
$$=\sum_{i=1}^{N}\int_{\partial\Omega}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}(x\cdot\nu)ds-\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx$$
$$-\sum_{i=1}^{N}\int_{\partial\Omega}\frac{1}{p_i}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}(x\cdot\nu)ds+\sum_{i=1}^{N}\int_{\Omega}\frac{N}{p_i}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx$$
$$=\sum_{i=1}^{N}\int_{\partial\Omega}\frac{p_i-1}{p_i}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}(x\cdot\nu)ds-\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx+\sum_{i=1}^{N}\int_{\Omega}\frac{N}{p_i}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx. \tag{17}$$

Finally substituting (13) and (17) into (12) and combining (11) yields that anisotropic Pohožaev identity (4). Since $\Omega \subset \mathbb{R}^N$ is strictly star-shaped with respect to the origin, it shows $x \cdot \nu \geqslant 0$ for $x \in \partial\Omega$. If $u \neq 0$, then from (4),

$$\begin{aligned}
0 &\leqslant \sum_{i=1}^{N} \int_{\partial\Omega} \frac{p_i - 1}{p_i} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} (x \cdot \nu)\, ds \\
&= \sum_{i=1}^{N} \int_{\Omega} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx + N \int_{\Omega} \left(\frac{\lambda}{p_1}|u|^{p_1} + \frac{1}{p_N}|u|^{p_N}\right) dx - \sum_{i=1}^{N} \int_{\Omega} \frac{N}{p_i} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx \\
&\leqslant \sum_{i=1}^{N} \int_{\Omega} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx + N \int_{\Omega} \left(\frac{\lambda}{p_1}|u|^{p_1} + \frac{1}{p_N}|u|^{p_N}\right) dx - \frac{N}{p_N} \sum_{i=1}^{N} \int_{\Omega} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx \\
&= \frac{p_N - N}{p_N} \sum_{i=1}^{N} \int_{\Omega} \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx + N \int_{\Omega} \left(\frac{\lambda}{p_1}|u|^{p_1} + \frac{1}{p_N}|u|^{p_N}\right) dx \\
&= \frac{p_N - N}{p_N} \int_{\Omega} (\lambda|u|^{p_1} + |u|^{p_N})\, dx + N \int_{\Omega} \left(\frac{\lambda}{p_1}|u|^{p_1} + \frac{1}{p_N}|u|^{p_N}\right) dx \\
&= \int_{\Omega} \left(\lambda \left(\frac{p_N - N}{p_N} + \frac{N}{p_1}\right)|u|^{p_1} + |u|^{p_N}\right) dx
\end{aligned}$$

which contradicts with (5), then the problem (1) has no nontrivial weak solutions under the condition (5).

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