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On Complete and Complete Moment Convergence for Weighted Sums of Widely Orthant Dependent Random Variables

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Abstract. In this article, the complete convergence for weighted sums of widely orthant dependent (WOD, in short) random variables without identical distribution is investigated. In addition, the complete moment convergence for weighted sums of WOD random variables is also obtained. The results obtained in the paper generalize some corresponding ones for some dependent random variables.

1. Introduction

It is well known that the complete convergence plays an important role in probability limit theory and mathematical statistics, especially in establishing the convergence rate for sums or weighted sums of random variables. The concept of complete convergence was introduced by Hsu and Robbins [1] as follows.

Definition 1.1. *A sequence of random variables $\{U_n, n \geq 1\}$ is said to converge completely to a constant a if for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} P(|U_n - a| > \varepsilon) < \infty.$$

In view of the Borel-Cantelli Lemma, this implies that $U_n \to a$ almost surely (a.s.). The converse is true if $\{U_n, n \geq 1\}$ are independent. Hsu and Robbins [1] proved that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to expected value if the variance of the summands is finite. Erdös [2] proved the converse. The result of Hsu-Robbins-Erdös is a fundamental theorem in probability theory and has been generalized and extended in several directions by many authors. One of the most important generalizations is Baum and Katz [3] for the strong law of large numbers as follows.

Theorem 1.1. *Let $1/2 < \alpha \leq 1$ and $\alpha p > 1$, let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables. Assume further that $EX_1 = 0$ if $\alpha \leq 1$. Then the following statements are equivalent:*

(i) $E|X_1|^p < \infty$;

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(ii) $\sum_{n=1}^{\infty} n^{\alpha p-2} P(\max_{1\le k\le n} |\sum_{i=1}^{k} X_i| > \varepsilon n^{\alpha}) < \infty$ *for all* $\varepsilon > 0$.

Since then, many authors were devoted to studying the Baum-Katz type strong law of large numbers for dependent random variables, such as Sung [4] for ρ^*-mixing sequences, Wang and Hu [5] for martingale difference sequences, Zhang [6] for estended negatively orthant dependent sequences, and so on. The main purpose of the article is to study the Baum-Katz type result for weighted sums of widely orthant dependent random variables.

The concept of widely orthant dependence structure was introduced by Wang et al. [7] as follows.

Definition 1.2. *For the random variables* $\{X_n, n \ge 1\}$, *if there exists a finite positive sequence* $\{g_U(n), n \ge 1\}$ *satisfying for each* $n \ge 1$ *and for all* $x_i \in (-\infty, \infty)$, $1 \le i \le n$,

$$P(X_1 > x_1, X_2 > x_2, \cdots, X_n > x_n) \le g_U(n) \prod_{i=1}^{n} P(X_i > x_i), \tag{1.1}$$

then we say that the random variables $\{X_n, n \ge 1\}$ *are widely upper orthant dependent (WUOD, in short); if there exists a finite positive sequence* $\{g_L(n), n \ge 1\}$ *satisfying for each* $n \ge 1$ *and for all* $x_i \in (-\infty, \infty)$, $1 \le i \le n$,

$$P(X_1 \le x_1, X_2 \le x_2, \cdots, X_n \le x_n) \le g_L(n) \prod_{i=1}^{n} P(X_i \le x_i), \tag{1.2}$$

then we say that the $\{X_n, n \ge 1\}$ *are widely lower orthant dependent (WLOD, in short); if they are both WUOD and WLOD, then we say that the* $\{X_n, n \ge 1\}$ *are widely orthant dependent (WOD, in short), and* $g_U(n)$, $g_L(n)$, $n \ge 1$, *are called dominating coefficients.*

An array $\{X_{ni}, i \ge 1, n \ge 1\}$ *of random variables is called rowwise WOD random variables if for every* $n \ge 1$, $\{X_{ni}, i \ge 1\}$ *is a sequence of WOD random variables.*

From (1.1) and (1.2), we can see that $g_U(n) \ge 1$ and $g_L(n) \ge 1$. It is easily seen that if both (1.1) and (1.2) hold for $g_L(n) = g_U(n) = M$ for any $n \ge 1$, where M is a positive constant, then the random variables $\{X_n, n \ge 1\}$ are called extended negatively dependent (END, in short). This is the definition of END sequence which was introduced by Liu [8]. If both (1.1) and (1.2) hold for $g_L(n) = g_U(n) = 1$ for any $n \ge 1$, then the random variables $\{X_n, n \ge 1\}$ are called negatively orthant dependent (NOD, in short), which was introduced by Ebrahimi and Ghosh [9]. It is well known that negatively associated (NA, in short) random variables are NOD. Hu [10] pointed out that negatively superadditive dependent (NSD, in short) random variables are NOD. Hence, the class of WOD random variables includes independent sequence, NA sequence, NSD sequence, NOD sequence and END sequence as special cases. Studying the probability limit theory and its applications for WOD random variables is of great interest.

Since Wang et al. [7] introduced the concept of WOD random variables, many authors were devoted to the study of limit behavior of WOD random variables. Wang et al. [7] provided some examples which showed that the class of WOD random variables contains some common negatively dependent random variables, some positively dependent random variables and some others; in addition, they studied the uniform asymptotics for the finite-time ruin probability of a new dependent risk model with a constant interest rate. He et al. [11] provided the asymptotic lower bounds of precise large deviations with nonnegative and dependent random variables. Wang et al. [12] investigated the asymptotics of the finite-time ruin probability for a generalized renewal risk model with independent strong subexponential claim sizes and widely lower orthant dependent inter-occurrence times. Shen [13] established the Bernstein type inequality for WOD random variables and gave some applications. Qiu and Chen [14] obtained some results on complete convergence and complete moment convergence for weighted sums of WOD random variables. Wang et al. [15] established some results on complete convergence for arrays of rowwise WOD random variables with application to complete consistency for the estimator in a nonparametric regression model based on WOD errors. Wang and Hu [16] studied the consistency of the nearest neighbor estimator of the density function based on WOD samples. Yang et al. [17] presented the Bahadur representation of sample quantiles for WOD random variables. Chen et al. [18] established an inequality of WOD random variables and gave some applications, including the strong law of large numbers, the complete convergence, the a.s. elementary renewal theorem, and the weighted elementary renewal theorem. Wang et al. [19]

obtained some results on complete convergence for arrays of rowwise WOD random variables and gave some applications. Xia et al. [20] established the complete consistency and strong convergence rate for the weighted estimator of nonparametric regression model based on WOD errors.

In this article, we will further study the complete convergence and complete moment convergence for weighted sums of WOD random variables under some mild conditions.

The concept of complete moment convergence was introduced by Chow [21] as follows: let $\{Z_n, n \geq 1\}$ be a sequence of random variables and $a_n > 0$, $b_n > 0$, $q > 0$. If

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|Z_n| - \epsilon\}_+^q < \infty \text{ for some or all } \epsilon > 0,$$

then the above result was called the complete moment convergence.

It can be easily verified that complete moment convergence implies complete convergence; thus, complete moment convergence is much stronger than complete convergence. For more details about complete moment convergence, one can refer to Liang et al. [22], Wu et al. [23], Shen et al. [24] and Wu et al. [25] for instance.

The following definitions of slowly varying function and stochastic domination play important roles throughout the article.

Definition 1.3. *A real-valued function $l(x)$, positive and measurable on $(0, \infty)$, is said to be slowly varying if*

$$\lim_{x\to\infty} \frac{l(x\lambda)}{l(x)} = 1$$

for each $\lambda > 0$.

Definition 1.4. *A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that*

$$P(|X_n| > x) \leq CP(|X| > x)$$

for all $x \geq 0$ and $n \geq 1$.

Throughout the article, C denotes a positive constant which may be different in various places. Denote $X^+ = \max(X, 0)$, $X^- = \max(-X, 0)$ and $\log x = \ln\max(x, e)$. Let $I(A)$ be the indicator function of the set A and $\lfloor x \rfloor$ be the integer part of x.

2. Preliminaries

In this section, we will present some important lemmas, which will be used to prove the main results of this work. The first one is a basic property for WOD random variables, which can be found in Wang et al. [7].

Lemma 2.1. *Let $\{X_n, n \geq 1\}$ be WLOD (WUOD) with dominating coefficients $g_L(n), n \geq 1$ $(g_U(n), n \geq 1)$. If $\{f_n(\cdot), n \geq 1\}$ are all nondecreasing, then $\{f_n(X_n), n \geq 1\}$ are still WLOD (WUOD) with dominating coefficients $g_L(n), n \geq 1$ $(g_U(n), n \geq 1)$; if $\{f_n(\cdot), n \geq 1\}$ are all nonincreasing, then $\{f_n(X_n), n \geq 1\}$ are WUOD (WLOD) with dominating coefficients $g_L(n), n \geq 1$ $(g_U(n), n \geq 1)$.*

The next one is the moment inequality for WOD random variables, which was established by Wang et al. [15].

Lemma 2.2. *Let $q > 1$ and $\{X_n, n \geq 1\}$ be a sequence of mean zero WOD random variables with dominating coefficients $g(n) = \max\{g_U(n), g_L(n)\}$. If $E|X_n|^q < \infty$ for any $n \geq 1$, then there exist positive constants $C_1(q)$ and $C_2(q)$ depending on q such that for each $n \geq 1$,*

$$E\left|\sum_{i=1}^{n} X_i\right|^q \leq [C_1(q) + C_2(q)g(n)] \sum_{i=1}^{n} E|X_i|^q, \textit{ for } 1 < q \leq 2,$$

and

$$E\left|\sum_{i=1}^{n} X_i\right|^q \le C_1(q)\sum_{i=1}^{n} E|X_i|^q + C_2(q)g(n)\left(\sum_{i=1}^{n} EX_i^2\right)^{\frac{q}{2}},\ \textit{for } q \ge 2.$$

The following one is a basic property for slowing varying function, where (i) is Proposition 1.3.6 (v) and (ii) is Theorem 1.5.11 in Bingham et al. [26], and (iii) can be found in Bai and Su [27].

Lemma 2.3. *If $l(x) > 0$ is a slowly varying function, then*

(i) $\lim_{x\to\infty} x^{\delta} l(x) = \infty$, $\lim_{x\to\infty} x^{-\delta} l(x) = 0$ for each $\delta > 0$;

(ii) $C_1 2^{kr} l(\varepsilon 2^k) \le \sum_{j=1}^{k} 2^{jr} l(\varepsilon 2^j) \le C_2 2^{kr} l(\varepsilon 2^k)$ for every $r > 0$, $\varepsilon > 0$, positive integer k and some $C_1 > 0$, $C_2 > 0$;

(iii) $C_3 2^{kr} l(\varepsilon 2^k) \le \sum_{j=k}^{\infty} 2^{jr} l(\varepsilon 2^j) \le C_4 2^{kr} l(\varepsilon 2^k)$ for every $r < 0$, $\varepsilon > 0$, positive integer k and some $C_3 > 0$, $C_4 > 0$.

The last one is a basic property for stochastic domination, which can be referred to Wu [28].

Lemma 2.4. *Let $\{X_n, n \ge 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. Then for any $\alpha > 0$ and $b > 0$,*

$$E|X_n|^{\alpha} I(|X_n| \le b) \le C_1[E|X|^{\alpha} I(|X| \le b) + b^{\alpha} P(|X| > b)],$$

and

$$E|X_n|^{\alpha} I(|X_n| > b) \le C_2 E|X|^{\alpha} I(|X| > b),$$

where C_1 and C_2 are positive constants. Consequently, $E|X_n|^{\alpha} \le CE|X|^{\alpha}$.

3. Main Results

Our main results are as follows.

Theorem 3.1. *Let $0 < p < 2$, $\alpha p > 1$, and $\{X_n, n \ge 1\}$ be a sequence of WOD random variables with dominating coefficients $g(n) = O(n^{\delta})$ for some $0 \le \delta < 2\alpha - \alpha p$, which is stochastically dominated by a random variable X. Let $l(x)$ be a slowly varying function and $l(x) \uparrow$ if $p = 1$. Assume further that $EX_n = 0$ if $p \ge 1$, and $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of real numbers such that*

$$\sum_{i=1}^{n} a_{ni}^2 = O(n). \tag{3.1}$$

If

$$E[|X|^{p+\frac{\delta}{\alpha}} l(|X|^{\frac{1}{\alpha}})] < \infty, \tag{3.2}$$

then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\left(\left|\sum_{i=1}^{n} a_{ni} X_i\right| > \varepsilon n^{\alpha}\right) < \infty. \tag{3.3}$$

For $\alpha p = 2$, we have the following corollary by Theorem 3.1.

Corollary 3.1. *Let $0 < p < 2$ and $\{X_n, n \ge 1\}$ be a sequence of WOD random variables with dominating coefficients $g(n) = O(n^{\delta})$ for some $0 \le \delta < \frac{4}{p} - 2$, which is stochastically dominated by a random variable X. Let $l(x)$ be a slowly varying function. Assume further that $EX_n = 0$ if $p > 1$, and $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of real numbers satisfying (3.1). If*

$$E\left[|X|^{p+\frac{p\delta}{2}} l(|X|^{\frac{p}{2}})\right] < \infty,$$

then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} l(n) P\left(\frac{1}{n^{\frac{2}{p}}}\left|\sum_{i=1}^{n} a_{ni} X_i\right| > \varepsilon\right) < \infty.$$

Theorem 3.2. *Suppose that the conditions of Theorem 3.1 hold and $1 \le p < 2$.*
(i) If $1 < p < 2$, then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) E\left(\left|\sum_{i=1}^{n} a_{ni} X_i\right| - \varepsilon n^{\alpha}\right)^{+} < \infty; \tag{3.4}$$

(ii) if $p = 1$ and

$$E[|X|^{1+\frac{\delta}{\alpha}} l(|X|^{\frac{1}{\alpha}}) \log |X|] < \infty, \tag{3.5}$$

then for any $\varepsilon > 0$, (3.4) stills holds.
Remark 3.1. *Denote $S_n = \sum_{i=1}^{n} a_{ni} X_i$. We point out that (3.4) implies (3.3). This can be obtained by the following inequality:*

$$\begin{aligned} \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n) E\left(\left|\sum_{i=1}^{n} a_{ni} X_i\right| - \varepsilon n^{\alpha}\right)^{+} &= \sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} l(n) \int_0^{\infty} P(|S_n| - \varepsilon n^{\alpha} > t) dt \\ &\ge \sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} l(n) \int_0^{\varepsilon n^{\alpha}} P(|S_n| - \varepsilon n^{\alpha} > t) dt \\ &\ge \varepsilon \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P(|S_n| > 2\varepsilon n^{\alpha}). \end{aligned}$$

Remark 3.2. *In Theorem 3.1, if $\delta = 0$, namely $g(n) = O(1)$, then the WOD random variables reduce to END random variables. So the results of Theorems 3.1-3.2 and Corollary 3.1 also hold for END random variables.*
Remark 3.3. *There are many examples of slowly varying functions which are positive and monotone nondecreasing, such as $l(x) = 1$, $l(x) = \log x$, and so on.*

4. Proofs of the Main Results

In this section, we will present the proofs of the main results obtained in Section 3.
Proof of Theorem 3.1.
Without loss of generality, we assume that

$$\sum_{i=1}^{n} a_{ni}^2 \le Cn, \tag{4.1}$$

$a_{ni} \ge 0$ for all $1 \le i \le n$ and $n \ge 1$ (Otherwise, we use a_{ni}^+ and a_{ni}^- instead of a_{ni}, respectively and note that $a_{ni} = a_{ni}^+ - a_{ni}^-$).
It follows by (4.1) and Hölder's inequality that

$$\sum_{i=1}^{n} |a_{ni}| \le \left(n \sum_{i=1}^{n} a_{ni}^2\right)^{\frac{1}{2}} \le Cn. \tag{4.2}$$

For fixed $n \ge 1$, denote for $1 \le i \le n$ that

$$X_i^{(n)} = -n^{\alpha} I(X_i < -n^{\alpha}) + X_i I(|X_i| \le n^{\alpha}) + n^{\alpha} I(X_i > n^{\alpha}),$$

$$T^{(n)} = n^{-\alpha} \sum_{i=1}^{n} a_{ni}(X_i^{(n)} - EX_i^{(n)}).$$

It is easily checked that for any $\varepsilon > 0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\left(\left|\sum_{i=1}^{n} a_{ni} X_i\right| > \varepsilon n^{\alpha}\right) \\
\leq\ & \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \sum_{i=1}^{n} P(|X_i| > n^{\alpha}) + \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P\left(|T^{(n)}| > \varepsilon - n^{-\alpha}\left|\sum_{i=1}^{n} E a_{ni} X_i^{(n)}\right|\right) \\
\triangleq\ & I_1 + I_2. \qquad (4.3)
\end{aligned}$$

In order to prove (3.3), we just need to show that $I_1 < \infty$ and $I_2 < \infty$. By Lemma 2.3, we can get that

$$\begin{aligned}
I_1 &\leq C\sum_{n=1}^{\infty} n^{\alpha p-1} l(n) \sum_{j=n}^{\infty} P(j^{\alpha} < |X| \leq (j+1)^{\alpha}) \\
&= C\sum_{j=1}^{\infty} P(j^{\alpha} < |X| \leq (j+1)^{\alpha}) \sum_{n=1}^{j} n^{\alpha p-1} l(n) \\
&\leq C\sum_{j=1}^{\infty} P(j < |X|^{\frac{1}{\alpha}} \leq (j+1)) \sum_{i=1}^{\lfloor \log_2^j \rfloor+1} \sum_{n=2^{i-1}}^{2^i} n^{\alpha p-1} l(n) \\
&\leq C\sum_{j=1}^{\infty} P(j < |X|^{\frac{1}{\alpha}} \leq (j+1)) \sum_{i=1}^{\lfloor \log_2^j \rfloor+1} 2^{i\alpha p} l(2^i) \\
&\leq C\sum_{j=1}^{\infty} P(j < |X|^{\frac{1}{\alpha}} \leq (j+1)) 2^{(\lfloor \log_2^j \rfloor+1)\alpha p} l(2^{\lfloor \log_2^j \rfloor+1}) \\
&\leq C\sum_{j=1}^{\infty} P(j < |X|^{\frac{1}{\alpha}} \leq (j+1)) j^{\alpha p} l(j) \\
&\leq C\sum_{j=1}^{\infty} j^{\alpha p} l(j) E(\frac{|X|^{\frac{1}{\alpha}}}{j})^{\delta} I(j < |X|^{\frac{1}{\alpha}} \leq (j+1)) \\
&\leq CE[|X|^{p+\frac{\delta}{\alpha}} l(|X|^{\frac{1}{\alpha}})] \ < \ \infty. \qquad (4.4)
\end{aligned}$$

In the following, we will prove that $I_2 < \infty$. First, we show that

$$n^{-\alpha}\left|\sum_{i=1}^{n} E a_{ni} X_i^{(n)}\right| \to 0, \text{ as } n \to \infty. \qquad (4.5)$$

By (3.2), one can get that for any $0 < \gamma < p + \frac{\delta}{\alpha}$,

$$E|X|^{p+\frac{\delta}{\alpha}-\gamma} < \infty. \qquad (4.6)$$

We consider the following two cases.

Case 1: $0 < \alpha \leq 1$

Noting that $\alpha p > 1$, we have that $p \geq 1$. If $p > 1$, taking γ such that $0 < \gamma < \frac{\alpha p+\delta-1}{\alpha}$, we have by $EX_n = 0$, (4.2), (4.6) and Lemma 2.4 that

$$\begin{aligned}
n^{-\alpha}\left|\sum_{i=1}^{n} Ea_{ni}X_i^{(n)}\right| &\le n^{-\alpha}\left|\sum_{i=1}^{n} Ea_{ni}X_iI(|X_i \le n^{\alpha})\right| + \sum_{i=1}^{n}|a_{ni}|P(|X| > n^{\alpha}) \\
&\le n^{1-\alpha}E|X|I(|X| > n^{\alpha}) + CnP(|X| > n^{\alpha}) \\
&\le Cn^{1-\alpha}E|X|I(|X| > n^{\alpha}) \\
&= Cn^{1-\alpha}E|X|^{p+\frac{\delta}{\alpha}-\gamma}|X|^{1-p-\frac{\delta}{\alpha}+\gamma}I(|X| > n^{\alpha}) \\
&\le Cn^{1-\alpha}n^{\alpha-\alpha p-\delta+\alpha\gamma}E|X|^{p+\frac{\delta}{\alpha}-\gamma} \\
&\le Cn^{1-\alpha p-\delta+\alpha\gamma}E|X|^{p+\frac{\delta}{\alpha}-\gamma} \to 0, \text{ as } n \to \infty.
\end{aligned}$$

If $p = 1$, we have by $EX_n = 0$, (4.2) and Lemma 2.4 again that

$$\begin{aligned}
n^{-\alpha}\left|\sum_{i=1}^{n} Ea_{ni}X_i^{(n)}\right| &\le Cn^{1-\alpha}E|X|I(|X| > n^{\alpha}) \\
&\le Cn^{1-\alpha p-\delta}E|X|^{p+\frac{\delta}{\alpha}}I(|X| > n^{\alpha}) \to 0, \text{ as } n \to \infty.
\end{aligned}$$

Case 2: $\alpha > 1$

Take γ such that $1 + \alpha\gamma - \alpha p - \delta < 0$. It follows by Lemma 2.4 again that

$$\begin{aligned}
n^{-\alpha}\left|\sum_{i=1}^{n} Ea_{ni}X_i^{(n)}\right| &\le n^{1-\alpha}E|X|I(|X| \le n^{\alpha}) + CnP(|X| > n^{\alpha}) \\
&= n^{1-\alpha}\sum_{k=1}^{n} E[|X|I((k-1)^{\alpha} < |X| \le k^{\alpha})] + CnP(|X| > n^{\alpha}).
\end{aligned} \tag{4.7}$$

It follows by (4.6) that

$$\begin{aligned}
\sum_{k=1}^{\infty} k^{1-\alpha}E|X|I((k-1)^{\alpha} < |X| \le k^{\alpha}) &\le \sum_{k=1}^{\infty} kP((k-1)^{\alpha} < |X| \le k^{\alpha}) \\
&= \sum_{i=1}^{\infty}\sum_{k=i}^{\infty} P((k-1)^{\alpha} < |X| \le k^{\alpha}) \\
&= \sum_{i=0}^{\infty} P(|X| > i^{\alpha}) \\
&\le 1 + \sum_{i=1}^{\infty} \frac{E|X|^{p+\frac{\delta}{\alpha}-\gamma}}{i^{\alpha(p+\frac{\delta}{\alpha}-\gamma)}} < \infty,
\end{aligned}$$

which together with Kronecker's Lemma yields that

$$n^{1-\alpha}\sum_{k=1}^{n} E[|X|I((k-1)^{\alpha} < |X| \le k^{\alpha})] \to 0, \text{ as } n \to \infty. \tag{4.8}$$

According to (4.6) and Markov's inequality, we have

$$nP(|X| > n^{\alpha}) \le n^{1-\alpha p-\delta+\alpha\gamma}E|X|^{p+\frac{\delta}{\alpha}-\gamma} \to 0, \text{ as } n \to \infty. \tag{4.9}$$

In this case, (4.5) follows by (4.7)-(4.9) immediately. Hence, for all n large enough, we obtain

$$n^{-\alpha}\left|\sum_{i=1}^{n} Ea_{ni}X_i^{(n)}\right| < \frac{\varepsilon}{2},$$

which implies that

$$I_2 \leq C \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P(|T^{(n)}| > \frac{\varepsilon}{2}).$$

By Lemma 2.1, we can see that for fixed $n \geq 1$, $\{a_{ni}(X_i^{(n)} - EX_i^{(n)}),\ 1 \leq i \leq n\}$ are still WOD random variables. Hence we have by Markov's inequality, Lemma 2.2 (taking $q = 2$), (4.1) and Lemma 2.4 that

$$\begin{aligned}
I_2 &\leq C \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) E|T^{(n)}|^2 \\
&= C \sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha} l(n) E\left|\sum_{i=1}^{n} a_{ni}(X_i^{(n)} - EX_i^{(n)})\right|^2 \\
&\leq C \sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha} l(n) n^{\delta} \sum_{i=1}^{n} E\left|a_{ni}(X_i^{(n)} - EX_i^{(n)})\right|^2 \\
&\leq C \sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha+\delta} l(n) \sum_{i=1}^{n} a_{ni}^2 E|X_i^{(n)}|^2 \\
&\leq C \sum_{n=1}^{\infty} n^{\alpha p-1+\delta} l(n) P(|X| > n^{\alpha}) + C \sum_{n=1}^{\infty} n^{\alpha p-1+\delta-2\alpha} l(n) E[X^2 I(|X| \leq n^{\alpha})] \\
&\triangleq I_3 + I_4.
\end{aligned} \tag{4.10}$$

In order to prove $I_2 < \infty$, we just need to show that $I_3 < \infty$ and $I_4 < \infty$. Similar to the proof of (4.4), we can get that

$$\begin{aligned}
I_3 &\leq C \sum_{n=1}^{\infty} n^{\alpha p-1+\delta} l(n) \sum_{j=n}^{\infty} P(j^{\alpha} < |X| \leq (j+1)^{\alpha}) \\
&= C \sum_{j=1}^{\infty} P(j^{\alpha} < |X| \leq (j+1)^{\alpha}) \sum_{n=1}^{j} n^{\alpha p-1+\delta} l(n) \\
&\leq C \sum_{j=1}^{\infty} j^{\alpha p+\delta} l(j) P(j^{\alpha} < |X| \leq (j+1)^{\alpha}) \\
&\leq C \sum_{j=1}^{\infty} E|X|^{p+\frac{\delta}{\alpha}} l(|X|^{\frac{1}{\alpha}}) I(j^{\alpha} < |X| \leq (j+1)^{\alpha}) \\
&\leq CE[|X|^{p+\frac{\delta}{\alpha}} l(|X|^{\frac{1}{\alpha}})] < \infty.
\end{aligned} \tag{4.11}$$

By Lemma 2.3 and (3.2) again, we have

$$\begin{aligned}
I_4 &= C \sum_{n=1}^{\infty} n^{\alpha p-1+\delta-2\alpha} l(n) E[X^2 I(|X| \leq n^{\alpha})] \\
&= C \sum_{j=0}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} n^{\alpha p-1+\delta-2\alpha} l(n) E[X^2 I(|X| \leq n^{\alpha})] \\
&\leq C \sum_{j=0}^{\infty} 2^{j(\alpha p+\delta-2\alpha)} l(2^j) E[X^2 I(|X| \leq 2^{\alpha(j+1)})]
\end{aligned} \tag{4.12}$$

$$
\begin{aligned}
&\le C\sum_{j=1}^{\infty}2^{j(\alpha p+\delta-2\alpha)}l(2^j)\sum_{k=1}^{j}E[X^2I(2^{\alpha k}<|X|\le 2^{\alpha(k+1)})]\\
&= C\sum_{k=1}^{\infty}E[X^2I(2^{\alpha k}<|X|\le 2^{\alpha(k+1)})]\sum_{j=k}^{\infty}2^{j(\alpha p+\delta-2\alpha)}l(2^j)\\
&\le C\sum_{k=1}^{\infty}2^{k(\alpha p+\delta-2\alpha)}l(2^k)E[X^2I(2^{\alpha k}<|X|\le 2^{\alpha(k+1)})]\\
&\le C\sum_{k=1}^{\infty}2^{k\alpha p+k\delta}l(2^k)P(2^{\alpha k}<|X|\le 2^{\alpha(k+1)})\\
&\le C\sum_{k=1}^{\infty}E[|X|^{p+\frac{\delta}{\alpha}}l(|X|^{\frac{1}{\alpha}})I(2^{\alpha k}<|X|\le 2^{\alpha(k+1)})]\\
&\le CE[|X|^{p+\frac{\delta}{\alpha}}l(|X|^{\frac{1}{\alpha}})]<\infty. \qquad (4.13)
\end{aligned}
$$

This completes the proof of the theorem. □

Proof of Theorem 3.2.

We only give the proof for case (i), since the proof for case (ii) is similar to that of case (i). Let $1 < p < 2$, $S_n = \sum_{i=1}^{n} a_{ni}X_i$ and assume that $a_{ni} \ge 0$. For any $\varepsilon > 0$, we have by Theorem 3.1 that

$$
\begin{aligned}
&\sum_{n=1}^{\infty}n^{\alpha p-\alpha-2}l(n)E(|S_n|-\varepsilon n^{\alpha})^{+}\\
&= \sum_{n=1}^{\infty}n^{\alpha p-\alpha-2}l(n)\int_0^{\infty}P(|S_n|-\varepsilon n^{\alpha}>t)dt\\
&= \sum_{n=1}^{\infty}n^{\alpha p-\alpha-2}l(n)\int_0^{n^{\alpha}}P(|S_n|-\varepsilon n^{\alpha}>t)dt\\
&\quad+\sum_{n=1}^{\infty}n^{\alpha p-\alpha-2}l(n)\int_{n^{\alpha}}^{\infty}P(|S_n|-\varepsilon n^{\alpha}>t)dt\\
&\le \sum_{n=1}^{\infty}n^{\alpha p-2}l(n)P(|S_n|>\varepsilon n^{\alpha})+\sum_{n=1}^{\infty}n^{\alpha p-\alpha-2}l(n)\int_{n^{\alpha}}^{\infty}P(|S_n|>t)dt\\
&\le C\sum_{n=1}^{\infty}n^{\alpha p-\alpha-2}l(n)\int_{n^{\alpha}}^{\infty}P(|S_n|>t)dt. \qquad (4.14)
\end{aligned}
$$

Hence, we just need to show that

$$
H \triangleq \sum_{n=1}^{\infty}n^{\alpha p-\alpha-2}l(n)\int_{n^{\alpha}}^{\infty}P(|S_n|>t)dt<\infty. \qquad (4.15)
$$

For fixed $t > 0$, denote

$$
\begin{aligned}
Z_{ti} &= -tI(X_i<-t)+X_iI(|X_i|\le t)+tI(X_i>t),\ i=1,2,\cdots,\\
U_{ti} &= tI(X_i<-t)+X_iI(|X_i|>t)-tI(X_i>t),\ i=1,2,\cdots.
\end{aligned}
$$

It is easy to see that $X_i = U_{ti} + Z_{ti} = U_{ti} + EZ_{ti} + Z_{ti} - EZ_{ti}$, so we have

$$\begin{aligned} H &\leq \sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} l(n) \int_{n^\alpha}^{\infty} P\left(\left|\sum_{i=1}^{n} a_{ni} U_{ti}\right| > t/3\right) dt \\ &\quad + \sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} l(n) \int_{n^\alpha}^{\infty} P\left(\left|\sum_{i=1}^{n} a_{ni} EZ_{ti}\right| > t/3\right) dt \\ &\quad + \sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} l(n) \int_{n^\alpha}^{\infty} P\left(\left|\sum_{i=1}^{n} a_{ni} (Z_{ti} - EZ_{ti})\right| > t/3\right) dt \\ &\triangleq H_1 + H_2 + H_3. \end{aligned} \tag{4.16}$$

In order to prove (3.4), it suffices to show $H_1 < \infty$, $H_2 < \infty$ and $H_3 < \infty$. Noting that $|U_{ti}| \leq 2|X_i| I(|X_i| > t)$, and similar to the proof of (4.4), we have by Markov's inequality, (4.2), Lemma 2.3 and Lemma 2.4 that

$$\begin{aligned} H_1 &\leq C \sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} l(n) \int_{n^\alpha}^{\infty} t^{-1} E\left|\sum_{i=1}^{n} a_{ni} U_{ti}\right| dt \\ &\leq C \sum_{n=1}^{\infty} n^{\alpha p-\alpha-1} l(n) \int_{n^\alpha}^{\infty} t^{-1} E[|X| I(|X| > t)] dt \\ &= C \sum_{n=1}^{\infty} n^{\alpha p-\alpha-1} l(n) \sum_{m=n}^{\infty} \int_{m^\alpha}^{(m+1)^\alpha} t^{-1} E[|X| I(|X| > t)] dt \\ &\leq C \sum_{n=1}^{\infty} n^{\alpha p-\alpha-1} l(n) \sum_{m=n}^{\infty} m^{-1} E[|X| I(|X| > m^\alpha)] \\ &= C \sum_{m=1}^{\infty} m^{-1} E[|X| I(|X| > m^\alpha)] \sum_{n=1}^{m} n^{\alpha p-\alpha-1} l(n) \\ &\leq C \sum_{m=1}^{\infty} m^{-1} E[|X| I(|X| > m^\alpha)] m^{\alpha p-\alpha} l(m) \\ &= C \sum_{n=1}^{\infty} n^{\alpha p-\alpha-1} l(n) E[|X| I(|X| > n^\alpha)] \\ &= C \sum_{n=1}^{\infty} n^{\alpha p-\alpha-1} l(n) \sum_{m=n}^{\infty} E[|X| I(m < |X|^{\frac{1}{\alpha}} \leq (m+1))] \\ &= C \sum_{m=1}^{\infty} E[|X| I(m < |X|^{\frac{1}{\alpha}} \leq (m+1))] \sum_{n=1}^{m} n^{\alpha p-\alpha-1} l(n) \\ &\leq C \sum_{m=1}^{\infty} m^{\alpha p-\alpha} l(m) E[|X| I(m < |X|^{\frac{1}{\alpha}} \leq (m+1))] \\ &\leq CE[|X|^p l(|X|^{\frac{1}{\alpha}})] \\ &< \infty. \end{aligned} \tag{4.17}$$

According to the proof of (4.16), we have by Markov's inequality, (3.2) and Lemma 2.3 that

$$\begin{aligned} H_2 &= \sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} l(n) \int_{n^\alpha}^{\infty} P\left(\left|\sum_{i=1}^{n} a_{ni} EZ_{ti}\right| > t/3\right) dt \\ &\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} l(n) \int_{n^\alpha}^{\infty} t^{-1} \left|\sum_{i=1}^{n} a_{ni} EZ_{ti}\right| dt \\ &\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} l(n) \int_{n^\alpha}^{\infty} t^{-1} \sum_{i=1}^{n} E[|a_{ni}X_i| I(|X_i| > t)] dt \\ &\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1} l(n) \int_{n^\alpha}^{\infty} t^{-1} E[|X| I(|X| > t)] dt \\ &< \infty. \end{aligned} \tag{4.18}$$

For fixed $t > 0$ and $n \geq 1$, it is easily seen that $\{a_{ni}(Z_{ti} - EZ_{ti}),\ i \geq 1\}$ are still WOD random variables by Lemma 2.1. Hence, we have by Markov's inequality, Lemma 2.2 (taking $q = 2$), Lemma 2.4 and (4.1) that

$$\begin{aligned} H_3 &= \sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} l(n) \int_{n^\alpha}^{\infty} P\left(\left|\sum_{i=1}^{n} a_{ni}(Z_{ti} - EZ_{ti})\right| > t/3\right) dt \\ &\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} l(n) \int_{n^\alpha}^{\infty} t^{-2} E\left|\sum_{i=1}^{n} a_{ni}(Z_{ti} - EZ_{ti})\right|^2 dt \\ &\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} l(n) \int_{n^\alpha}^{\infty} t^{-2} n^{\delta} \sum_{i=1}^{n} E|a_{ni}(Z_{ti} - EZ_{ti})|^2 dt \\ &\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2+\delta} l(n) \int_{n^\alpha}^{\infty} t^{-2} \sum_{i=1}^{n} a_{ni}^2 E|Z_{ti}|^2 dt \\ &= C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2+\delta} l(n) \int_{n^\alpha}^{\infty} t^{-2} \sum_{i=1}^{n} a_{ni}^2 [EX_i^2 I(|X_i| \leq t) + t^2 P(|X_i| > t)] dt \\ &\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1+\delta} l(n) \int_{n^\alpha}^{\infty} t^{-2} E[X^2 I(|X| \leq t)] dt \\ &\quad + \sum_{n=1}^{\infty} n^{\alpha p-\alpha-1+\delta} l(n) \int_{n^\alpha}^{\infty} P(|X| > t) dt \\ &\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1+\delta} l(n) \int_{n^\alpha}^{\infty} t^{-2} E[X^2 I(|X| \leq t)] dt \\ &\quad + C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1+\delta} l(n) \int_{n^\alpha}^{\infty} t^{-1} E[|X| I(|X| > t)] dt \\ &\triangleq W_1 + W_2. \end{aligned} \tag{4.19}$$

Similar to the proof of (4.16), we have

$$\begin{aligned} W_2 &= C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1+\delta} l(n) \int_{n^\alpha}^{\infty} t^{-1} E[|X| I(|X| > t)] dt \\ &\leq C\sum_{m=1}^{\infty} m^{\alpha p-\alpha+\delta} l(m) E[|X| I(m < |X|^{\frac{1}{\alpha}} \leq (m+1))] \end{aligned} \tag{4.20}$$

$$
\begin{aligned}
&\leq CE[|X|^{p+\frac{\delta}{\alpha}}l(|X|^{\frac{1}{\alpha}})] \\
&< \infty.
\end{aligned} \tag{4.21}
$$

So we just need to show $W_1 < \infty$. According to Lemma 2.3, Lemma 2.4, (3.2) and (4.12), we have

$$
\begin{aligned}
W_1 &= C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1+\delta}l(n)\int_{n^\alpha}^{\infty} t^{-2}E[X^2 I(|X|\leq t)]dt \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1+\delta}l(n)\sum_{m=n}^{\infty}\int_{m^\alpha}^{(m+1)^\alpha} t^{-2}E[X^2 I(|X|\leq t)]dt \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1+\delta}l(n)\sum_{m=n}^{\infty} m^{-\alpha-1}E[X^2 I(|X|\leq (m+1)^\alpha] \\
&= C\sum_{m=1}^{\infty} m^{-\alpha-1}E[X^2 I(|X|\leq (m+1)^\alpha]\sum_{n=1}^{m} n^{\alpha p-\alpha-1+\delta}l(n) \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-2\alpha-1+\delta}l(n)E[X^2 I(|X|\leq (n+1)^\alpha] \\
&= C\sum_{n=1}^{\infty} n^{\alpha p-2\alpha-1+\delta}l(n)E[X^2 I(n^\alpha<|X|\leq (n+1)^\alpha] \\
&\quad +C\sum_{n=1}^{\infty} n^{\alpha p-2\alpha-1+\delta}l(n)E[X^2 I(|X|\leq n^\alpha] \\
&\leq C\sum_{n=1}^{\infty} n^{-1}E[|X|^{p+\frac{\delta}{\alpha}}l(|X|^{\frac{1}{\alpha}})I(n^\alpha<|X|\leq (n+1)^\alpha] \\
&\quad +CE[|X|^{p+\frac{\delta}{\alpha}}l(|X|^{\frac{1}{\alpha}})] \\
&\leq CE[|X|^{p+\frac{\delta}{\alpha}}l(|X|^{\frac{1}{\alpha}})] \\
&< \infty.
\end{aligned} \tag{4.22}
$$

This completes the proof of the theorem. □

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