Filomat 32:15 (2018), 5329–5335
https://doi.org/10.2298/FIL1815329Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

An Extension of the Schwarz Inequality in Inner Product Spaces

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Abstract. We extend the improved Schwarz inequality of Dragomir [1, Theorem 2] to any power $p \geq 2$,

$$\|x\|^p\|y\|^p - |\langle x,y\rangle|^p \geq \left|\det\begin{bmatrix} |\langle x,e\rangle| & (\|x\|^p - |\langle x,e\rangle|^p)^{1/p} \\ |\langle y,e\rangle| & (\|y\|^p - |\langle y,e\rangle|^p)^{1/p}\end{bmatrix}\right|^p$$

for any vectors $x, y, e \in \mathbb{C}^n$ with $\|e\| = 1$. Applications to n-tuples of complex numbers are also included.

1. Introduction

Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space over a number field ($\mathbb{R}$ or $\mathbb{C}$). The Schwarz inequality states that

$$\|x\|^2\|y\|^2 \geq |\langle x,y\rangle|^2, \text{ for any } x, y \in V. \tag{1}$$

Equality holds if and only if x and y are linearly dependent. A large number of generalizations and refinements of the Schwarz inequality have been investigated in the literature.

In 1985, Dragomir [2] established the following inequality related to Schwarz inequality

$$\begin{aligned}&\left(\|x\|^2\|z\|^2 - |\langle x,z\rangle|^2\right)\left(\|y\|^2\|z\|^2 - |\langle y,z\rangle|^2\right)\\ &\geq \left|\langle x,y\rangle\|z\|^2 - \langle x,z\rangle\langle z,y\rangle\right|^2\\ &\geq \left||\langle x,y\rangle|\|z\|^2 - |\langle x,z\rangle||\langle z,y\rangle|\right|^2.\end{aligned} \tag{2}$$

If we take the square root of (2) and divide it by $\|x\|\|y\|\|z\|^2 \neq 0$, then

$$\left|\frac{|\langle x,y\rangle|}{\|x\|\|y\|} - \frac{|\langle x,z\rangle|}{\|x\|\|z\|}\frac{|\langle z,y\rangle|}{\|z\|\|y\|}\right| \leq \sqrt{1 - \frac{|\langle x,z\rangle|^2}{\|x\|^2\|z\|^2}}\sqrt{1 - \frac{|\langle y,z\rangle|^2}{\|y\|^2\|z\|^2}}. \tag{3}$$

Also we can get

$$\frac{|\langle x,e\rangle|^2}{\|x\|^2} + \frac{|\langle y,e\rangle|^2}{\|y\|^2} + \frac{|\langle x,y\rangle|^2}{\|x\|^2\|y\|^2} - 2\frac{|\langle x,e\rangle|}{\|x\|}\frac{|\langle y,e\rangle|}{\|y\|}\frac{|\langle x,y\rangle|}{\|x\|\|y\|} - 1 \leq 0, \tag{4}$$

2010 *Mathematics Subject Classification*. Primary 46C05; Secondary 26D15
Keywords. Inner product spaces, Schwarz inequality
Received: 04 April 2018; Accepted: 23 July 2018
Communicated by Fuad Kittaneh
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for any $x, y \neq 0$ with $\|e\| = 1$.

Recently, the following refinements of the Schwarz inequality were proposed [1]. Let $x, y, e \in V$ with $\|e\| = 1$ and $p \geq 2$, then

$$\|x\|^p\|y\|^p - |\langle x, y\rangle|^p \geq \left|\det\begin{bmatrix} \|x\| & (\|x\|^p - |\langle x, e\rangle|^p)^{1/p} \\ \|y\| & (\|y\|^p - |\langle y, e\rangle|^p)^{1/p} \end{bmatrix}\right|^p \tag{5}$$

and

$$\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2 \geq \left(\det\begin{bmatrix} |\langle x, e\rangle| & (\|x\|^2 - |\langle x, e\rangle|^2)^{1/2} \\ |\langle y, e\rangle| & (\|y\|^2 - |\langle y, e\rangle|^2)^{1/2} \end{bmatrix}\right)^2. \tag{6}$$

In this paper, we shall prove that the inequality (6) can be extended for any power $p \geq 2$,

$$\|x\|^p\|y\|^p - |\langle x, y\rangle|^p \geq \left|\det\begin{bmatrix} |\langle x, e\rangle| & (\|x\|^p - |\langle x, e\rangle|^p)^{1/p} \\ |\langle y, e\rangle| & (\|y\|^p - |\langle y, e\rangle|^p)^{1/p} \end{bmatrix}\right|^p.$$

Moreover, we show that the above inequality is stronger than (5).

2. Some Lemmas

In the following, we prove some basic lemmas that will be needed in the proof of the main results.

Lemma 2.1. *[3] Let n be an integer greater than one and let $\boldsymbol{A} = (a_{ij})$ be an $n \times n$ real symmetric positive semidefinite matrix with nonnegative entries. If p is a real number such that $p \geq n - 2$, then $\boldsymbol{A}^{(p)} \equiv (a_{ij}^p)$ is positive semidefinite.*

Lemma 2.2. *Let $a, b \geq 0$, $p \geq 2$, then*

$$|a - b|^p \leq a^p - 2a^{p/2}b^{p/2} + b^p = (a^{p/2} - b^{p/2})^2. \tag{7}$$

Proof. Without loss of generality, we may assume that $a \geq b \geq 0$. Equivalently, we need to prove

$$(a - b)^{2p} \leq (a^p - b^p)^2, \quad p \geq 1,$$

or

$$(a - b)^p \leq a^p - b^p, \quad p \geq 1.$$

Let $t = \dfrac{b}{a} \in [0, 1]$, $a \neq 0$, it remains to prove

$$(1 - t)^p \leq 1 - t^p, \quad p \geq 1,$$

which is trivial since

$$(1 - t)^p \leq 1 - t \leq 1 - t^p, \quad p \geq 1.$$

□

Lemma 2.3. *Let $a, b \in [0, 1]$, $p \geq 2$, then*

$$\left|a\sqrt[p]{1 - b^p} - b\sqrt[p]{1 - a^p}\right| \geq \left|\sqrt[p]{1 - b^p} - \sqrt[p]{1 - a^p}\right|.$$

Proof. It is trivial for $a = 1$ or $b = 1$. Assume that $a, b \neq 1$ and $a \geq b$. Fix $p \geq 2$ and set $f(t) = \dfrac{\sqrt[p]{1-t^p}}{1-t}$ for $t \in [0, 1)$, then

$$\frac{d}{dt} f(t) = \frac{(1-t^p)^{\frac{1}{p}-1}(1-t^{p-1})}{(1-t)^2} \geq 0.$$

Since $f(t)$ is increasing, if $a, b \in [0, 1)$ and $a \geq b$, then

$$f(a) = \frac{\sqrt[p]{1-a^p}}{1-a} \geq \frac{\sqrt[p]{1-b^p}}{1-b} = f(b),$$

in another form

$$a\sqrt[p]{1-b^p} - b\sqrt[p]{1-a^p} \geq \sqrt[p]{1-b^p} - \sqrt[p]{1-a^p},$$

which leads to the desired result. □

3. Main Results

In this section, we generalize the inequality (6) (see also [1, Theorem 2]) to any power $p \geq 2$.

Theorem 3.1. *Let $x, y, e \in V$ with $\|e\| = 1$, $p \geq 2$. Then the following refinement of the Schwarz inequality holds*

$$\|x\|^p\|y\|^p - |\langle x, y\rangle|^p \geq \left|\det\begin{bmatrix} |\langle x, e\rangle| & (\|x\|^p - |\langle x, e\rangle|^p)^{1/p} \\ |\langle y, e\rangle| & (\|y\|^p - |\langle y, e\rangle|^p)^{1/p}\end{bmatrix}\right|^p. \tag{8}$$

Proof. It is trivial for $x = 0$ or $y = 0$. Assume that $x \neq 0$ and $y \neq 0$, then (8) is equivalent to

$$1 - \frac{|\langle x, y\rangle|^p}{\|x\|^p\|y\|^p} \geq \left|\frac{|\langle x, e\rangle|}{\|x\|}\sqrt[p]{1 - \frac{|\langle y, e\rangle|^p}{\|y\|^p}} - \frac{|\langle y, e\rangle|}{\|y\|}\sqrt[p]{1 - \frac{|\langle x, e\rangle|^p}{\|x\|^p}}\right|^p. \tag{9}$$

For convenience, taking

$$a = \frac{|\langle x, e\rangle|}{\|x\|}, \quad b = \frac{|\langle y, e\rangle|}{\|y\|}, \quad c = \frac{|\langle x, y\rangle|}{\|x\|\|y\|},$$

then $0 \leq a, b, c \leq 1$ and $a^2 + b^2 + c^2 - 2abc - 1 \leq 0$ according to (4).

Let $G = \begin{bmatrix} 1 & c & a \\ c & 1 & b \\ a & b & 1\end{bmatrix}$, it is clear that G is positive semidefinite. With Lemma 2.1, $G^{(p)} = \begin{bmatrix} 1 & c^p & a^p \\ c^p & 1 & b^p \\ a^p & b^p & 1\end{bmatrix}$ is also positive semidefinite for $p \geq 1$. Hence

$$1 + 2a^pb^pc^p - (a^{2p} + b^{2p} + c^{2p}) \geq 0, \quad p \geq 1, \tag{10}$$

equivalently,

$$|c^p - a^pb^p| \leq \sqrt{1-a^{2p}}\sqrt{1-b^{2p}}, \quad p \geq 1. \tag{11}$$

Using the above inequalities (10) and (11), it follows from Lemma 2.2 that

$$
\begin{aligned}
&\left|\frac{|\langle x,e\rangle|}{\|x\|}\sqrt[p]{1-\frac{|\langle y,e\rangle|^p}{\|y\|^p}}-\frac{|\langle y,e\rangle|}{\|y\|}\sqrt[p]{1-\frac{|\langle x,e\rangle|^p}{\|x\|^p}}\right|^p\\
&=\left|a\sqrt[p]{1-b^p}-b\sqrt[p]{1-a^p}\right|^p\\
&\le a^p(1-b^p)-2a^{p/2}b^{p/2}\sqrt{1-b^p}\sqrt{1-a^p}+b^p(1-a^p)\\
&=a^p+b^p-2a^pb^p-2a^{p/2}b^{p/2}\sqrt{1-a^p}\sqrt{1-b^p}\\
&\le a^p+b^p-2a^pb^p-2a^{p/2}b^{p/2}\left|c^{p/2}-a^{p/2}b^{p/2}\right|\\
&=a^p+b^p-2a^{p/2}b^{p/2}\left(a^{p/2}b^{p/2}+\left|c^{p/2}-a^{p/2}b^{p/2}\right|\right)\\
&\le a^p+b^p-2a^{p/2}b^{p/2}c^{p/2}\\
&\le 1-c^p\\
&=1-\frac{|\langle x,y\rangle|^p}{\|x\|^p\|y\|^p},\quad p\ge 2
\end{aligned}
$$

which proves (9). □

Remark 3.2. *The result (8) is stronger than (5), since the inequality (5) (see also [1, Theorem 1]) is equivalent to*

$$1-\frac{|\langle x,y\rangle|^p}{\|x\|^p\|y\|^p}\ge\left|\sqrt[p]{1-\frac{|\langle y,e\rangle|^p}{\|y\|^p}}-\sqrt[p]{1-\frac{|\langle x,e\rangle|^p}{\|x\|^p}}\right|^p,$$

for any x, $y \neq 0$. By Lemma 2.3, we have

$$\left|\frac{|\langle x,e\rangle|}{\|x\|}\sqrt[p]{1-\frac{|\langle y,e\rangle|^p}{\|y\|^p}}-\frac{|\langle y,e\rangle|}{\|y\|}\sqrt[p]{1-\frac{|\langle x,e\rangle|^p}{\|x\|^p}}\right|\ge\left|\sqrt[p]{1-\frac{|\langle y,e\rangle|^p}{\|y\|^p}}-\sqrt[p]{1-\frac{|\langle x,e\rangle|^p}{\|x\|^p}}\right|.$$

Remark 3.3. *It is also true that*

$$\|x\|^p\|y\|^p-|Re\langle x,y\rangle|^p\ge\left|\det\begin{bmatrix}|Re\langle x,e\rangle| & (\|x\|^p-|Re\langle x,e\rangle|^p)^{1/p}\\ |Re\langle y,e\rangle| & (\|y\|^p-|Re\langle y,e\rangle|^p)^{1/p}\end{bmatrix}\right|^p,$$

for $x, y, e \in V$ with $\|e\|=1$, $p\ge 2$.

The following inequality has been obtained by Wang and Zhang in [4] (see also [5, p. 195])

$$\sqrt{1-\frac{|\langle x,y\rangle|^2}{\|x\|^2\|y\|^2}}\le\sqrt{1-\frac{|\langle x,z\rangle|^2}{\|x\|^2\|z\|^2}}+\sqrt{1-\frac{|\langle y,z\rangle|^2}{\|y\|^2\|z\|^2}},$$

for any $x, y, z \in V \setminus \{0\}$. In [6], Lin showed that it can be extended for any power $p > 2$,

$$\sqrt[p]{1-\frac{|\langle x,y\rangle|^p}{\|x\|^p\|y\|^p}}\le\sqrt[p]{1-\frac{|\langle x,z\rangle|^p}{\|x\|^p\|z\|^p}}+\sqrt[p]{1-\frac{|\langle y,z\rangle|^p}{\|y\|^p\|z\|^p}}.$$

For any nonzero vectors x and y, define the angle between x and y by [5, p. 30]

$$\Psi_{xy}=\cos^{-1}\frac{|\langle x,y\rangle|}{\|x\|\|y\|}.$$

Now we obtain the following result when $\Psi_{xy}, \Psi_{xz}, \Psi_{yz}\in[0,\frac{\pi}{4}]$.

Theorem 3.4. *Let $x, y, z \in V \setminus \{0\}$, $p \geq 2$, if $\Psi_{xy}, \Psi_{xz}, \Psi_{yz} \in [0, \frac{\pi}{4}]$, then we have*

$$\sqrt[p]{1 - \frac{|\langle x, y\rangle|^p}{||x||^p||y||^p}} \leq \frac{|\langle y, z\rangle|}{||y||||z||}\sqrt[p]{1 - \frac{|\langle x, z\rangle|^p}{||x||^p||z||^p}} + \frac{|\langle x, z\rangle|}{||x||||z||}\sqrt[p]{1 - \frac{|\langle y, z\rangle|^p}{||y||^p||z||^p}}. \tag{12}$$

Proof. The idea of proof is similar to that in [6, Proposition 2]. Using the fact that $\Psi_{xy} \leq \Psi_{xz} + \Psi_{zy}$ for any $x, y, z \neq 0$ [6], we obtain

$$\sin \Psi_{xy} \leq \sin (\Psi_{xz} + \Psi_{zy}) = \cos \Psi_{zy} \sin \Psi_{xz} + \cos \Psi_{xz} \sin \Psi_{zy},$$

since the function sin is increasing on $[0, \frac{\pi}{2}]$. Taking

$$a = \cos \Psi_{xz} = \frac{|\langle x, z\rangle|}{||x||||z||}, \quad b = \cos \Psi_{yz} = \frac{|\langle y, z\rangle|}{||y||||z||}, \quad c = \cos \Psi_{xy} = \frac{|\langle x, y\rangle|}{||x||||y||},$$

then we have

$$0 \leq \sqrt{1 - c^2} \leq a\sqrt{1 - b^2} + b\sqrt{1 - a^2} \leq 1\,,$$

similarly,

$$\sqrt{1 - a^2} \leq a\sqrt{1 - b^2} + b\sqrt{1 - a^2}\,,$$
$$\sqrt{1 - b^2} \leq a\sqrt{1 - b^2} + b\sqrt{1 - a^2}$$

for $\Psi_{xy}, \Psi_{xz}, \Psi_{yz} \in [0, \frac{\pi}{4}]$.

Fix $p > 2$ and set$f(t) = \left(1 - (1 - t^2)^{p/2}\right)^{1/p}$ for $t \in [0, 1]$. Then

$$\frac{d}{dt} f(t) = \left(1 - (1 - t^2)^{p/2}\right)^{1/p-1} (1 - t^2)^{p/2-1} t \geq 0$$

and

$$\frac{d}{dt}\frac{f(t)}{t} = t^{-2}\left(1 - (1 - t^2)^{p/2}\right)^{1/p-1}\left((1 - t^2)^{p/2-1} - 1\right) \leq 0.$$

Since $f(t)$ is increasing and $\frac{f(t)}{t}$ is decreasing, we can get

$$\begin{aligned}
& f(\sqrt{1 - c^2}) \leq f(a\sqrt{1 - b^2} + b\sqrt{1 - a^2}) \\
&= \frac{a\sqrt{1 - b^2} f(a\sqrt{1 - b^2} + b\sqrt{1 - a^2})}{a\sqrt{1 - b^2} + b\sqrt{1 - a^2}} + \frac{b\sqrt{1 - a^2} f(a\sqrt{1 - b^2} + b\sqrt{1 - a^2})}{a\sqrt{1 - b^2} + b\sqrt{1 - a^2}} \\
&\leq \frac{a\sqrt{1 - b^2} f(\sqrt{1 - b^2})}{\sqrt{1 - b^2}} + \frac{b\sqrt{1 - a^2} f(\sqrt{1 - a^2})}{\sqrt{1 - a^2}} \\
&= af(\sqrt{1 - b^2}) + bf(\sqrt{1 - a^2}),
\end{aligned}$$

which leads to the desired result. □

Remark 3.5. *If we define the angle between x and y by $\Phi_{xy} = \cos^{-1}\frac{Re|\langle x, y\rangle|}{||x||||y||}$, $x, y \neq 0$, the corresponding result for the angle Φ_{xy} still holds*

$$\sqrt[p]{1 - \frac{|Re\langle x, y\rangle|^p}{||x||^p||y||^p}} \leq \frac{|Re\langle y, z\rangle|}{||y||||z||}\sqrt[p]{1 - \frac{|Re\langle x, z\rangle|^p}{||x||^p||z||^p}} + \frac{|Re\langle x, z\rangle|}{||x||||z||}\sqrt[p]{1 - \frac{|Re\langle y, z\rangle|^p}{||y||^p||z||^p}}, \tag{13}$$

for $\Phi_{xy}, \Phi_{xz}, \Phi_{yz} \in [0, \frac{\pi}{4}]$.

4. Applications

In this section, we show some applications of Theorem 3.1 to n-Tuples of complex numbers. Let $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n)$, $e = (e_1, \cdots, e_n) \in \mathbb{C}^n$ with $\sum_{k=1}^n |e_k|^2 = 1$. $\mathbb{C}^n$ is an inner product space over $\mathbb{C}$ with the inner product $\langle x, y\rangle = y^* x = \sum_{k=1}^n x_k \bar{y}_k$. Then the inequality (8) can be written as

$$\begin{aligned}&\left(\sum_{k=1}^n |x_k|^2\right)^{p/2}\left(\sum_{k=1}^n |y_k|^2\right)^{p/2} - \left|\sum_{k=1}^n x_k\bar{y}_k\right|^p \qquad (14)\\ &\geq \left|\det\begin{bmatrix} |\sum_{k=1}^n x_k\bar{e}_k| & \left((\sum_{k=1}^n |x_k|^2)^{p/2} - |\sum_{k=1}^n x_k\bar{e}_k|^p\right)^{1/p} \\ |\sum_{k=1}^n y_k\bar{e}_k| & \left((\sum_{k=1}^n |y_k|^2)^{p/2} - |\sum_{k=1}^n y_k\bar{e}_k|^p\right)^{1/p}\end{bmatrix}\right|^p\end{aligned}$$

for any power $p \geq 2$.

(i) Taking $e = (0, \cdots, 0, 1, 0, \cdots, 0)$, which has a 1 as its mth entry and 0s elsewhere, then

$$\begin{aligned}&\left(\sum_{k=1}^n |x_k|^2\right)^{p/2}\left(\sum_{k=1}^n |y_k|^2\right)^{p/2} - \left|\sum_{k=1}^n x_k\bar{y}_k\right|^p\\ &\geq \max_{m\in\{1,\cdots,n\}}\left|\det\begin{bmatrix} |x_m| & \left((\sum_{k=1}^n |x_k|^2)^{p/2} - |x_m|^p\right)^{1/p} \\ |y_m| & \left((\sum_{k=1}^n |y_k|^2)^{p/2} - |y_m|^p\right)^{1/p}\end{bmatrix}\right|^p.\end{aligned}$$

For $p = 2$, we can get a simpler inequality

$$\begin{aligned}&\sum_{k=1}^n |x_k|^2\sum_{k=1}^n |y_k|^2 - \left|\sum_{k=1}^n x_k\bar{y}_k\right|^2\\ &\geq \max_{m\in\{1,\cdots,n\}}\left(\det\begin{bmatrix} |x_m| & \left(\sum_{1\leq k\neq m\leq n} |x_k|^2\right)^{1/2} \\ |y_m| & \left(\sum_{1\leq k\neq m\leq n} |y_k|^2\right)^{1/2}\end{bmatrix}\right)^2.\end{aligned}$$

When $n = 2$, this is as follows

$$(|x_1|^2 + |x_2|^2)(|y_1|^2 + |y_2|^2) \geq |x_1\bar{y}_1 + x_2\bar{y}_2|^2 + (|x_1||y_2| - |x_2||y_1|)^2.$$

Equality holds for real numbers.

(ii) Taking $e = (\frac{1}{\sqrt{n}}, \cdots, \frac{1}{\sqrt{n}})$, by (14) we get

$$\begin{aligned}&\left(\sum_{k=1}^n |x_k|^2\right)^{p/2}\left(\sum_{k=1}^n |y_k|^2\right)^{p/2} - \left|\sum_{k=1}^n x_k\bar{y}_k\right|^p\\ &\geq n^p\left|\det\begin{bmatrix} |\frac{1}{n}\sum_{k=1}^n x_k| & \left((\frac{1}{n}\sum_{k=1}^n |x_k|^2)^{p/2} - |\frac{1}{n}\sum_{k=1}^n x_k|^p\right)^{1/p} \\ |\frac{1}{n}\sum_{k=1}^n y_k| & \left((\frac{1}{n}\sum_{k=1}^n |y_k|^2)^{p/2} - |\frac{1}{n}\sum_{k=1}^n y_k|^p\right)^{1/p}\end{bmatrix}\right|^p.\end{aligned}$$

For $p = 2$, we can get an interesting result mentioned by S. S. Dragomir [1],

$$\sum_{k=1}^{n}|x_k|^2\sum_{k=1}^{n}|y_k|^2-\left|\sum_{k=1}^{n}x_k\bar{y}_k\right|^2$$
$$\geq n^2\left(\det\begin{bmatrix}|\frac{1}{n}\sum_{k=1}^{n}x_k| & \left((\frac{1}{n}\sum_{k=1}^{n}|x_k|^2)-|\frac{1}{n}\sum_{k=1}^{n}x_k|^2\right)^{1/2}\\ |\frac{1}{n}\sum_{k=1}^{n}y_k| & \left((\frac{1}{n}\sum_{k=1}^{n}|y_k|^2)-|\frac{1}{n}\sum_{k=1}^{n}y_k|^2\right)^{1/2}\end{bmatrix}\right)^2,$$

which is equivalent to

$$|E(XY)|^2\leq E(X^2)E(Y^2)-\left(|E(X)|\sqrt{Var(Y)}-|E(Y)|\sqrt{Var(X)}\right)^2$$

for real numbers in the probability version [7].

Acknowledgements

The author would like to thank M. Lin for helpful discussions.

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