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Optimal Control of Feedback Control Systems Governed by Systems of Evolution Hemivariational Inequalities

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Abstract. In this paper, we introduce and consider a feedback control system governed by the system of evolution hemivariational inequalities. Several sufficient conditions are formulated by virtue of the properties of multimaps and partial Clarke's subdifferentials such that the existence result of feasible pairs of the feedback control systems is guaranteed. Moreover, an existence result of optimal control pairs for an optimal control system is also established.

1. Introduction

It is well-known that hemivariational inequalities have played an important role in many applications, such as mechanics and engineering, especially in nonsmooth analysis and optimization (see [1, 8–10, 17–20, 22, 24, 25]). Some existence theorems and well-posedness results for hemivariational inequalities have been obtained in the literature; see e.g., [2–6, 27, 30–38] and references therein. Recently, some researchers devoted to consider the optimal control problems for hemivariational inequalities. In [12], Haslinger and Panagiotopoulos proved the existence of optimal control pairs for a class of coercive hemivariational inequalities. In [21], Migorski and Ochal discussed the optimal control problems for the parabolic hemivariational inequalities. Park and Park [24, 25] extended the existence of optimal control pairs to the hyperbolic linear systems. Furthermore, Tolstonogov [28, 29] made efforts to probe into the optimal control problems for differential inclusions with subdifferential type.

Our main purpose here is to study the existence result of feasible pairs of feedback optimal control systems for systems of evolution hemivariational inequalities. To begin with, let us recall several existing results. Throughout, we assume that H is a separable Hilbert space with the inner product $\langle\cdot,\cdot\rangle_H$ and norm $\|\cdot\|_H$, and $A : D(A) \subseteq H \to H$ is the infinitesimal generator of a compact C_0-semigroup $\{T(t)\}_{t\geq 0}$ on H. Let E be a reflexive Banach space, $u : [0,T] \to E$ a control function and $B : E \to H$ a bounded linear operator. Denote by $J^\circ(t,\cdot;\cdot)$ the Clarke's generalized directional derivative ([7]) of a locally Lipschitz functional $J(t,\cdot) : H \to \mathbf{R}$. Denote by $P(E)$ the collection of all nonempty subsets of E. Define two symbols: $P_{f(c)}(E) := \{\Omega \subseteq E : \Omega \text{ is nonempty, closed (convex)}\}$ and $P_{(w)k(c)}(E) := \{\Omega \subseteq E : \Omega \text{ is nonempty, (weakly) compact (convex)}\}$.

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Now, we focus on the following evolution hemivariational inequality problem

$$\begin{cases} \langle -x'(t) + Ax(t) + Bu(t), v\rangle_H + J^\circ(t, x(t); v) \geq 0, \text{ a.e. } t \in [0, T], \ \forall v \in H, \\ x(0) = x_0 \in H. \end{cases} \tag{1}$$

and the following feedback control problem

$$\begin{cases} \langle -x'(t) + Ax(t) + Bu(t), v\rangle_H + J^\circ(t, x(t); v) \geq 0, \text{ a.e. } t \in [0, T], \ \forall v \in H, \\ u(t) \in \mathcal{U}(t, x(t)), \\ x(0) = x_0 \in H, \end{cases} \tag{2}$$

where $\mathcal{U} : [0, T] \times H \to P(E)$ is a multimap.

In [14], Huang, Liu and Zeng proved the existence of solutions of the evolution hemivariational inequality problem (1) and the existence of feasible pairs of the feedback control problem (2). To the best of our knowledge, feedback control problems are ubiquitous around us, including trajectory planning of a robot manipulator, guidance of a tactical missile toward a moving target, regulation of room temperature, and control of string vibrations. Li and Yong [16] studied the optimal feedback control of semilinear evolution equations in Banach spaces. Huang, Liu and Zeng [14] studied the above feedback control problem (2) governed by evolution hemivariational inequality. By using the properties of multimaps and Clarke's generalized subdifferential, they formulated some sufficient conditions to guarantee the existence result of feasible pairs of feedback control problem. Moreover, they also established an existence result of optimal control pairs for an optimal control problem.

However, there is little study for the optimal control of feedback control problems described by evolution hemivariational inequalities in the literature. It is worth pointing out that the study for the optimal control of feedback control systems described by systems of evolution hemivariational inequalities is still untreated topic in the literature and this fact is the motivation of the present work.

For our purpose, we assume that V_i is a separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle_{V_i}$ and norm $\| \cdot \|_{V_i}$, and $A_i : D(A_i) \subseteq V_i \to V_i$ is the infinitesimal generator of a compact C_0-semigroup $\{T_i(t)\}_{t\geq 0}$ on V_i, where $i \in \{1, 2\}$ and $I = [0, T]$ for some $0 < T < \infty$. Let U_i be a reflexive Banach space, $u_i : [0, T] \to U_i$ a control function and $B_i : U_i \to V_i$ a bounded linear operator. For $i, j = 1, 2$ and $i \neq j$, the notation $J_i^\circ(t, x_1, x_2; v_i)$ stands for the partial Clarke generalized directional derivative (cf. [33]) of a locally Lipschitz functional $J(t, \cdot, \cdot) : V_1 \times V_2 \to \mathbf{R}$ with respect to the ith variable at x_i in the direction v_i for the given x_j.

In the present paper, we aim to study the existence of solutions of the following system of evolution hemivariational inequalities:

$$\begin{cases} \langle -x_1'(t) + A_1x_1(t) + B_1u_1(t), v_1\rangle_{V_1} + J_1^\circ(t, x_1(t), x_2(t); v_1) \geq 0, \text{ a.e. } t \in I, \forall v_1 \in V_1, \\ \langle -x_2'(t) + A_2x_2(t) + B_2u_2(t), v_2\rangle_{V_2} + J_2^\circ(t, x_1(t), x_2(t); v_2) \geq 0, \text{ a.e. } t \in I, \forall v_2 \in V_2, \\ x_i(0) = x_i^0 \in V_i, \ i = 1, 2. \end{cases} \tag{3}$$

In what follows we are concerned with the existence of feasible pairs of the following feedback control systems:

$$\begin{cases} \langle -x_1'(t) + A_1x_1(t) + B_1u_1(t), v_1\rangle_{V_1} + J_1^\circ(t, x_1(t), x_2(t); v_1) \geq 0, \text{ a.e. } t \in I, \forall v_1 \in V_1, \\ \langle -x_2'(t) + A_2x_2(t) + B_2u_2(t), v_2\rangle_{V_2} + J_2^\circ(t, x_1(t), x_2(t); v_2) \geq 0, \text{ a.e. } t \in I, \forall v_2 \in V_2, \\ u_i(t) \in \mathcal{U}_i(t, x_1(t), x_2(t)), \ i = 1, 2, \\ x_i(0) = x_i^0 \in V_i, \ i = 1, 2, \end{cases} \tag{4}$$

where $\mathcal{U}_i : [0, T] \times V_1 \times V_2 \to P(U_i)$ is a multimap for $i = 1, 2$.

The rest of this paper is organized as follows. In the next section, we will introduce some useful preliminaries and physical models. In Section 3, some sufficient conditions and techniques are established for the existence of feasible pairs of system (4). We first study the existence of solutions of (3) by a fixed point theorem of multimaps. In Section 4, we will study the optimal control system (4).

2. Preliminaries and Physical Models

In this section, we first introduce some basic preliminaries which are used throughout this paper. Some terminologies are borrowed from [14]. Let $i \in \{1, 2\}$ and $I = [0, T]$. The norm of the Hilbert space V_i will be denoted by $\|\cdot\|_{V_i}$. Let $V = V_1 \times V_2$. Endowed with the norm defined by $\|\mathbf{x}\|_V := \|x_1\|_{V_1} + \|x_2\|_{V_2}$ for all $\mathbf{x} = (x_1, x_2) \in V$, V is a reflexive Banach space ([6]). For a C_0-semigroup $\{T_i(t)\}_{t\geq 0}$, there exist constants ω_i and $\rho_i > 0$ such that $\|T_i(t)\| \leq \rho_i e^{\omega_i t}$ for $0 \leq t < \infty$ and we set $\sup_{t\in I} \|T_i(t)\| \leq \sup_{t\in I} \rho_i e^{\omega_i t} \leq M_i$ with $M_i > 0$ ([26]). Let $C(I, V_i)$ denote the Banach space of all continuous functions from I into V_i with the norm $\|x_i\|_{C(I,V_i)} = \sup_{t\in I} \|x_i(t)\|_{V_i}$, $L^2(I, V_i)$ denote the Banach space of all Bochner L^2-integrable functions from I into V_i with the norm $\|x_i\|_{L^2(I,V_i)} = (\int_0^T \|x_i(s)\|^2_{V_i} ds)^{\frac{1}{2}}$.

Let X and Y be two topological vector spaces. Denote by $P(Y)$ $[C(Y), K(Y), K_v(Y)]$ the collection of all nonempty [respectively, nonempty closed, nonempty compact, nonempty compact convex] subsets of Y. A multimap $F : I \to C(X)$ is said to be measurable, if $F^{-1}(Q) := \{t \in I : F(t) \cap Q \neq \emptyset\} \in \Sigma$ for every closed set $Q \subset X$, where Σ denotes the σ-field of Lebesgue measurable sets on $I = [0, T]$. Every measurable multimap F admits a measurable selection $f : I \to X$, i.e., f is measurable and $f(t) \in F(t)$ for a.e. $t \in I$. A multimap $F : X \to C(Y)$ is said to be upper semicontinuous (for short, u.s.c.), if for every open subset $D \subset Y$ the set $F_+^{-1}(D) = \{x \in X : F(x) \subset D\}$ is open in X; weakly u.s.c., if $F : X \to C(Y_w)$ is u.s.c., where Y_w is the space Y equipped with a weak topology. A multimap $F : X \to C(Y)$ is said to be closed if its graph $\mathrm{Gr}(F) := \{(x, y) \in X \times Y : x \in X, y \in F(x)\}$ is a closed subset of $X \times Y$; compact, if F maps bounded sets of X into relatively compact sets in Y.

We have the following important property for multimaps.

Lemma 2.1. *([15]). Let X and Y be metric spaces and $F : X \to K(Y)$ a closed compact multimap. Then F is u.s.c.*

Definition 2.2. *([16]). Let X be a Banach space and Y be a metric space. Let $F : X \to P(Y)$ be a multimap. We say that F possesses the Cesari property at x_0, if $\bigcap_{\delta>0} \overline{\mathrm{co}}F(O_\delta(x_0)) = F(x_0)$, where $\overline{\mathrm{co}}D$ is the closed convex hull of D, $O_\delta(x_0)$ is the δ-neighborhood of x_0. If F has the Cesari property at every point $x \in Z \subset X$, we simply say that F has the Cesari property on Z.*

Lemma 2.3. *([16]). Let X be a Banach space and Y be a metric space. Let $F : X \to P(Y)$ be u.s.c. with convex and closed values. Then F has the Cesari property on X.*

Now, let us proceed to the definition of the Clarke's subdifferential for a locally Lipschitz functional $h : X \to \mathbf{R}$, where X is a Banach space, X^* is the dual space of X and $\langle \cdot, \cdot \rangle_X$ is the duality pairing between X^* and X. We denote by $h^\circ(x; v)$ the Clarke's generalized directional derivative of h at the point $x \in X$ in the direction $v \in X$, that is

$$h^\circ(x; v) := \limsup_{\lambda\to 0^+, y\to x} \frac{h(y + \lambda v) - h(y)}{\lambda}.$$

Recall also that the Clarke's subdifferential or generalized gradient of h at $x \in X$, denoted by $\partial h(x)$, is a subset of X^* given by $\partial h(x) := \{x^* \in X^* : h^\circ(x; v) \geq \langle x^*, v\rangle_X,\ \forall\, v \in X\}$.

Lemma 2.4. *([22]). If $h : X \to \mathbf{R}$ is a locally Lipschitz functional on an open subset Ω of X, then*

(i) the function $(x, v) \mapsto h^\circ(x; v)$ is u.s.c. from $\Omega \times X$ into $\mathbf{R}$, i.e., for all $x \in \Omega$, $v \in X$, $\{x_n\} \subset \Omega$, $\{v_n\} \subset X$ such that $x_n \to x$ in Ω and $v_n \to v$ in X, we have $\limsup_{n\to\infty} h^\circ(x_n; v_n) \leq h^\circ(x; v)$;

(ii) for every $x \in \Omega$ the gradient $\partial h(x)$ is a nonempty, convex and weakly* compact subset of X^*, and $\|x^*\|_{X^*} \leq \ell_0$ for any $x^* \in \partial h(x)$ (where $\ell_0 > 0$ is the Lipschitz constant of h near x);

(iii) the graph of ∂h is closed in $X \times X^*_{w^*}$;

(iv) the multimap ∂h is u.s.c. from Ω into $X^*_{w^*}$;

(v) for every $v \in X$, one has $h^\circ(x; v) = \max\{\langle x^*, v\rangle_X : x^* \in \partial h(x)\}$.

The key tool in one of our main results is the following fixed point theorem.

Theorem 2.5. *([11]). Let X be a Banach space, C a closed convex subset of X, D an open subset of C (relative to C) and $0 \in D$. Suppose that $\mathcal{F} : \overline{D} \to K_v(C)$ is an u.s.c. and compact multimap. Then either (i) $\mathcal{F}$ has a fixed point in D, or (ii) there are $x \in \partial D$ and $\lambda \in (0,1)$ with $x \in \lambda\mathcal{F}(x)$.*

In the sequel, we shall study the existence of mild solutions of the following system of semilinear evolutionary inclusions, which is specified as follows: Find $(x_1, x_2) \in C(I, V_1) \times C(I, V_2)$ such that

$$\begin{cases} x_1'(t) \in A_1x_1(t) + B_1u_1(t) + \partial_1 J(t, x_1(t), x_2(t)), \text{ a.e. } t \in I = [0,T], \\ x_2'(t) \in A_2x_2(t) + B_2u_2(t) + \partial_2 J(t, x_1(t), x_2(t)), \text{ a.e. } t \in I = [0,T], \\ x_i(0) = x_i^0 \in V_i, \quad i = 1,2, \end{cases} \tag{5}$$

where, for $i = 1,2$, $A_i : D(A_i) \subseteq V_i \to V_i$ is the infinitesimal generator of a C_0-semigroup $T_i(t)$ $(t \geq 0)$ on a separable Hilbert space V_i. For $i, j = 1,2$ and $i \neq j$, the notation $\partial_i J(t, x_1, x_2)$ stands for the partial Clarke generalized gradient (cf. [33]) of a locally Lipschitz functional $J(t, \cdot, \cdot) : V_1 \times V_2 \to \mathbf{R}$ with respect to the ith variable at x_i for the given x_j. The control function u_i takes values in $L^2(I, U_i)$, the admissible controls set U_i is a Hilbert space, and B_i is a bounded linear operator from U_i into V_i.

We say that $(x_1, x_2) \in W^{1,2}(I, V_1) \times W^{1,2}(I, V_2)$ is a solution of (5) if there exists a pair of functions $(f_1, f_2) \in L^2(I, V_1) \times L^2(I, V_2)$ such that $(f_1, f_2) \in \partial_1 J(t, x_1(t), x_2(t)) \times \partial_2 J(t, x_1(t), x_2(t))$ and

$$\begin{cases} x_1'(t) = A_1x_1(t) + B_1u_1(t) + f_1(t), \text{ a.e. } t \in I, \\ x_2'(t) = A_2x_2(t) + B_2u_2(t) + f_2(t), \text{ a.e. } t \in I, \\ x_i(0) = x_i^0 \in V_i, \ i = 1,2, \end{cases}$$

which implies

$$\begin{cases} \langle -x_1'(t) + A_1x_1(t) + B_1u_1(t), v_1\rangle_{V_1} + \langle f_1(t), v_1\rangle_{V_1} = 0, \text{ a.e. } t \in I, \ \forall v_1 \in V_1, \\ \langle -x_2'(t) + A_2x_2(t) + B_2u_2(t), v_2\rangle_{V_2} + \langle f_2(t), v_2\rangle_{V_2} = 0, \text{ a.e. } t \in I, \ \forall v_2 \in V_2, \\ x_i(0) = x_i^0 \in V_i, \ i = 1,2. \end{cases}$$

Since for $i = 1,2$, $f_i(t) \in \partial_i J(t, x_1(t), x_2(t))$ and $\langle f_i(t), v_i\rangle_{V_i} \leq J_i^\circ(t, x_1(t), x_2(t); v_i)$, we obtain

$$\begin{cases} \langle -x_1'(t) + A_1x_1(t) + B_1u_1(t), v_1\rangle_{V_1} + J_1^\circ(t, x_1(t), x_2(t); v_1) \geq 0, \text{ a.e. } t \in I, \ \forall v_1 \in V_1, \\ \langle -x_2'(t) + A_2x_2(t) + B_2u_2(t), v_2\rangle_{V_2} + J_2^\circ(t, x_1(t), x_2(t); v_2) \geq 0, \text{ a.e. } t \in I, \ \forall v_2 \in V_2, \\ x_i(0) = x_i^0 \in V_i, \ i = 1,2. \end{cases}$$

Hence, any solutions of system (5) are also solutions of system (3).

Similarly, the feedback control system (4) of evolution hemivariational inequalities can be reduced to the following feedback control system with partial Clarke's subdifferentials:

$$\begin{cases} x_1'(t) \in A_1x_1(t) + B_1u_1(t) + \partial_1 J(t, x_1(t), x_2(t)), \text{ a.e. } t \in I, \\ x_2'(t) \in A_2x_2(t) + B_2u_2(t) + \partial_2 J(t, x_1(t), x_2(t)), \text{ a.e. } t \in I, \\ u_i(t) \in \mathcal{U}_i(t, x_1(t), x_2(t)), \ i = 1,2, \\ x_i(0) = x_i^0 \in V_i, \ i = 1,2. \end{cases} \tag{6}$$

Therefore, in order to study the system (3) of evolution hemivariational inequalities and the feedback control system (4) of evolution hemivariational inequalities, we only need to deal with the system (5) of semilinear evolutionary inclusions, and the feedback control system (6) with partial Clarke's subdifferentials.

We note that system (3) arises in many important models for distributed parameter control problems and that a large class of identification problems enter our formulation. Let us indicate a problem which is one of the motivations for the study of the system (3) of evolution hemivariational inequalities ([21]).

We consider the following system of heat initial-boundary value problems:

$$\begin{cases} \frac{\partial}{\partial t}x_1(t,y) = \frac{\partial^2}{\partial y^2}x_1(t,y) + B_1u_1(t,y) + f_1(x_1,x_2), \ (t,y) \in (0,1)\times(0,\pi), \\ \frac{\partial}{\partial t}x_2(t,y) = \frac{\partial^2}{\partial y^2}x_2(t,y) + B_2u_2(t,y) + f_2(x_1,x_2), \ (t,y) \in (0,1)\times(0,\pi), \\ x_i(t,0) = x_i(t,\pi) = 0, \ t \in (0,1), \ i = 1,2, \\ x_i(0,y) = x_i^0(y), \ y \in (0,\pi), \ i = 1,2. \end{cases} \tag{7}$$

This system represents the heat flows with temperature-dependent sources. Let $i \in \{1,2\}$. $x_i = x_i(t,y)$ represents the temperature at the time $t \in (0,1)$ and point $y \in (0,\pi)$. The temperatures of boundaries are zero and the initial temperature is $x_i^0(y) \quad (y \in (0,\pi))$. u_i is a control function. $f_i(x_1,x_2)$ is a heat source dependent of temperatures.

We suppose that for $i = 1,2$, the control u_i is a feedback control by the temperatures such that

$$u_i \in U_i(t,x_1,x_2), \ \text{a.e. } (t,y) \in (0,1)\times(0,\pi), \tag{8}$$

f_i is a known function of the temperatures of the following form

$$f_i(t,y) \in \partial_i J(t,y,x_1(t,y),x_2(t,y)), \ \text{a.e. } (t,y) \in (0,1)\times(0,\pi), \tag{9}$$

where for $i,j = 1,2$ and $i \neq j$, $\partial_i J(t,y,\eta_1,\eta_2)$ denotes the partial Clarke's generalized gradient of a locally Lipschitz functional $J(t,y,\cdot,\cdot) : \mathbf{R}^2 \to \mathbf{R}$ with respect to the ith variable at η_i for the given η_j. The multivalued mapping $\partial_i J(t,y,\cdot,\cdot) : \mathbf{R}^2 \to 2^{\mathbf{R}}$ is generally nonmonotone and it includes the vertical jumps which means that the law is characterized by the partial Clarke's generalized gradient of a nonsmooth potential J.

Let $i \in \{1,2\}$. Take $V_i = L^2(0,\pi)$, $x_i(t)(\cdot) = x_i(t,\cdot)$ and the operator $A_i : D(A_i) \subset V_i \to V_i$ is defined by $A_ix_i = x_i''$, where the domain $D(A_i)$ is given by $\{x_i \in V_i : x_i', x_i'' \in V_i, \ x_i(0) = x_i(\pi)\}$. Then, A_i can be represented as $A_ix_i = -\sum_{n=1}^{\infty} n^2\langle x_i, e_n\rangle e_n, \ x_i \in D(A_i)$. where $e_n(y) = \sqrt{2/\pi}\sin ny \ (n = 1,2,...)$ is an orthonormal basis of V_i. We knew that A_i generates a strongly continuous semigroup $T_i(t) \quad (t > 0)$ in V_i, which is compact and analytic (see [26]), given by $T_i(t)x_i = \sum_{n=1}^{\infty} e^{-n^2t}\langle x_i, e_n\rangle e_n, \ x_i \in V_i$, and $\|T_i(t)\| \le e^{-1} < 1 = M_i$.

Now let the function $J : (0,1)\times V_1 \times V_2 \to \mathbf{R}$ be given by

$$J(t,x_1,x_2) = \int_0^1 (j(t,y,x_1(y)) + j(t,y,x_2(y)))dy, \ t \in (0,1), \ (x_1,x_2) \in V_1 \times V_2,$$

where $j(t,y,z) = \int_0^z \phi(t,y,\theta)d\theta, \ (t,y) \in (0,1)\times(0,\pi), \ z \in \mathbf{R}$.

Let $\phi : (0,1)\times(0,\pi)\times\mathbf{R} \to \mathbf{R}$ be a function satisfying the following assumptions:

(i) for all $y \in (0,\pi), z \in \mathbf{R}$, $\phi(\cdot,y,z) : (0,1) \to \mathbf{R}$ is measurable;
(ii) for all $t \in (0,1), z \in \mathbf{R}$, $\phi(t,\cdot,z) : (0,\pi) \to \mathbf{R}$ is continuous;
(iii) for all $z \in \mathbf{R}$ there exists a constant $c_1 > 0$ such that $|\phi(\cdot,\cdot,z)| \le c_1(1+|z|)$ for $z \in \mathbf{R}$;
(iv) for every $z \in \mathbf{R}$, $\phi(\cdot,\cdot,z \pm 0)$ exists.

If ϕ satisfies (iii), then $\partial j(z) \subset [\underline{\phi}(z), \overline{\phi}(z)]$ for $z \in \mathbf{R}$ (we omit (t,y) here), where $\underline{\phi}(z)$ and $\overline{\phi}(z)$ denote the essential supermum and essential infimum of ϕ at z ([7]), respectively.

If ϕ satisfies (i)-(iv), then the function $j(\cdot,\cdot,\cdot)$ defined above has the following properties:

(i) for all $y \in (0,\pi), z \in \mathbf{R}$, $j(\cdot,y,z)$ is measurable and $j(\cdot,\cdot,0) \in L^2((0,1)\times(0,\pi))$;
(ii) for all $t \in (0,1), z \in \mathbf{R}$, $j(t,\cdot,z) : (0,\pi) \to \mathbf{R}$ is continuous;
(iii) for all $(t,y) \in (0,1)\times(0,\pi)$, $j(t,y,\cdot) : \mathbf{R} \to \mathbf{R}$ is locally Lipschitz;
(iv) there exists a constant $c_2 > 0$ such that $|\zeta| \le c_2(1+|z|)$ for all $\zeta \in \partial j(t,y,z)$, $(t,y) \in (0,1)\times(0,\pi)$;
(v) there exists a constant $c_3 > 0$ such that $j^\circ(t,y,z;-z) \le c_3(1+|z|)$ for all $(t,y) \in (0,1)\times(0,\pi)$.

Assume that for $i = 1, 2$, U_i is a reflexive Banach space, $u_i : (0, 1) \to U_i$ a control function and $B_i : U_i \to \mathbf{R}$ a bounded linear operator. Thus, combining (8)-(9), system (7) turns to be system (6).

Therefore, the variational formulation of the above system leads to the system (3) of evolution hemivariational inequalities and is met, for example, in the nonmonotone nonconvex interior semipermeability problems. For the latter, Panagiotopoulos [23] considered a temperature control problem in which they regulated the temperature to deviate as little as possible from a given interval. We remark that the monotone semipermeability problems, leading to variational inequalities, have been studied by Duvaut and Lions under the assumption that $J(t, y, \cdot)$ is a proper, lower semicontinuous, convex function which means that $\partial J(t, y, \cdot)$ is a maximal monotone operator in $\mathbf{R}^2$.

3. The Existence of Feasible Pairs

In this section we study the existence of feasible pairs for system (6).

At the first, we study the existence of solutions of system (6). We will make the following conditions.

(HT): $T_i(t) \quad (t > 0)$ is a compact operator for $i = 1, 2$.

Let $J : I \times V_1 \times V_2 \to \mathbf{R}$ be a functional satisfying the following conditions:

(HJ1) the function $t \mapsto J(t, x_1, x_2)$ is measurable for all $(x_1, x_2) \in V_1 \times V_2$;

(HJ2) the function $(x_1, x_2) \to J(t, x_1, x_2)$ is locally Lipschitz on $V_1 \times V_2$ for a.e. $t \in I$;

(HJ3) for $i = 1, 2$, there exist a function $\phi_i \in L^2(I, \mathbf{R}^+)$ and a constant $L_i > 0$ such that

$$\|\partial_i J(t, x_1, x_2)\| = \sup\{\|\zeta_i\|_{V_i} : \zeta_i \in \partial_i J(t, x_1, x_2)\} \leq \phi_i(t) + L_i\|x_i\|_{V_i}$$

for a.e. $t \in I$ and all $(x_1, x_2) \in V_1 \times V_2$;

(HJ4) $J(t, x_1, x_2) + J(t, y_1, y_2) = J(t, x_1, y_2) + J(t, y_1, x_2)$ for a.e. $t \in I$ and all $(x_1, x_2), (y_1, y_2) \in V_1 \times V_2$.

Lemma 3.1. *([30]). Suppose that the functional $J : I \times V_1 \times V_2 \to \mathbf{R}$ satisfies the hypotheses (HJ2), (HJ4). Then, for any sequence $\mathbf{x}^n = (x_1^n, x_2^n) \in V = V_1 \times V_2$ converging strongly to $\mathbf{x} = (x_1, x_2) \in V$ and $y_i^n \in V_i$ converging strongly to $y_i \in V_i$, one has $\limsup_{n\to\infty} J_i^\circ(t, x_1^n, x_2^n; y_i^n) \leq J_i^\circ(t, x_1, x_2; y_i)$,* a.e. *$t \in I$, where $i = 1, 2$.*

Lemma 3.2. *([22]). Let E be a separable reflexive Banach space, $0 < T < \infty$ and $h : (0, T) \times E \to \mathbf{R}$ be a function such that $h(\cdot, x)$ is measurable for all $x \in E$ and $h(t, \cdot)$ is locally Lipschitz on E for all $t \in (0, T)$. Then the multifunction $(t, x) \in (0, T) \times E \mapsto \partial h(t, x) \subset E^*$ is measurable, where ∂h denotes the Clarke generalized gradient of $h(t, \cdot)$.*

Lemma 3.3. *([22]). Let X and Y be two topological spaces, $F : X \to 2^Y$ a multivalued mapping.*

(i) If F is u.s.c. and closed-valued, then F is closed;

(ii) If F is compact-valued, then F is u.s.c. at $x \in X$ if and only if for any net $\{x_\alpha\} \subseteq X$ with $x_\alpha \to x$ and for any net $\{y_\alpha\} \subseteq Y$ with $y_\alpha \in F(x_\alpha)$ for all α, there exist $y \in F(x)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y$;

(iii) F is l.s.c. at $x \in X$ if and only if for any $y \in F(x)$ and for any net $\{x_\alpha\}$ with $x_\alpha \to x$, there exists a net $\{y_\alpha\}$ with $y_\alpha \in F(x_\alpha)$ for all α such that $y_\alpha \to y$.

We shall make use of the following well-known results in this paper.

Theorem 3.4. *([13]). If (Ω, Σ) is a measurable space, X is a Polish space (i.e., separable completely metric space) and $F : \Omega \to P_f(X)$ is measurable, then $F(\cdot)$ admits a measurable selection (i.e., there exists $f : \Omega \to X$ measurable such that for every $x \in \Omega$, $f(x) \in F(x)$).*

Lemma 3.5. *([22]). Let (Ω, Σ, μ) be a σ-finite measure space, E be a Banach space and $1 \leq p < \infty$. If $f_n, f \in L^p(\Omega, E)$, $f_n \to f$ weakly in $L^p(\Omega, E)$ and $f_n(x) \in G(x)$ for μ-a.e. $x \in \Omega$ and all $n \in \mathbf{N}$ where $G(x) \in P_{wk}(E)$ for μ-a.e. $x \in \Omega$, then $f(x) \in \overline{\mathrm{conv}}(w\text{-}\limsup\{f_n(x)\}_{n\in\mathbf{N}})$ for μ-a.e. on $x \in \Omega$, where $\overline{\mathrm{conv}}$ denotes the closed convex hull of a set.*

By the symbol of S^2_Ψ we will denote the set of all Bochner L^2-integrable selections of the multimap $\Psi : I \to P(H)$, i.e., $S^2_\Psi = \{\psi \in L^2(I, H) : \psi(t) \in \Psi(t) \text{ for a.e. } t \in I\}$. Next, for $i = 1, 2$, we define the superposition multioperator $\mathcal{P}^i_J : C(I, V_1) \times C(I, V_2) \to P(L^2(I, V_i))$ as follows

$$\mathcal{P}^i_J(x_1, x_2) = S^2_{\partial_i J(\cdot, x_1(\cdot), x_2(\cdot))} = \{w_i \in L^2(I, V_i) : w_i(t) \in \partial_i J(t, x_1(t), x_2(t)) \text{ a.e. } t \in I\},$$

for all $(x_1, x_2) \in C(I, V_1) \times C(I, V_2)$.

Let $C = C(I, V_1) \times C(I, V_2)$. Endowed with the norm defined by $\|\mathbf{x}\|_C := \|x_1\|_{C(I,V_1)} + \|x_2\|_{C(I,V_2)}$ for all $\mathbf{x} = (x_1, x_2) \in C = C(I, V_1) \times C(I, V_2)$, C is a reflexive Banach space.

We have the following property for the operator $\mathcal{P}^i_J$ for $i = 1, 2$.

Lemma 3.6. *Let $i \in \{1, 2\}$. If conditions (HJ1)-(HJ4) are satisfied, then for every $(x_1, x_2) \in C(I, V_1) \times C(I, V_2)$, the set $\mathcal{P}^i_J(x_1, x_2)$ has nonempty, convex and weakly compact values.*

Proof. First of all, for $i = 1, 2$, from the reflexivity of V_i and Lemma 2.4(ii), we know that for every $(t, x_1, x_2) \in I \times V_1 \times V_2$, the set $\partial_i(t, x_1, x_2)$ is nonempty, convex and weakly compact in V_i and the multifunction $\partial_i J$ is $P_{wkc}(V_i)$-valued. Therefore, it is not difficult to check that for $i = 1, 2$, $\mathcal{P}^i_J(x_1, x_2)$ has convex and weakly compact values. Next, we show that for $i = 1, 2$, $\mathcal{P}^i_J(x_1, x_2)$ is nonempty. Indeed, let $(x_1, x_2) \in C(I, V_1) \times C(I, V_2)$. Then for $i = 1, 2$, there exists a sequence $\{\varphi^n_i\} \subseteq C(I, V_i)$ of step functions such that

$$\varphi^n_i(t) \to x_i(t), \text{ in } C(I, V_i), \text{ a.e. } t \in I. \tag{10}$$

From hypotheses (HJ1), (HJ2) and Lemma 3.2, it follows that for $i = 1, 2$, the multifunction $(t, x_1, x_2) \mapsto \partial_i J(t, x_1, x_2)$ is measurable. Thus, for $i = 1, 2$, $t \mapsto \partial_i J(t, \varphi^n_1(t), \varphi^n_2(t))$ is measurable from I into $P_{fc}(V_i)$. For $i = 1, 2$, applying Theorem 3.4, for every $n \ge 1$, there exists a measurable function $\zeta^n_i : I \to V_i$ such that $\zeta^n_i(t) \in \partial_i J(t, \varphi^n_1(t), \varphi^n_2(t))$ a.e. $t \in I$. Next, from hypothesis (HJ3), we obtain that for $i = 1, 2$,

$$\|\zeta^n_i\|_{L^2(I,V_i)} \le \|\phi_i\|_{L^2(I,\mathbf{R}^+)} + L_i \|\varphi^n_i\|_{L^2(I,V_i)}.$$

Hence, for $i = 1, 2$, $\{\zeta^n_i\}$ remains in a bounded subset of $L^2(I, V_i)$. Thus, for $i = 1, 2$, by passing to a subsequence if necessary, we may suppose that $\zeta^n_i \to \zeta_i$ weakly in $L^2(I, V_i)$ with $\zeta_i \in L^2(I, V_i)$. Then it follows from Lemma 3.5 that for $i = 1, 2$,

$$\zeta_i(t) \in \overline{\mathrm{co}}(w - \limsup \{\zeta^n_i(t)\}_{n \ge 1}), \text{ a.e. } t \in I. \tag{11}$$

We claim that for a.e. $t \in I$, the multifunction $(x_1, x_2) \mapsto \partial_i J(t, x_1, x_2)$ is u.s.c. from $V_1 \times V_2$ into $(V_i)_w$, where for $i = 1, 2$, $(V_i)_w$ is the space furnished with the w-topology of V_i.

Indeed, for any sequence $\{(x^n_1, x^n_2)\} \subseteq V_1 \times V_2$ with $(x^n_1, x^n_2) \to (x_1, x_2)$ in $V = V_1 \times V_2$ and for any sequence $\{y^n_i\} \subseteq V_i$ with $y^n_i \in \partial_i J(t, x^n_1, x^n_2)$ for all $n \ge 1$, we know by the definition of the partial Clarke generalized gradient $\partial_i J(t, x^n_1, x^n_2)$ of a locally Lipschitz functional $J(t, \cdot, \cdot) : V_1 \times V_2 \to \mathbf{R}$, that $\langle y^n_i, v_i \rangle_{V_i} \le J^\circ_i(t, x^n_1, x^n_2; v_i)$, $\quad \forall v_i \in V_i$. Since $(x^n_1, x^n_2) \to (x_1, x_2)$ in $V_1 \times V_2$, from Hypotheses (HJ2), (HJ4) and Lemma 3.1 it follows that for $i = 1, 2$,

$$\limsup_{n \to \infty} \langle y^n_i, v_i \rangle_{V_i} \le \limsup_{n \to \infty} J^\circ_i(t, x^n_1, x^n_2; v_i) \le J^\circ_i(t, x_1, x_2; v_i), \quad \forall v_i \in V_i. \tag{12}$$

Also, for $i = 1, 2$, from hypotheses (HJ3) and $y^n_i \in \partial_i J(t, x^n_1, x^n_2)$, we get $\|y^n_i\|_{V_i} \le \phi_i(t) + L_i \|x^n_i\|_{V_i}$, which together with the boundedness of $\{x^n_i\}$, implies that $\{y^n_i\}$ is bounded. Taking into account the reflexivity of V_i, we know that there exists a subsequence $\{y^{n_k}_i\}$ of $\{y^n_i\}$ such that $y^{n_k}_i \to y_i$ weakly in V_i. So, it follows from (12) that $\langle y_i, v_i \rangle = \lim_{k \to \infty} \langle y^{n_k}_i, v_i \rangle_{V_i} \le \limsup_{n \to \infty} \langle y^n_i, v_i \rangle_{V_i} \le J^\circ_i(t, x_1, x_2; v_i)$, $\forall v_i \in V_i$, which yields $y_i \in \partial_i J(t, x_1, x_2)$. Therefore, in terms of Lemma 3.3 (ii), we deduce that for a.e. $t \in I$, the multifunction $(x_1, x_2) \mapsto \partial_i J(t, x_1, x_2)$ is u.s.c. from $V_1 \times V_2$ into $(V_i)_w$, where $i = 1, 2$.

Recalling that the graph of an u.s.c. multifunction with closed values is closed (due to Lemma 3.3 (i), we obtain that for $i = 1, 2$ and a.e. $t \in I$, if $f^n_i \in \partial_i J(t, \zeta^n_1, \zeta^n_2)$, $f^n_i \in V_i$, $f^n_i \to f_i$ weakly in V_i, $\zeta^n_i \in C(I, V_i)$, $\zeta^n_i \to \zeta_i$ in $C(I, V_i)$, then $f_i \in \partial_i J(t, \zeta_1, \zeta_2)$.

Therefore, by (12), we have

$$w - \limsup \partial_i J(t, \zeta_1^n(t), \zeta_2^n(t)) \subset \partial_i J(t, x_1(t), x_2(t)) \text{ a.e. } t \in I, \tag{13}$$

where the Kuratowski limit superior (cf. Definition 3.14 of [22]) is given by

$$w - \limsup \partial_i J(t, \varphi_1^n(t), \varphi_2^n(t)) = \{\zeta_i \in V_i : \zeta_i = w - \lim \zeta_i^{n_k},\ \zeta_i^{n_k} \in \partial_i J(t, \varphi_1^{n_k}(t), \varphi_2^{n_k}(t)),\ n_1 < \cdots < n_k < \cdots\}.$$

So, from (11) and (13), we get for $i = 1, 2$,

$$\zeta_i(t) \in \overline{\text{co}}(w - \limsup\{\zeta_i^n(t)\}_{n\geq 1}) \subset \overline{\text{co}}(w - \limsup \partial_i J(t, \varphi_1^n(t), \varphi_2^n(t))) \subset \partial_i J(t, x_1(t), x_2(t)), \text{ a.e. } t \in I.$$

Since for $i = 1, 2$, $\zeta_i \in L^2(I, V_i)$ and $\zeta_i(t) \in \partial_i J(t, x_1(t), x_2(t))$ a.e. $t \in I$, it is clear that $\zeta_i \in \mathcal{P}_J^i(x_1, x_2)$. Therefore, $\mathcal{P}_J^i(x_1, x_2)$ is nonempty. The proof is completed. □

Lemma 3.7. *Let $i \in \{1, 2\}$. If conditions (HJ1)-(HJ4) are satisfied, then operator $\mathcal{P}_J^i$ is closed in $C \times L_w^2(I, V_i)$, where $L_w^2(I, V_i)$ is the space furnished with the w-topology of $L^2(I, V_i)$.*

Proof. By Lemma 3.6, we know that for each $\mathbf{x} = (x_1, x_2) \in C = C(I, V_1) \times C(I, V_2)$, the set $\mathcal{P}_J^i(x_1, x_2)$ has nonempty, convex and weakly compact values for $i = 1, 2$. Utilizing the similar arguments to those in the proof of Lemma 3.6, we can prove that for $i = 1, 2$ the operator $\mathcal{P}_J^i : C = C(I, V_1) \times C(I, V_2) \to P(L^2(I, V_i))$ is u.s.c. from $C = C(I, V_1) \times C(I, V_2)$ into $L_w^2(I, V_i)$ where for $i = 1, 2$. So, it follows from Lemma 3.5(ii) that for $i = 1, 2$, the graph of the u.s.c. multifunction $\mathcal{P}_J^i$ with closed values is closed (due to Lemma 3.3 (i), which hence implies that for $i = 1, 2$, if $\mathbf{x}^n = (x_1^n, x_2^n) \to \mathbf{x} = (x_1, x_2)$ in $C = C(I, V_1) \times C(I, V_2)$, $w_i^n \to w_i$ weakly in $L^2(I, V_i)$ and $w_i^n \in \mathcal{P}_J^i(x_1^n, x_2^n)$, then $w_i \in \mathcal{P}_J^i(x_1, x_2)$. The proof is completed. □

Lemma 3.8. *Let $i \in \{1, 2\}$. If the conditions (HJ1)-(HJ4) are satisfied, then for a.e. $t \in I$, the multimap $\partial_i J(t, \cdot, \cdot) : V_1 \times V_2 \to P(V_i)$ has the Cesari property, i.e., $\bigcap_{\delta>0} \overline{\text{co}}\partial_i J(t, O_\delta(\mathbf{x})) = \partial_i J(t, \mathbf{x})$, for all $\mathbf{x} = (x_1, x_2) \in V = V_1 \times V_2$.*

Proof. On one hand, it is clear that for any $\delta > 0$, $\partial_i J(t, \mathbf{x}) \subset \overline{\text{co}}\partial_i J(t, O_\delta(\mathbf{x}))$, for all $\mathbf{x} = (x_1, x_2) \in V = V_1 \times V_2$ and a.e. $t \in I$. Therefore, $\partial_i J(t, \mathbf{x}) \subset \bigcap_{\delta>0} \overline{\text{co}}\partial_i J(t, O_\delta(\mathbf{x}))$, for all $\mathbf{x} = (x_1, x_2) \in V = V_1 \times V_2$ and a.e. $t \in I$.

On the other hand, by the proof of Lemma 3.6, we know that for a.e. $t \in I$, the multifunction $(x_1, x_2) \mapsto \partial_i J(t, x_1, x_2)$ is u.s.c. from $V_1 \times V_2$ into $(V_i)_w$. Let $t \in I$, $\mathbf{x} = (x_1, x_2) \in V$ be fixed. For any neighborhood $\Omega_i \supset \partial_i J(t, \mathbf{x})$ (in the sense of weak topology of V_i), there exists a $\delta > 0$ such that $\partial_i J(t, O_\delta(\mathbf{x})) \subset \Omega_i$. Since $(V_i)_w$ is locally convex, we can choose Ω_i to be convex. Therefore, $\overline{\text{co}}\partial_i J(t, O_\delta(\mathbf{x})) \subset \overline{\Omega_i}$.

Now, we show that $\partial_i J(t, \mathbf{x}) = \bigcap \overline{\Omega_i}$ for all neighborhood Ω_i of $\partial_i J(t, \mathbf{x})$. To the contrary, there exists $y_i \in \bigcap \overline{\Omega_i}$ and $y_i \notin \partial_i J(t, \mathbf{x})$. Then there exists a closed set $D_{y_i} \ni y_i$ such that $D_{y_i} \cap \partial_i J(t, \mathbf{x}) = \emptyset$. By the separation property, there exist a neighborhood N_i of D_{y_i} and a neighborhood Ω_i' of $\partial_i J(t, \mathbf{x})$ such that $N_i \cap \Omega_i' = \emptyset$. This shows $y_i \notin \Omega_i'$ which is a contradiction. Therefore, $\bigcap_{\delta>0} \overline{\text{co}}\partial_i J(t, O_\delta(\mathbf{x})) \subset \bigcap \overline{\Omega_i} = \partial_i J(t, \mathbf{x})$. The proof is complete. □

Definition 3.9. *A pair of functions $\mathbf{x} = (x_1, x_2) \in C = C(I, V_1) \times C(I, V_2)$ is said to be a mild solution of system* (5) *on the interval $I = [0, T]$ if*

$$\begin{cases} x_1(t) = T_1(t)x_1^0 + \int_0^t T_1(t-s)(B_1 u_1(s) + f_1(s))ds,\ t \in I, \\ x_2(t) = T_2(t)x_2^0 + \int_0^t T_2(t-s)(B_2 u_2(s) + f_2(s))ds,\ t \in I, \end{cases}$$

where $(f_1, f_2) \in \mathcal{P}_J^1(\mathbf{x}) \times \mathcal{P}_J^2(\mathbf{x})$.

Lemma 3.10. *([16]). Let $i \in \{1, 2\}$. If condition (HT) holds, then operator $G_i : L^p(I, V_i) \to C(I, V_i)$ for some $p > 1$, given by $(G_i f_i)(\cdot) = \int_0^\cdot T_i(\cdot - s) f_i(s)ds$, is compact for $f_i \in L^p(I, V_i)$.*

Now we can obtain the following result.

Theorem 3.11. *If the conditions (HT) and (HJ1)-(HJ4) are satisfied, then for any* $\mathbf{u} = (u_1, u_2) \in L^2(I, U_1) \times L^2(I, U_2)$, *system* (5) *has at least one mild solution in* $C = C(I, V_1) \times C(I, V_2)$.

Proof. Consider the multimap $\mathcal{F} : C \to K_v(C)$ defined by

$$\mathcal{F}(\mathbf{x}) = \{\mathbf{y} = (y_1, y_2) \in C : \ \exists \mathbf{f} = (f_1, f_2) \in \mathcal{P}^1_J(\mathbf{x}) \times \mathcal{P}^2_J(\mathbf{x}) \text{ s.t. } y_1(t) = T_1(t)x^0_1 + \int_0^t T_1(t-s)(B_1u_1(s) + f_1(s))ds,$$

$$y_2(t) = T_2(t)x^0_2 + \int_0^t T_2(t-s)(B_2u_2(s) + f_2(s))ds\}, \textit{for all } \mathbf{x} = (x_1, x_2) \in C.$$

Now, we verify that $\mathcal{F}$ has a fixed point in C. First, $\mathcal{F}(x)$ is convex by the convexity of $\mathcal{P}^i_J(\mathbf{x})$ which follows from Lemma 3.6. Next, we subdivide the proof into five steps.

Step 1. $\mathcal{F}$ maps bounded sets into bounded sets in C.

For $\forall k_0 > 0$, let $\mathcal{B}_{k_0} = \{\mathbf{x} = (x_1, x_2) \in C : \|\mathbf{x}\|_C \leq k_0\}$. Actually, it is enough to show that there exists a positive constant ℓ such that for each $\varphi = (\varphi_1, \varphi_2) \in \mathcal{F}(\mathbf{x})$, $\mathbf{x} \in \mathcal{B}_{k_0}$, $\|\varphi\|_C \leq \ell$. If $\varphi = (\varphi_1, \varphi_2) \in \mathcal{F}(\mathbf{x})$, then there exists $\mathbf{f} = (f_1, f_2) \in \mathcal{P}^1_J(\mathbf{x}) \times \mathcal{P}^2_J(\mathbf{x})$ such that for every $t \in I$,

$$\begin{cases} \varphi_1(t) = T_1(t)x^0_1 + \int_0^t T_1(t-s)f_1(s)ds + \int_0^t T_1(t-s)B_1u_1(s)ds, \\ \varphi_2(t) = T_2(t)x^0_2 + \int_0^t T_2(t-s)f_2(s)ds + \int_0^t T_2(t-s)B_2u_2(s)ds. \end{cases}$$

By (HJ3) and the Holder inequality, we obtain that for $i = 1, 2$ and every $t \in I$,

$$\begin{aligned} \|\varphi_i(t)\|_{V_i} &\leq \|T_i(t)x^0_i\|_{V_i} + \int_0^t \|T_i(t-s)f_i(s)\|_{V_i}ds + \int_0^t \|T_i(t-s)B_iu_i(s)\|_{V_i}ds \\ &\leq M_i\|x^0_i\|_{V_i} + M_i\int_0^t [\phi_i(s) + L_i\|x_i(s)\|_{V_i} + \|B_i\|\|u_i(s)\|_{U_i}]ds \\ &\leq M_i\|x^0_i\|_{V_i} + M_i(\|\phi_i\|_{L^2(I,\mathbf{R}^+)} + \|B_i\|\|u_i\|_{L^2(I,U_i)})T^{\frac{1}{2}} + M_iL_ik_0T := \ell_i. \end{aligned}$$

Let $\ell = \ell_1 + \ell_2$. Then it is easy to see that for each $\varphi = (\varphi_1, \varphi_2) \in \mathcal{F}(\mathbf{x})$, $\mathbf{x} \in \mathcal{B}_{k_0}$, $\|\varphi\|_C \leq \ell$. Thus, $\{\mathcal{F}(\mathbf{x}) : \mathbf{x} = (x_1, x_2) \in \mathcal{B}_{k_0}\}$ is bounded in C.

Step 2. $\mathcal{F}$ maps bounded sets into equicontinuous sets of C.

In the following, we will show that $\{\mathcal{F}(\mathbf{x}) : \mathbf{x} = (x_1, x_2) \in \mathcal{B}_{k_0}\}$ is a family of equicontinuous functions.

Indeed, for any $\mathbf{x} = (x_1, x_2) \in \mathcal{B}_{k_0}$, $\varphi = (\varphi_1, \varphi_2) \in \mathcal{F}(\mathbf{x})$, there exists $\mathbf{f} = (f_1, f_2) \in \mathcal{P}^1_J(x_1, x_2) \times \mathcal{P}^2_J(x_1, x_2)$ such that for every $t \in I$,

$$\begin{cases} \varphi_1(t) = T_1(t)x^0_1 + \int_0^t T_1(t-s)f_1(s)ds + \int_0^t T_1(t-s)B_1u_1(s)ds, \\ \varphi_2(t) = T_2(t)x^0_2 + \int_0^t T_2(t-s)f_2(s)ds + \int_0^t T_2(t-s)B_2u_2(s)ds. \end{cases}$$

On one hand, for any $\mathbf{x} = (x_1, x_2) \in \mathcal{B}_{k_0}$, when $t_1 = 0$, $\quad 0 < t_2 \leq \delta_0$ and δ_0 is small enough, we obtain that for $i = 1, 2$

$$\begin{aligned} \|\varphi_i(t_2) - \varphi_i(t_1)\|_{V_i} &\leq \|T_i(t_2)x^0_i - x^0_i\|_{V_i} + \|\int_0^{t_2} T_i(t_2 - s)(B_iu_i(s) + f_i(s))ds\|_{V_i} \\ &\leq \|T_i(t_2)x^0_i - x^0_i\|_{V_i} + M_i(\|\phi_i\|_{L^2(I,\mathbf{R}^+)} + \|B_i\|\|u_i\|_{L^2(I,U_i)})\delta_0{}^{\frac{1}{2}} + M_iL_ik_0\delta_0. \end{aligned}$$

Then, we can easily see that $\|\varphi_i(t_2) - \varphi_i(t_1)\|_{V_i}$ tends to zero independently of $\mathbf{x} \in \mathcal{B}_{k_0}$ as $\delta_0 \to 0$.

On the other hand, for any $\mathbf{x} = (x_1, x_2) \in \mathcal{B}_{k_0}$ and $\frac{\delta_0}{2} \le t_1 < t_2 \le T$, we obtain

$$\begin{aligned}
\|\varphi_i(t_2) - \varphi_i(t_1)\|_{V_i} &\le \|T_i(t_2)x_i^0 - T_i(t_1)x_i^0\|_{V_i} + \|\int_0^{t_2} T_i(t_2 - s)(B_i u_i(s) + f_i(s))ds \\
&\quad - \int_0^{t_1} T_i(t_1 - s)(B_i u_i(s) + f_i(s))ds\|_{V_i} \\
&\le \|T_i(t_2)x_i^0 - T_i(t_1)x_i^0\|_{V_i} + \|\int_{t_1}^{t_2} T_i(t_2 - s)(B_i u_i(s) + f_i(s))ds\|_{V_i} \\
&\quad + \|\int_0^{t_1} [T_i(t_2 - s) - T_i(t_1 - s)](B_i u_i(s) + f_i(s))ds\|_{V_i} \\
&:= Q_{i,1} + Q_{i,2} + Q_{i,3}.
\end{aligned}$$

By (HJ3), we have $Q_{i,2} \le M_i(\|\phi_i\|_{L^2(I,\mathbf{R}^+)} + \|B_i\|\|u_i\|_{L^2(I,U_i)})(t_2 - t_1)^{\frac{1}{2}} + M_i L_i k_0 (t_2 - t_1)$. For $t_1 \ge \frac{\delta_0}{2} > 0$ and $\delta > 0$ being small enough, we obtain

$$\begin{aligned}
Q_{i,3} &\le [\|\int_0^{t_1-\delta} [T_i(t_2 - s) - T_i(t_1 - s)](B_i u_i(s) + f_i(s))ds\|_{V_i} + \|\int_{t_1-\delta}^{t_1} [T_i(t_2 - s) - T_i(t_1 - s)](B_i u_i(s) + f_i(s))ds\|_{V_i}] \\
&\le ((\|\phi_i\|_{L^2(I,\mathbf{R}^+)} + \|B_i\|\|u_i\|_{L^2(I,U_i)})(t_1 - \delta)^{\frac{1}{2}} + L_i k_0 (t_1 - \delta)) \times \sup_{s \in [0, t_1 - \delta]} \|T_i(t_2 - s) - T_i(t_1 - s)\| \\
&\quad + 2M_i(\|\phi_i\|_{L^2(I,\mathbf{R}^+)} + \|B_i\|\|u_i\|_{L^2(I,U_i)})\delta^{\frac{1}{2}} + 2M_i L_i k_0 \delta.
\end{aligned}$$

Since (HT) implies the continuity of $T_i(t)$ $(t > 0)$ in t in the uniform operator topology, it is easily seen that $Q_{i,3}$ tends to zero independently of $\mathbf{x} = (x_1, x_2) \in \mathcal{B}_{k_0}$ as $t_2 \to t_1$, $\delta \to 0$. It is clear that $Q_{i,1}$ and $Q_{i,2}$ both tend to zero as $t_2 \to t_1$ does not depend on particular choice of $\mathbf{x} = (x_1, x_2)$. Thus, for $i = 1, 2$, we get that $\|\varphi_i(t_2) - \varphi_i(t_1)\|_{V_i}$ tends to zero independently of $\mathbf{x} = (x_1, x_2) \in \mathcal{B}_{k_0}$ as $\delta_0 \to 0$. Therefore, $\{\mathcal{F}(\mathbf{x}) : \mathbf{x} = (x_1, x_2) \in \mathcal{B}_{k_0}\}$ is equicontinuous.

Step 3. $\mathcal{F}$ is a compact multivalued map.

We show that for any $t \in I$, $\Lambda(t) := \{\varphi(t) : \varphi = (\varphi_1, \varphi_2) \in \mathcal{F}(\mathcal{B}_{k_0})\}$ is relatively compact in $V = V_1 \times V_2$.

Clearly, $\Lambda(0) = \{\mathbf{x}^0\}$ with $\mathbf{x}^0 = (x_1^0, x_2^0)$ is compact. So, it is only necessary to consider $t > 0$. Let $0 < t \le T$ be fixed. For any $\mathbf{x} = (x_1, x_2) \in \mathcal{B}_{k_0}$, $\varphi = (\varphi_1, \varphi_2) \in \mathcal{F}(\mathbf{x})$, there exists $\mathbf{f} = (f_1, f_2) \in \mathcal{P}_J^1(\mathbf{x}) \times \mathcal{P}_J^2(\mathbf{x})$ such that for every $t \in I$,

$$\begin{cases} \varphi_1(t) = T_1(t)x_1^0 + \int_0^t T_1(t - s)f_1(s)ds + \int_0^t T_1(t - s)B_1 u_1(s)ds, \\ \varphi_2(t) = T_2(t)x_2^0 + \int_0^t T_2(t - s)f_2(s)ds + \int_0^t T_2(t - s)B_2 u_2(s)ds. \end{cases}$$

Let $0 < t \le T$ be fixed. For $\forall \varepsilon \in (0, t)$, define $\varphi^\varepsilon(t) = (\varphi_1^\varepsilon(t), \varphi_2^\varepsilon(t))$, where for $i = 1, 2$,

$$\begin{aligned}
\varphi_i^\varepsilon(t) &= T_i(t)x_i^0 + \int_0^{t-\varepsilon} T_i(t - s)(B_i u_i(s) + f_i(s))ds \\
&= T_i(t)x_i^0 + \int_0^{t-\varepsilon} T_i(\varepsilon)T_i(t - s - \varepsilon)(B_i u_i(s) + f_i(s))ds \\
&= T_i(\varepsilon)T_i(t - \varepsilon)x_i^0 + T_i(\varepsilon)\int_0^{t-\varepsilon} T_i(t - s - \varepsilon)(B_i u_i(s) + f_i(s))ds \\
&:= T_i(\varepsilon)z_i(t, \varepsilon).
\end{aligned}$$

where $\mathbf{x} = (x_1, x_2) \in \mathcal{B}_{k_0}$ and $\mathbf{f} = (f_1, f_2) \in \mathcal{P}_J^1(\mathbf{x}) \times \mathcal{P}_J^2(\mathbf{x})$. For $i = 1, 2$, from the boundedness of $\int_0^{t-\varepsilon} T_i(t - s - \varepsilon)(B_i u_i(s) + f_i(s))ds$ and the compactness of $T_i(t)$ $(t > 0)$, we obtain that the set $\{\varphi_i^\varepsilon(t) : \varphi \in \mathcal{F}(\mathcal{B}_{k_0})\}$ is relatively compact in V_i for each $\varepsilon \in (0,t)$. So, it follows that $\Lambda_\varepsilon(t) := \{\varphi^\varepsilon(t) : \varphi \in \mathcal{F}(\mathcal{B}_{k_0})\}$ is relatively compact in $V = V_1 \times V_2$ for each $\varepsilon \in (0,t)$. Moreover, for every $\varphi = (\varphi_1, \varphi_2) \in \mathcal{F}(\mathbf{x})$ with $\mathbf{x} = (x_1, x_2)$, we have for $i = 1, 2$

$$\begin{aligned}||\varphi_i(t) - \varphi_i^\varepsilon(t)||_{V_i} &= ||\int_{t-\varepsilon}^t T_i(t-s)(B_iu_i(s) + f_i(s))ds||_{V_i}\\ &\le M_i(||\phi_i||_{L^2(I,\mathbf{R}^+)} + ||B_i||||u_i||_{L^2(I,U_i)})\varepsilon^{\frac{1}{2}} + M_iL_ik_0\varepsilon,\end{aligned}$$

which implies that for $i = 1, 2$, the set $\{\varphi_i(t) : \varphi \in \mathcal{F}(\mathcal{B}_{k_0})\}$ $(t > 0)$ is totally bounded, i.e., relatively compact in V_i. So, it follows that $\Lambda(t)$ $(t > 0)$ is totally bounded, i.e., relatively compact in $V = V_1 \times V_2$. Therefore, from Steps 1-3, $\{\mathcal{F}(\mathbf{x}) : \mathbf{x} = (x_1, x_2) \in \mathcal{B}_{k_0}\}$ is relative compact by the generalized Ascoli-Arzela theorem. Thus, $\mathcal{F}$ is a compact multivalued map.

Step 4. $\mathcal{F}$ has a closed graph.

Let $\mathbf{x}^n = (x_1^n, x_2^n) \to \mathbf{x}^* = (x_1^*, x_2^*)$ in C, $\varphi^n = (\varphi_1^n, \varphi_2^n) \in \mathcal{F}(\mathbf{x}^n)$ and $\varphi^n \to \varphi^* = (\varphi_1^*, \varphi_2^*)$ in C. We will show that $\varphi^* \in \mathcal{F}(\mathbf{x}^*)$. Indeed, $\varphi^n \in \mathcal{F}(\mathbf{x}^n)$ means that there exists a $\mathbf{f}^n = (f_1^n, f_2^n) \in \mathcal{P}_J^1(\mathbf{x}^n) \times \mathcal{P}_J^2(\mathbf{x}^n)$ such that for each $t \in I$,

$$\begin{cases}\varphi_1^n(t) = T_1(t)x_1^0 + \int_0^t T_1(t-s)(B_1u_1(s) + f_1^n(s))ds,\\ \varphi_2^n(t) = T_2(t)x_2^0 + \int_0^t T_2(t-s)(B_2u_2(s) + f_2^n(s))ds.\end{cases}$$

For $i = 1, 2$, from (HJ3) it is not difficult to show that $\{f_i^n\}_{n\ge1} \subseteq L^2(I, V_i)$ is bounded, and hence we may assume that $f_i^n \rightharpoonup f_i^*$ for some $f_i^* \in L^2(I, V_i)$. Define the continuous linear operator $G_i : L^2(I, V_i) \to C(I, V_i)$ as follows $(G_if_i)(\cdot) = \int_0^\cdot T_i(\cdot - s)f_i(s)ds$, for $f_i \in L^2(I, V_i)$. Since $\mathbf{x}^n \to \mathbf{x}^*$ in C with $\mathbf{x}^n = (x_1^n, x_2^n)$ and $\mathbf{x}^* = (x_1^*, x_2^*)$, it follows from Lemmas 3.7 and 3.10 that

$$\begin{cases}\varphi_1^*(t) = T_1(t)x_1^0 + \int_0^t T_1(t-s)(B_1u_1(s) + f_1^*(s))ds,\\ \varphi_2^*(t) = T_2(t)x_2^0 + \int_0^t T_2(t-s)(B_2u_2(s) + f_2^*(s))ds.\end{cases}$$

and $\mathbf{f}^* := (f_1^*, f_2^*) \in \mathcal{P}_J^1(\mathbf{x}^*) \times \mathcal{P}_J^2(\mathbf{x}^*)$, i.e., $\mathcal{F}$ has a closed graph. Therefore, since $\mathcal{F}$ takes compact values, from Lemma 2.1 we deduce that $\mathcal{F}$ is u.s.c.

Step 5. According to Theorem 2.5, it is sufficient to show that there exists an open set $D \subset C$ such that there is no $\mathbf{x} \in \partial D$ satisfying $\mathbf{x} \in \lambda\mathcal{F}(\mathbf{x})$ for $\forall\lambda \in (0, 1)$. Indeed, let $\mathbf{x} = (x_1, x_2) \in \lambda\mathcal{F}(\mathbf{x})$ for some $\lambda \in (0, 1)$. Then, there exists $\mathbf{f} = (f_1, f_2) \in \mathcal{P}_J^1(\mathbf{x}) \times \mathcal{P}_J^2(\mathbf{x})$ such that for each $t \in I$,

$$\begin{cases}x_1(t) = \lambda T_1(t)x_1^0 + \lambda \int_0^t T_1(t-s)(B_1u_1(s) + f_1(s))ds,\\ x_2(t) = \lambda T_2(t)x_2^0 + \lambda \int_0^t T_2(t-s)(B_2u_2(s) + f_2(s))ds.\end{cases}$$

Then by the assumptions, we get that for $i = 1, 2$ and each $t \in I$

$$\begin{aligned}||x_i(t)||_{V_i} &\le ||T_i(t)x_i^0||_{V_i} + ||\int_0^t T_i(t-s)f_i(s)ds||_{V_i} + ||\int_0^t T_i(t-s)B_iu_i(s)ds||_{V_i}\\ &\le M_i||x_i^0||_{V_i} + M_i\int_0^t (\phi_i(s) + L_i||x_i(s)||_{V_i} + ||B_i||||u_i(s)||_{U_i})ds\\ &\le \rho_i + M_iL_i\int_0^t ||x_i(s)||_{V_i}ds,\end{aligned}$$

where $\rho_i = M_i[||x_i^0||_{V_i} + (||\phi_i||_{L^2(I,\mathbf{R}^+)} + ||B_i||||u_i||_{L^2(I,U_i)})T^{\frac{1}{2}}]$. It follows from the standard Gronwall inequality [?] that for $i = 1, 2$ and each $t \in I$ $||x_i(t)||_{V_i} \le \rho_ie^{M_iL_iT}$. Hence,

$$\begin{aligned}||\mathbf{x}||_C &= ||x_1||_{C(I,V_1)} + ||x_2||_{C(I,V_2)}\\ &= \sup_{t\in I}||x_1(t)||_{V_1} + \sup_{t\in I}||x_2(t)||_{V_2}\\ &\le \rho_1e^{M_1L_1T} + \rho_2e^{M_2L_2T} := \ell,\end{aligned}$$

Set $D = \{\mathbf{x} = (x_1, x_2) \in C : \|\mathbf{x}\|_C < \ell + 1\}$. Clearly, D is an open subset of C, $\mathcal{F} : \overline{D} \to K_v(C)$ is u.s.c. and compact. From the choice of D, there is no $\mathbf{x} \in \partial D$ satisfying $\mathbf{x} \in \lambda \mathcal{F}(\mathbf{x})$ for some $\lambda \in (0, 1)$. Therefore, by Theorem 2.5 we deduce that $\mathcal{F}$ has a fixed point $\mathbf{x}^*$ in C. Consequently, the system (5) has at least one mild solution in C. The proof is completed. □

Now, we give the following definition.

Definition 3.12. *A pair* $(\mathbf{x}, \mathbf{u})$ *is said to be feasible if* $(\mathbf{x}, \mathbf{u})$ *satisfies system* (6) *for* $t \in I$.

For convenience, we denote $U[0, T] = \{\mathbf{u} : [0, T] \to U \mid \mathbf{u}(\cdot) \text{ is measurable}\}$ where $U = U_1 \times U_2$ is the reflexive Banach space endowed by the norm $\|\mathbf{u}\|_U = \|u_1\|_{U_1} + \|u_2\|_{U_2}$ for all $\mathbf{u} = (u_1, u_2) \in U = U_1 \times U_2$, $V[0, T] = \{(\mathbf{x}, \mathbf{u}) \in C \times U[0, T] \mid (\mathbf{x}, \mathbf{u}) \text{ is feasible}\}$.

Now, we study the existence result of feasible pairs for system (6). We assume that for $i = 1, 2$, the feedback multimap $\mathcal{U}_i : I \times V_1 \times V_2 \to P(U_i)$ satisfies the following conditions:

(U1) there exist a function $\phi_i^1 \in L^2(I, \mathbf{R}^+)$ and a constant $L_i^1 > 0$, such that

$$\|\mathcal{U}_i(t, x_1, x_2)\| = \sup_{z_i \in \mathcal{U}_i(t, x_1, x_2)} \|z_i\|_{U_i} \leq \phi_i^1(t) + L_i^1 \|x_i\|_{V_i}$$

for all $(t, x_1, x_2) \in I \times V_1 \times V_2$;

(U2) for a.e. $t \in I$, $\mathbf{x} = (x_1, x_2) \in V = V_1 \times V_2$, the set $\mathcal{U}_i(t, \mathbf{x})$ satisfies the following

$$\bigcap_{\delta > 0} \overline{\mathrm{co}}\, \mathcal{U}_i(O_\delta(t, \mathbf{x})) = \mathcal{U}_i(t, \mathbf{x}).$$

Remark 3.13. *([14]). Let* $i \in \{1, 2\}$. *Then by Lemma 2.1, condition (U2) is fulfilled if* $\mathcal{U}_i$ *is u.s.c. with convex and closed values.*

Now, we are in a position to present the main result of this section.

Theorem 3.14. *If conditions (HT), (HJ1)-(HJ4) and (U1)-(U2) are satisfied, then the set* $V[0, T]$ *is nonempty.*

Proof. For any integer $k > 0$, let $t_j = j\frac{T}{k}$, $0 \leq j \leq k - 1$. We define $\mathbf{u}^k = (u_1^k, u_2^k)$ where $i = 1, 2$,

$$u_i^k(t) = \sum_{j=0}^{k-1} u_{i,j} \chi_{[t_j, t_{j+1})}(t), \; t \in I,$$

where $\chi_{[t_j, t_{j+1})}$ is the character function of interval $[t_j, t_{j+1})$. The sequence $\{u_{i,j}\}_{j=0}^{k-1}$ is constructed as follows.

Firstly, for $i = 1, 2$, we take $u_{i,0} \in \mathcal{U}_i(0, \mathbf{x}^0)$ with $\mathbf{x}^0 = (x_1^0, x_2^0)$. By Theorem 3.11, there exists $\mathbf{x}^k(\cdot) = (x_1^k(\cdot), x_2^k(\cdot))$ given by

$$\begin{cases} x_1^k(t) = T_1(t)x_1^0 + \int_0^t T_1(t - s)(B_1 u_{1,0}(s) + f_1^k(s))ds, \; t \in [0, \frac{T}{k}], \\ x_2^k(t) = T_2(t)x_2^0 + \int_0^t T_2(t - s)(B_2 u_{2,0}(s) + f_2^k(s))ds, \; t \in [0, \frac{T}{k}]. \end{cases}$$

where $\mathbf{f}^k = (f_1^k, f_2^k) \in \mathcal{P}_J^1(\mathbf{x}^k) \times \mathcal{P}_J^2(\mathbf{x}^k)$. Then take $u_{i,1} \in \mathcal{U}_i(\frac{T}{k}, \mathbf{x}^k(\frac{T}{k}))$ for $i = 1, 2$. We can repeat this procedure to obtain $\mathbf{x}^k = (x_1^k, x_2^k)$ on $[\frac{T}{k}, \frac{2T}{k}]$, etc. By induction, we end up with the following: for $i = 1, 2$,

$$\begin{cases} x_i^k(t) = T_i(t)x_i^0 + \int_0^t T_i(t - s)(B_i u_i^k(s) + f_i^k(s))ds, \; t \in I, \\ u_i^k(t) \in \mathcal{U}_i(\frac{jT}{k}, \mathbf{x}^k(\frac{jT}{k})), \; t \in [\frac{jT}{k}, \frac{(j+1)T}{k}), \; 0 \leq j \leq k - 1, \end{cases}$$

where $\mathbf{f}^k = (f_1^k, f_2^k) \in \mathcal{P}_J^1(\mathbf{x}^k) \times \mathcal{P}_J^2(\mathbf{x}^k)$. From the proof of Theorem 3.11, it is easy to prove that for $i = 1, 2$, there exists $r_i^0 > 0$ such that $\|x_i^k\|_{C(I, V_i)} \leq r_i^0$. Moreover, it follows from (HJ3) and (U1) that for $i = 1, 2$ there exist $r_i^1, r_i^2 > 0$ such that $\|u_i^k(\cdot)\|_{L^2(I,U_i)} \leq r_i^1,\ \|f_i^k(\cdot)\|_{L^2(I,V_i)} \leq r_i^2$. Therefore, there are subsequences of $\{\mathbf{u}^k(\cdot)\}$ and $\{\mathbf{f}^k(\cdot)\}$, denoted by $\{\mathbf{u}^k(\cdot)\}$ and $\{\mathbf{f}^k(\cdot)\}$ again, such that for $i = 1, 2$,

$$u_i^k(\cdot) \rightharpoonup u_i^*(\cdot) \text{ in } L^2(I, U_i),\ f_i^k(\cdot) \rightharpoonup f_i^*(\cdot) \text{ in } L^2(I, V_i). \tag{14}$$

From (HT), by Lemma 3.10 we have that for $i = 1, 2$ and any $t \in I$

$$\int_0^t T_i(t-s)(B_i u_i^k(s) + f_i^k(s))ds \to \int_0^t T_i(t-s)(B_i u_i^*(s) + f_i^*(s))ds.$$

For $i = 1, 2$, we let $x_i^*(t) = T_i(t)x_i^0 + \int_0^t T_i(t-s)(B_i u_i^*(s) + f_i^*(s))ds,\ t \in I$. Then, for $i = 1, 2$, $x_i^k(t) \to x_i^*(t)$, uniformly in $t \in I$, i.e., $x_i^k(\cdot) \to x_i^*(\cdot)$ in $C(I, V_i)$. So it follows that

$$\mathbf{x}^k(\cdot) \to \mathbf{x}^*(\cdot) \text{ in } C.$$

Hence, for any $\delta > 0$, there exists an integer $k_0 > 0$ such that

$$\mathbf{x}^k(t) \in O_\delta(\mathbf{x}^*(t)),\ t \in I,\ k \geq k_0. \tag{15}$$

On the other hand, by the definition of $\mathbf{u}^k(\cdot)$ for k large enough, we have

$$u_i^k(t) \in \mathcal{U}_i(t_j, \mathbf{x}^k(t_j)) \subset \mathcal{U}_i(O_\delta(t, \mathbf{x}^*(t))),\ \forall t \in [\frac{jT}{k}, \frac{(j+1)T}{k}),\ 0 \leq j \leq k-1. \tag{16}$$

Secondly, by (14) and Mazur Theorem [16], let $a_{m,l}, b_{m,l} \geq 0$ and $\sum_{m\geq1} a_{m,l} = \sum_{m\geq1} b_{m,l} = 1$ such that for $i = 1, 2$,

$$\phi_i^l(\cdot) = \sum_{m\geq1} a_{m,l} u_i^{m+l}(\cdot) \to u_i^*(\cdot) \text{ in } L^2(I, U_i),\ \psi_i^l(\cdot) = \sum_{m\geq1} b_{m,l} f_i^{m+l}(\cdot) \to f_i^*(\cdot) \text{ in } L^2(I, V_i).$$

Then, there are subsequences of $\{\phi^l\}$ and $\{\psi^l\}$ with $\phi^l = (\phi_1^l, \phi_2^l)$ and $\psi^l = (\psi_1^l, \psi_2^l)$, denoted by $\{\phi^l\}$ and $\{\psi^l\}$ again, such that for $i = 1, 2$, $\phi_i^l(t) \to u_i^*(t)$ in U_i, $\psi_i^l(t) \to f_i^*(t)$ in V_i, a.e. $t \in I$. Hence, from (15) and (16), for l large enough, $\phi_i^l(t) \in \mathrm{co}\mathcal{U}_i(O_\delta(t, \mathbf{x}^*(t))),\ \psi_i^l(t) \in \mathrm{co}\partial_i J(t, O_\delta(\mathbf{x}^*(t))),$ a.e. $t \in I$. Thus, for any $\delta > 0$,

$$u_i^*(t) \in \overline{\mathrm{co}}\mathcal{U}_i(O_\delta(t, \mathbf{x}^*(t))),\ f_i^*(t) \in \overline{\mathrm{co}}\partial_i J(t, O_\delta(\mathbf{x}^*(t))), \text{ a.e. } t \in I.$$

From (U2) and Lemma 3.8, we have $u_i^*(t) \in \mathcal{U}_i(t, \mathbf{x}^*(t)),\ f_i^*(t) \in \partial_i J(t, \mathbf{x}^*(t)),$ a.e. $t \in I$. Therefore, $(\mathbf{x}^*, \mathbf{u}^*)$ is a feasible pair on I. The proof is completed. □

Remark 3.15. *It is worth pointing out that the system* (5) *of semilinear inclusions and the feedback control system* (6) *reduce to the semilinear inclusion (2.1) and the feedback control problem (2.2) in [14], respectively, via choosing appropriately the operators $A_i, B_i, J, \mathcal{U}_i$ and u_i, x_i for $i = 1, 2$. By introducing the operators $\mathcal{P}_J^i : C = C(I, V_1) \times C(I, V_2) \to P(L^2(I, V_i)), i = 1, 2$ and using Theorem 2.5 for the existence of fixed points of u.s.c. and compact multimaps, we prove Theorem 3.11 for the existence of mild solutions of system* (5)*, which generalizes and extends Theorem 3.5 in [14] from the semilinear inclusion (2.1) in [14] to the system* (5) *of semilinear inclusions. By using Theorem 3.11 and the conditions (U1)-(U2) on the feedback multimaps $\mathcal{U}_i : I \times V_1 \times V_2 \to P(U_i), i = 1, 2$, we establish Theorem 3.14 for the existence of feasible pairs of system* (6)*, which generalizes and extends Theorem 3.8 in [14] from the feedback control problem (2.2) in [14] to the feedback control system* (6)*.*

4. Existence of Optimal State-Control Pairs

In this section, we consider the optimal control system stated as follows.
System (φ): find a pair $(\mathbf{x}^*, \mathbf{u}^*) \in V[0, T]$ such that

$$\begin{cases} \varphi_1(x_1^*, u_1^*) \leq \varphi_1(x_1, u_1), \\ \varphi_2(x_2^*, u_2^*) \leq \varphi_2(x_2, u_2), \end{cases}$$

for all $(\mathbf{x},\mathbf{u}) \in V[0,T]$, where $\varphi_i(x_i,u_i) = \int_0^T f_i^0(t,x_i(t),u_i(t))dt$ for $i=1,2$.

We make the following assumptions on $\mathbf{f}^0 = (f_1^0, f_2^0)$:

($\mathbf{f}^0$1) the functional $f_i^0 : I \times V_i \times U_i \to \mathbf{R} \cup \{\pm\infty\}$ is Borel measurable in (t,x_i,u_i) for $i=1,2$;

($\mathbf{f}^0$2) $f_i^0(t,\cdot,\cdot)$ is lower semicontinuous on $V_i \times U_i$ a.e. $t \in I$ for $i=1,2$ (i.e., for all $x_i \in V_i,\ u_i \in U_i,\ \{x_i^n\} \subset V_i,\ \{u_i^n\} \subset U_i$ such that $x_i^n \to x_i$ in V_i and $u_i^n \to u_i$ in U_i, we have $\liminf_{n\to\infty} f_i^0(t,x_i^n,u_i^n) \geq f_i^0(t,x_i,u_i)$) and there exists a constant $M_{i,1} > 0$ such that $f_i^0(t,x_i,u_i) \geq -M_{i,1},\ (t,x_i,u_i) \in I \times V_i \times U_i$.

For any $(t,\mathbf{x}) \in I \times V$ with $\mathbf{x} = (x_1,x_2) \in V = V_1 \times V_2$, we set the set

$$\varepsilon_i(t,\mathbf{x}) = \{(z_i^0,z_i^1,z_i^2) \in \mathbf{R} \times V_i \times U_i\ z_i^0 \geq f_i^0(t,x_i,z_i^2),\ z_i^1 \in \partial_i J(t,\mathbf{x}),\ z_i^2 \in \mathcal{U}_i(t,\mathbf{x})\}.$$

To obtain the existence result of optimal state-control pairs for system (φ), we assume that:

(V_ε): for a.e. $t \in I$, the map $\varepsilon_i(t,\cdot,\cdot) : V_1 \times V_2 \to P(\mathbf{R} \times V_i \times U_i)$ has the Cesari property, i.e.,

$$\bigcap_{\delta>0} \overline{\mathrm{co}}\,\varepsilon_i(t,O_\delta(\mathbf{x})) = \varepsilon_i(t,\mathbf{x}),\ \forall \mathbf{x} \in V.$$

Theorem 4.1. *If conditions (HT), (HJ1)-(HJ4), (U1)-(U2), ($\mathbf{f}^0$1)-($\mathbf{f}^0$2), (V_ε) are satisfied, then system (φ) admits at least one optimal state-control pair.*

Proof. For $i=1,2$, without considering the situation $\inf\{\varphi_i(x_i,u_i)\ (\mathbf{x},\mathbf{u}) \in V[0,T]\} = +\infty$, we assume that $\inf\{\varphi_i(x_i,u_i)\ (\mathbf{x},\mathbf{u}) \in V[0,T]\} = m_i < +\infty$. By ($\mathbf{f}^0$2), we have $\varphi_i(x_i,u_i) \geq m_i \geq -M_{i,1}T > -\infty$ for $i=1,2$. Then there exists a sequence $\{(\mathbf{x}^n,\mathbf{u}^n)\}_{n\geq1} \subset V[0,T]$ such that for $i=1,2$

$$\varphi_i(x_i^n,u_i^n) \to m_i.$$

From the proof of Theorem 3.14, without loss of generality, we obtain that for $i=1,2$

$$x_i^n(\cdot) \to x_i^*(\cdot) \text{ in } C(I,V_i),$$

and

$$u_i^n(\cdot) \rightharpoonup u_i^*(\cdot) \text{ in } L^2(I,U_i),\ f_i^n(\cdot) \rightharpoonup f_i^*(\cdot) \text{ in } L^2(I,V_i),$$

where $x_i^*(t) = T_i(t)x_i^0 + \int_0^t T_i(t-s)(B_iu_i^*(s) + f_i^*(s))ds,\ t \in I$.

By Mazur Theorem again, let $a_{m,l}, b_{m,l} \geq 0$ and $\sum_{m\geq1} a_{m,l} = \sum_{m\geq1} b_{m,l} = 1$ such that for $i=1,2$,

$$\phi_i^l(\cdot) = \sum_{m\geq1} a_{m,l}u_i^{m+l}(\cdot) \to u_i^*(\cdot) \text{ in } L^2(I,U_i),\ \psi_i^l(\cdot) = \sum_{m\geq1} b_{m,l}f_i^{m+l}(\cdot) \to f_i^*(\cdot) \text{ in } L^2(I,V_i).$$

For $i=1,2$, we let $\bar{\psi}_i^l(\cdot) = \sum_{k\geq1} b_{k,l}f_i^0(\cdot,x_i^{k+l}(\cdot),u_i^{k+l}(\cdot))$, and $\bar{f}_i^0(t) = \underline{\lim}_{l\to+\infty}\bar{\psi}_i^l(t) \leq -M_{i,1}$, a.e. $t \in I$. For any $\delta > 0$ and l large enough, from ($\mathbf{f}^0$2) we have for $i=1,2$,

$$(\bar{\psi}_i^l(t), \psi_i^l(t), \phi_i^l(t)) \in \varepsilon_i(t,O_\delta(\mathbf{x}^*(t))),\ \text{a.e. } t \in I,$$

with $\mathbf{x}^* = (x_1^*,x_2^*)$. Then for $i=1,2$, $(\bar{f}_i^0(t), f_i^*(t), u_i^*(t)) \in \overline{\mathrm{co}}\,\varepsilon_i(t,O_\delta(\mathbf{x}^*(t))),$ a.e. $t \in I$. From (V_ε), we have for $i=1,2$,

$$(\bar{f}_i^0(t), f_i^*(t), u_i^*(t)) \in \varepsilon_i(t,\mathbf{x}^*(t)),\ \text{a.e. } t \in I,$$

i.e.,

$$\begin{cases} \bar{f}_i^0(t) \geq f_i^0(t,x_i^*(t),u_i^*(t)),\ t \in I, \\ f_i^*(t) \in \partial_i J(t,\mathbf{x}^*(t)),\ t \in I, \\ u_i^*(t) \in \mathcal{U}_i(t,\mathbf{x}^*(t)),\ t \in I. \end{cases}$$

Therefore, $(\mathbf{x}^*, \mathbf{u}^*) \in V[0,T]$. By Fatou's Lemma, we obtain that for $i = 1, 2$,

$$\begin{aligned}\int_0^T \bar{f}_i^0(t)dt &= \int_0^T \underline{\lim}_{l\to+\infty} \bar{\psi}_i^l(t)dt \le \underline{\lim}_{l\to+\infty} \int_0^T \bar{\psi}_i^l(t)dt \\ &= \underline{\lim}_{l\to+\infty} \int_0^T \sum_{k\ge1} b_{k,l} f_i^0(t, x_i^{k+l}(t), u_i^{k+l}(t))dt \\ &= \underline{\lim}_{l\to+\infty} \sum_{k\ge1} b_{k,l} \int_0^T f_i^0(t, x_i^{k+l}(t), u_i^{k+l}(t))dt \\ &= \sum_{k\ge1} b_{k,l} \underline{\lim}_{l\to+\infty} \int_0^T f_i^0(t, x_i^{k+l}(t), u_i^{k+l}(t))dt \\ &= m_i.\end{aligned}$$

Therefore, for $i = 1, 2$,

$$m_i \le \varphi_i(x_i^*, u_i^*) = \int_0^T f_i^0(t, x_i^*(t), u_i^*(t))dt \le m_i,$$

i.e.,

$$\int_0^T f_i^0(t, x_i^*(t), u_i^*(t))dt = m_i = \inf_{(\mathbf{x},\mathbf{u})\in V[0,T]} \varphi_i(x_i, u_i)$$

with $\mathbf{x} = (x_1, x_2)$ and $\mathbf{u} = (u_1, u_2)$.

Thus, $(\mathbf{x}^*, \mathbf{u}^*)$ is an optimal state-control pair. The proof is completed. □

Remark 4.2. *It is worth pointing out that the optimal control system (φ) reduce to the optimal control problem (φ) in [14], via choosing appropriately the functions f_i^0, φ_i and u_i, x_i for $i = 1, 2$. By introducing the operators $\varepsilon_i : I \times V_1 \times V_2 \to P(\mathbf{R} \times V_i \times U_i), i = 1, 2$ and using Mazur Theorem and the proof of Theorem 3.14, we prove Theorem 4.1 for the existence of optimal state-control pairs of system (φ), which generalizes and extends Theorem 4.1 in [14] from the optimal control problem (φ) in [14] to the optimal control system (φ).*

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