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Symplectic Multiquadric Quasi-interpolation Approximations of KdV Equation

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Abstract. Radial basis functions quasi-interpolation is very useful tool for the numerical solution of differential equations, since it possesses shape-preserving and high-order approximation properties. Based on multiquadric quasi-interpolations, this study suggests a meshless symplectic procedure for KdV equation. The method has a number of advantages over existing approaches including no need to solve a resultant full matrix, accuracy and ease of implementation. We also present a theoretical framework to show the conservativeness and convergence of the proposed method. As the numerical experiments show, it not only offers a high order accuracy but also has a good property of long-time tracking capability.

1. Introduction

This paper aims to develop a meshless symplectic algorithm for Korteweg-de Vries (KdV) equation

$$u_t + \varepsilon u u_x + \mu u_{xxx} = 0 \tag{1}$$

where ε and μ are positive constants. The equation is usually employed to model soliton waves, which propagate without change of the shape and velocity properties [26]. The nonlinear term uu_x causes the steeping of the waveform, whereas the dispersion term u_{xxx} makes the waveform spread. The KdV type of equations have many real-life applications in science and engineering fields. These equations produce the ion acoustic solitons in plasma physics [5]; they describe a long wave in shallow seas and deep oceans in geophysical fluid dynamics [20, 21]; their strong presence is also exhibited in cluster physics, superdeformed nuclei, fission, thin films, radar and rheology [15, 22], for example. Some explicit exact methods have been developed, such as by [12, 14]. But it is still hard to obtain an analytic formula for KdV equation, since usually analytical solutions are not available. This leads to a practical need to develop numerical procedure to simulate the equation. For instance, the spectral/pseudospectral method by [10, 13]; the local discontinuous Galerkin method by [16, 17]; the exponential finite-difference method by [23].

On the other hand, KdV equation (1) can be treated as an infinite dimensional Hamiltonian system, which possesses a non-canonical Poisson bracket (symplectic structure) (cf. [3, 19, 25]). It is now well known from the development of algorithms for Hamiltonian systems that 'geometric integration' is an important guiding

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principle. Hence, robust numerical algorithms for KdV equation should preserve the symplectic structure. A standard method, to obtain symplectic algorithms for Hamiltonian partial differential equations (PDEs), involves two steps. First, discretization in space transforms Hamiltonian PDEs into a finite-dimensional Hamiltonian ordinary differential equations (ODEs). In the second step, the resulting system of ODEs is integrated by using symplectic integrators in time. In this numerical procedure, the key for success is to ensure that the obtained ODEs system is a finite-dimensional Hamiltonian system. Several spatial discretization approaches can be adopted, for example, finite difference method (FDM) by [3, 7, 8], finite element method (FEM) by [29], Fourier pseudospectral method by [19] and wavelet collocation method by [30]. However, most of those methods depend on a suitable generation of meshes, which is difficult for problems with very complicated and irregular geometries. To develop a highly accurate and flexible symplectic procedure for KdV equation motivates the current work.

It is well known the multiquadric (MQ) method is high-order accurate, flexible with respect to the geometry, computationally efficient, and easy to implement. Multiquadric kernels were proposed by Hardy [11]. Franke designed lots of numerical experiments, among of which multiquadrics performed best [9]. For the meshless collocation method by using multiquadric functions, one is required to solve a large scaled linear system of equations. Moreover, the coefficients matrix is usually very ill-conditioned and the results are sensitive to the shape parameter c [4, 24]. Therefore, multiquadric quasi-interpolation method has caught the attentions of many researchers. Beatson and Powell first presented some quasi-interpolation schemes by using (first-degree) multiquadric [2]. [27] improved these schemes and discussed their approximation order and the shape preserving properties. Lately [18] showed that multiquadric quasi-interpolation can approximate not only the function itself but also its high order derivatives. More generally, Bestson and Dyn gave the exact definition of MQ-B-Splines with generalized multiquadrics [1]. Based on the definition of MQ-B-Splines, Zhang and Wu developed a cubic MQ quasi-interpolation collocating with non-uniformly distributed data [28]. Indeed, the development of multiquadric quasi-interpolation methods as a meshfree numerical algorithm, has been widely used in science and engineering. Beatson even used the multiquadric quasi-interpolation as a computer aided design tool in the film ”The Lord of the Rings III”. More details about multiquadric method can be found in [1, 2, 4, 18, 27, 28].

In this paper, we will construct a meshless symplectic algorithm for KdV equation with the help of multiquadric quasi-interpolations. The method has a number of advantages over existing approaches, including:

(I). The method is highly accurate and flexible. Compared to traditional methods, multiquadric quasi-interpolations often offer a highly accurate approximation to the objective function. Moreover, multiquadric quasi-interpolation is meshfree, it can be used for the problem with the irregularly spaced points;

(II). The method possesses a long-time tracking capability for solving KdV equation. Since the method is symplectic, it can preserve structural properties of the original problem's flow as long as the intermediate problems' flow do;

(III). The method is computationally efficient. Because it does not require solving a resultant full matrix, the ill-conditioning problem arising when using the multiquadric as a global interpolant can be avoided.

The layout of the paper is as follows. Section 2 provides a brief introduction to multiquadric quasi-interpolations. Hamiltonian formulation of KdV equation is recalled in section 3. In this section, we also illustrate a spatial discretization approach by using multiquadric quasi-interpolations, and prove that the resulting semi-discrete equation is associated with a finite-dimensional Hamiltonian system. In section 4, by evolving the semi-discrete system with symplectic integrators, the expected symplectic algorithm can be obtained. In section 5, some numerical examples are given to verify the effectiveness of the method. The last section is dedicated to a brief conclusion.

2. Multiquadric Quasi-interpolations

In this section, a brief overview of multiquadric quasi-interpolations is given, details can be found in [1, 2, 27, 28]. Specifically, consider a points sequence

$$\mathbf{x} = \cdots < x_{j-1} < x_j < x_{j+1} < \cdots, \qquad h \triangleq \max_j (x_j - x_{j-1})$$

where $x_{\pm j} \to \pm\infty$ as $j \to \pm\infty$. The construction of multiquadric B-splines starts with a basic generalized multiquadric of order $2k$ ($2k-1$ degree)

$$\phi(x; 2k) = (x^2 + c^2)^{(2k-1)/2},$$

where c is a shape parameter, and the ϕ function centered at x_j is defined by

$$\phi_{j,2k}(x) = \phi(x - x_j; 2k).$$

Define the $\Psi-$spline $\Psi_{j,2k}$ as the weighted divided difference

$$\Psi_{j,2k} = \frac{x_{j+k} - x_{j-k}}{2}[x - x_{j-k}, x - x_{j-k+1}, \cdots, x - x_{j+k}]\phi(x; 2k).$$

Then multiquadric quasi-interpolations of a function $f : \mathbb{R} \mapsto \mathbb{R}$ on $\mathbf{x}$ are given

$$(\mathcal{L}f) = \sum f(x_j^*)\Psi_{j;2k}(x), \tag{2}$$

where $x_j^* = (x_{j-k+1} + \cdots + x_{j+k-1})/(2k-1)$. If $\mathbf{x}$ is uniformly distributed, we can replace $f(x_j^*)$ by $f(x_j)$, since $x_j^* = x_j$.

2.1. (First-degree) multiquadric quasi-interpolation

Take $k = 1$, we get multiquadric quasi-interpolation of a function $f(x)$

$$(\mathcal{L}f)(x) = \sum f(x_j)\Psi_j(x), \tag{3}$$

where $\Psi_j(x)$ are linear combinations of the first-degree multiquadric functions, that is

$$\Psi_j(x) = \Psi(x - x_j) = \frac{\phi_{j+1}(x) - \phi_j(x)}{2(x_{j+1} - x_j)} - \frac{\phi_j(x) - \phi_{j-1}(x)}{2(x_j - x_{j-1})},$$

with $\phi(x) = \sqrt{x^2 + c^2}$ and $\phi_j = \phi(x - x_j)$. As proved by [27], $(\mathcal{L}f)(x)$ converges to $f(x)$. Besides, as shown in [18], the multiquadric quasi-interpolation method provides an approximant to $m-$th derivatives $f^{(m)}(x)$ with an approximation order $O(h^{\frac{2}{m+1}})$, provided that $c = O(h^{\frac{1}{m+1}})$.

2.2. Cubic multiquadric quasi-interpolation

Take $k = 2$ and $x_j^* = (x_{j-1} + x_j + x_{j+1})/3$, cubic multiquadric quasi-interpolation of a function $f(x)$ can be obtained

$$(\mathcal{L}_2 f) = \sum f(x_j^*)\Psi_j(x), \tag{4}$$

where

$$\Psi_j(x) = \Psi(x - x_j) = \frac{\psi_{j+1}(x) - \psi_j(x)}{2(x_{j+2} - x_{j-1})} - \frac{\psi_j(x) - \psi_{j-1}(x)}{2(x_{j+1} - x_{j-2})}$$

and $\psi_j(x)$ are the following linear combinations of cubic MQ function, i.e.

$$\psi_j(x) = \psi(x - x_j) = \frac{(\phi_{j+1} - \phi_j)/(x_{j+1} - x_j) - (\phi_j - \phi_{j-1})/(x_j - x_{j-1})}{x_{j+1} - x_{j-1}},$$

with $\phi(x) = \sqrt{(x^2 + c^2)^3}$ and $\phi_j = \phi(x - x_j)$. As discussed in [1, 28], the above quasi-interpolation possesses shape-preserving and high-order approximation properties.

In this study, multiquadric quasi-interpolation is considered as an illustration for spatial discretization of KdV equation, while cubic multiquadric quasi-interpolation is employed in our numerical procedures.

3. Multiquadric Quasi-interpolation for Space Discretization

3.1. Hamiltonian Formulation of KdV Equation

An infinite-dimensional Hamiltonian system can be abstracted as follows: **(a).** a phase space $z \in Z$, **(b).** a Hamiltonian functional $\mathcal{H} : Z \mapsto \mathbb{R}$, and **(c).** a Poisson bracket $\{\mathcal{F}, \mathcal{G}\}$, which has to satisfy the skew-symmetry condition $\{\mathcal{F}, \mathcal{G}\} = -\{\mathcal{G}, \mathcal{F}\}$ and the Jacobi identity

$$\{\mathcal{F}, \{\mathcal{G}, \mathcal{H}\}\} + \{\mathcal{G}, \{\mathcal{H}, \mathcal{F}\}\} + \{\mathcal{H}, \{\mathcal{F}, \mathcal{G}\}\} = 0.$$

The infinite dimensional Hamiltonian thereby goes

$$z_t = \{z, \mathcal{H}\}.$$

KdV equation (1) is a non-trivial application of the Hamiltonian framework. It can be rewritten as

$$u_t = -\partial_x(\frac{\varepsilon}{2}u^2 + \mu u_{xx}) = -\partial_x \frac{\delta\mathcal{H}}{\delta u}. \tag{5}$$

The equation conserves the Hamiltonian functional (refers to the energy)

$$\mathcal{H} = \int \left[\frac{\varepsilon}{6}u^3 - \frac{\mu}{2}u_x^2\right] dx. \tag{6}$$

Since

$$\begin{aligned} \frac{d\mathcal{H}}{dt} &= \int \left[\frac{\varepsilon}{2}u^2 u_t - \mu u_x u_{xt}\right] dx = \int \left[(\frac{\varepsilon}{2}u^2 + \mu u_{xx})u_t\right] dx \\ &= -\int \left[(\frac{\varepsilon}{2}u^2 + \mu u_{xx})\partial_x(\frac{\varepsilon}{2}u^2 + \mu u_{xx})\right] dx = 0 \end{aligned}$$

under appropriate boundary conditions. Where $\frac{\delta\mathcal{H}}{\delta u}$ denotes variational derivative with respect to u, defined by

$$\int \frac{\delta\mathcal{H}}{\delta u} \cdot v dx \triangleq \lim_{\epsilon\to 0} \frac{\mathcal{H}(u+\epsilon v) - \mathcal{H}(u)}{\epsilon} = \int_0^L \left[\frac{\varepsilon}{2}u^2 \cdot v - \mu u_x \cdot v_x\right] dx = \int_0^L \left[(\frac{\varepsilon}{2}u^2 + \mu u_{xx}) \cdot v\right] dx,$$

for variations v with vanishing boundary variation, i.e, $v(0) = v(L) = 0$. This formulation suggests the Poisson bracket

$$\{\mathcal{F}, \mathcal{G}\} = -\int \frac{\delta\mathcal{F}}{\delta u}\frac{\partial}{\partial x}\frac{\delta\mathcal{G}}{\delta u} dx,$$

which indeed satisfies the required condition of skew-symmetry and Jacobi's identity (More details can be found in [3, 19, 25]).

3.2. Discrete Hamiltonian Approach with Multiquadric Quasi-interpolation

A discretization starts with a finite-dimensional approximation $Z_{\Delta x}$ to Z. Further, upon replacing exact integrals by some quadrature formulas, we obtain a finite-dimensional energy approximation $H_{\Delta x}$ and a bracket $\{\cdot,\cdot\}_{\Delta x}$. The finite-dimensional Hamiltonian system can be immediately deduced by

$$\mathbf{z}_t = \{\mathbf{z}, H_{\Delta x}\}_{\Delta x}, \mathbf{z} \in Z_{\Delta x},$$

if the 'numerical' bracket $\{\cdot,\cdot\}_{\Delta x}$ satisfies:

(a).skew-symplectic, $\{F_{\Delta x}, G_{\Delta x}\}_{\Delta x}$=-$\{G_{\Delta x}, F_{\Delta x}\}_{\Delta x}$,

(b). Jacobi identity, $\{F_{\Delta x}, \{G_{\Delta x}, H_{\Delta x}\}_{\Delta x}\}_{\Delta x}$ +$\{G_{\Delta x}, \{H_{\Delta x}, F_{\Delta x}\}_{\Delta x}\}_{\Delta x}$+$\{H_{\Delta x}, \{F_{\Delta x}, G_{\Delta x}\}_{\Delta x}\}_{\Delta x}$=0.

Now, we introduce a spatial discretization approach by using multiquadric quasi-interpolations. To illustrate the basic idea, we confine ourselves to (first-degree) multiquadric quasi-interpolation. One can generalize the discussion to the other multiquadric B-splines in a similar way.

First, we approximate the solution $u(x,t)$ of equation (5) with multiquadric quasi-interpolation,

$$u(x,t) \approx u^*(x,t) = (\mathcal{L}u)(x,t) = \sum_j u(x_j,t)\Psi_j(x).$$

Meanwhile, the first and second derivatives of $u(x,t)$ can be respectively approximated by

$$u_x(x,t) \approx u_x^*(x,t) = \sum_j u(x_j,t)\frac{\partial \Psi_j(x)}{\partial x}$$

and

$$u_{xx}(x,t) \approx u_{xx}^*(x,t) = \sum_j u(x_j,t)\frac{\partial^2 \Psi_j(x)}{\partial x^2}.$$

In addition, notice

$$\begin{aligned}\Psi_j(x) &= \frac{\phi_{j+1}(x)-\phi_j(x)}{2(x_{j+1}-x_j)} - \frac{\phi_j(x)-\phi_{j-1}(x)}{2(x_j-x_{j-1})} \\ &= \frac{x_{j+1}-x_{j-1}}{4}\phi''(x-\xi_j) \approx \Delta x_j \phi''(x-x_j)/2 = \Delta x_j \phi_j''(x)/2,\end{aligned}$$

where $\Delta x_j = \frac{x_{j+1}-x_{j-1}}{2}$ and $\xi_j \in [x_{j-1}, x_{j+1}]$. Therefore, the approximation can be written in the matrix form

$$U \approx \mathbf{\Phi}_2 \mathbf{\Delta} U,$$

and the derivatives approximations

$$U_x \approx \mathbf{\Phi}_3 \mathbf{\Delta} U, \quad U_{xx} \approx \mathbf{\Phi}_4 \mathbf{\Delta} U,$$

where $U = (\cdots, u(x_j,t), \cdots)^T$, $U_x = (\cdots, u_x(x_j,t), \cdots)^T$, $U_{xx} = (\cdots, u_{xx}(x_j,t), \cdots)^T$, $\mathbf{\Delta} = diag[\Delta x_j]$, and

$$\mathbf{\Phi}_2 = \left[\frac{\phi_j''(x_i)}{2}\right]_{i,j}, \qquad \mathbf{\Phi}_3 = \left[\frac{\phi_j^{(3)}(x_i)}{2}\right]_{i,j}, \qquad \mathbf{\Phi}_4 = \left[\frac{\phi_j^{(4)}(x_i)}{2}\right]_{i,j}.$$

Accordingly, Poisson mapping '∂_x' can be discretized as $\mathbf{\Phi}_3\mathbf{\Delta}$, while '$\partial_{xx}$' be discretized as $\mathbf{\Phi}_4\mathbf{\Delta}$. The equation (5) thereby converted into a semi-discrete system

$$U_t = -\mathbf{\Phi}_3(\varepsilon/2 \cdot \mathbf{\Delta} U^2 + \mu \cdot \mathbf{\Delta}\mathbf{\Phi}_4\mathbf{\Delta} U). \tag{7}$$

Moreover, a corresponding approximation to the Hamiltonian functional is introduced by

$$H_{\mathcal{L}} = \frac{\varepsilon}{6}\mathbf{1}^T \mathbf{\Delta} U^3 + \frac{\mu}{2} U^T \mathbf{\Delta}\mathbf{\Phi}_4\mathbf{\Delta} U, \tag{8}$$

where $\mathbf{1} = (\cdots, 1, \cdots)^T$. Somewhat more care is required to discretize the Poisson bracket. A possible choice is

$$\{F_{\mathcal{L}}, G_{\mathcal{L}}\}_{\mathcal{L}} = -\nabla F_{\mathcal{L}}^T \mathbf{\Phi}_3 \nabla G_{\mathcal{L}}$$

where $\nabla F_{\mathcal{L}} = (\cdots, \partial_{u_i} F_{\mathcal{L}}, \cdots)^T$ and $\nabla G_{\mathcal{L}} = (\cdots, \partial_{u_i} G_{\mathcal{L}}, \cdots)^T$. Because

$$\mathbf{\Phi}_3 = \left[\frac{\phi_j^{(3)}(x_i)}{2}\right]_{i,j}$$

is skew-symmetric (since $\frac{\phi^{(3)}(x)}{2} = -3c^2x/2(x^2+c^2)^{5/2}$), then the Jacobi identity is trivially satisfied. Consequently, the discrete Hamiltonian formulation of KdV equation is immediately deduced by

$$U_t = \{U, H_{\mathcal{L}}\}_{\mathcal{L}} = -\mathbf{\Phi}_3(\varepsilon/2 \cdot \mathbf{\Delta}U^2 + \mu \cdot \mathbf{\Delta\Phi}_4\mathbf{\Delta}U),$$

due to

$$\nabla H_{\mathcal{L}} = \varepsilon/2 \cdot \mathbf{\Delta}U^2 + \mu \cdot \mathbf{\Delta\Phi}_4\mathbf{\Delta}U.$$

Theorem 3.1. *In the limit $h \to 0$, we have*

$$\lim_{h\to 0} H_{\mathcal{L}} = \mathcal{H}(u) = \int \left[\frac{\varepsilon}{6}u^3 - \frac{\mu}{2}u_x^2\right] dx,$$

and Poisson bracket

$$\lim_{h\to 0}\{F_{\mathcal{L}}, G_{\mathcal{L}}\}_{\mathcal{L}} = \{\mathcal{F}, \mathcal{G}\} = \int \frac{\delta\mathcal{F}}{\delta u}\frac{\partial}{\partial x}\frac{\delta\mathcal{G}}{\delta u}dx.$$

Proof. According to the approximation $\partial_{xx} \approx \mathbf{\Phi}_4\mathbf{\Delta}$, then the Hamiltonian

$$\begin{aligned} H_{\mathcal{L}} &= \frac{\varepsilon}{6}\mathbf{1}^T\mathbf{\Delta}U^3 + \frac{\mu}{2}U^T\mathbf{\Delta\Phi}_4\mathbf{\Delta}U \approx \frac{\varepsilon}{6}\sum_j u_j^3\Delta x_j + \frac{\mu}{2}\sum_j u_j\Delta x_j\frac{\partial^2 u_j}{\partial x^2} \\ &\to \int\left[\frac{\varepsilon}{6}u^3 + \frac{\mu}{2}uu_{xx}\right]dx = \int\left[\frac{\varepsilon}{6}u^3 - \frac{\mu}{2}u_x^2\right]dx. \end{aligned}$$

In addition, notice the approximation $\partial_x \approx \mathbf{\Phi}_3\mathbf{\Delta}$, it follows that Poisson bracket

$$\{F_{\mathcal{L}}, G_{\mathcal{L}}\}_{\mathcal{L}} = -\nabla F_{\mathcal{L}}^T\mathbf{\Phi}_3\nabla G_{\mathcal{L}} \approx \sum_j \Delta x_j \frac{\delta\mathcal{F}}{\delta u}(x_j)\frac{\partial}{\partial x}\frac{\delta\mathcal{G}}{\delta u}(x_j) \to \int \frac{\delta\mathcal{F}}{\delta u}\frac{\partial}{\partial x}\frac{\delta\mathcal{G}}{\delta u}dx.$$

Indeed, the functional derivative $\frac{\delta\mathcal{F}}{\delta u}$ is defined by

$$\int \frac{\delta\mathcal{F}}{\delta u}\cdot v dx \triangleq \lim_{\epsilon\to 0}\frac{\mathcal{F}(u+\epsilon v) - \mathcal{F}(u)}{\epsilon} \approx \sum_j \frac{\partial}{\partial u_j}F_{\mathcal{L}}\cdot v_j$$

and we obtain

$$\Delta x_j\frac{\delta\mathcal{F}}{\delta u}(x_j) \approx \frac{\partial}{\partial u_j}F_{\mathcal{L}}.$$

□

Remark 3.2. *The spatial discretization approach by multiquadric quasi-interpolation is flexible, since it is simple to implement with the nonuniform knots $\{x_j\}$. The method is computationally efficient. Because $\mathbf{\Phi}_3$ and $\mathbf{\Phi}_4$ can be treated approximately as bounded matrices.*

4. Symplectic Integrator for Time Discretization

The finite-dimensional Hamiltonian system (7) can be discretized in time by many symplectic integrators. Our discussion is only restricted to second-order symplectic Runge-Kutta method (i.e. the Euler-centered scheme), it will be considered as an illustration for describing the convergence. Evolve the equation (7) by Euler-centered scheme, we can get a symplectic algorithm

$$\frac{U^{n+1} - U^n}{\tau} = -\mathbf{\Phi}_3\left[\varepsilon/2 \cdot \mathbf{\Delta}\left(\frac{U^n + U^{n+1}}{2}\right)^2 + \mu \cdot \mathbf{\Delta\Phi}_4\mathbf{\Delta}\frac{U^n + U^{n+1}}{2}\right], \tag{9}$$

where $U^n = \{u(x_j, t_n)\}$, $t_n = t_0 + n\tau$ and $\tau = \Delta t$. Define the truncation error

$$\begin{aligned} R^n &= \left[\frac{U^{n+1} - U^n}{\tau} + \mathbf{\Phi}_3\left(\varepsilon/2 \cdot \Delta(\frac{U^n + U^{n+1}}{2})^2 + \mu \cdot \mathbf{\Delta\Phi}_4\mathbf{\Delta}\frac{U^n + U^{n+1}}{2}\right)\right] \\ &- \left(U_t^{n+1/2} + \varepsilon U^{n+1/2}U_x^{n+1/2} + \mu U_{xxx}^{n+1/2}\right), \end{aligned}$$

and let $\|\cdot\|$ be the L^2 norm, then the following theorem can be obtained.

Theorem 4.1. *Suppose $u(x,t) \in H_0^1(\mathbb{R}) \cap H^2(\mathbb{R})$ for any $t \in [0,T]$, $u(x,t) \in C^4(\mathbb{R})$, $\forall x$. Then the estimate for the truncation error R^n is*

$$\|R^n\| \leq O(\tau^2 + h^\ell),$$

where ℓ is the approximation order to the second derivative of the function by multiquadric quasi-interpolation.

Proof. Based on Taylor expansion, we have

$$U^n + U^{n+1} = 2U^{n+1/2} + O(\tau^2),$$

and

$$\frac{U^{n+1} - U^n}{\tau} = U_t^{n+1/2} + O(\tau^2),$$

then

$$R^n = \left[U_t^{n+1/2} + \varepsilon/2\mathbf{\Phi}_3\mathbf{\Delta}(U^{n+1/2})^2 + \mu\mathbf{\Phi}_3\mathbf{\Delta\Phi}_4\mathbf{\Delta}U^{n+1/2}\right] - (U_t^{n+1/2} + \varepsilon U^{n+1/2}U_x^{n+1/2} + \mu U_{xxx}^{n+1/2}) + O(\tau^2).$$

Notice $\mathbf{\Phi}_3\mathbf{\Delta}$ and $\mathbf{\Phi}_4\mathbf{\Delta}$ are approximants of ∂_x and ∂_{xx} (the approximation order of the latter is ℓ and the former is larger than that). Hence, the truncation error goes as

$$\|R^n\| \leq O(\tau^2 + h^\ell).$$

□

Remark 4.2. *Compared with the classic symplectic algorithm by using FDM which usually possesses the error $O(\tau^2 + h^2)$, the proposed symplectic scheme possesses $O(\tau^2 + h^\ell)$, where ℓ is larger than 2 [27, 28].*

5. Numerical Examples

In this section, we give two examples to describe the efficiency of the multiquadric quasi-interpolation method for KdV equation, especially take cubic multiquadric quasi-interpolation (4) for numerical procedures. In our computations, the following approach is adopted to handle the boundary. For the data points $\{x_j\}_{j=0}^N$, take four extra artificial endpoints, satisfying

$$x_{-2} < x_{-1} < x_0 < x_1 < \cdots < x_N < x_{N+1} < x_{N+2},$$

the radial basis function is chosen to be

$$\phi_j(x) = \begin{cases} (x - x_j)^3, & for \quad -2 \leq j \leq 1, \\ \sqrt{((x - x_j)^2 + c^2)^3}, & for \quad 2 \leq j \leq N-2, \\ (x_j - x)^3, & for \quad N-1 \leq j \leq N+2. \end{cases}$$

Example 5.1. *Consider KdV equation* (1) *with a single solitary solution, where* $\varepsilon = 6$ *and* $\mu = 1$*. The initial condition is given by*

$$u(x,0) = \frac{r}{2} sech^2\left(\frac{\sqrt{r}}{2}x - 7\right), \quad r = 0.5, \tag{10}$$

and the analytic solution is [6]

$$\frac{r}{2} sech^2\left(\frac{\sqrt{r}}{2}(x - rt) - 7\right), \quad r = 0.5.$$

The problem is calculated till $t = 60$ in $x \in [0, 80]$. The root mean square error (RMS-error) and the max error (MAX-error) are defined as

$$\sqrt{\frac{1}{N}\sum_j \left(u_{ture}(x_j, t_n) - u(x_j, t_n)\right)^2}$$

and

$$\max_j |u_{ture}(x_j, t_n) - u(x_j, t_n)|,$$

where N is the number of collocation points. In Table.1 we report the results for the solution at $t = 1$ with $\tau = 0.001$ (in order to investigate the space accuracy of the algorithm, a very small τ is chosen), a shape parameter of $c = 0.3h$, N uniform points in $[0, 80]$. While the graphs of the exact and estimated solution at $t = 1$ with the nonuniform data points $\{x_j : x_j = 40\cos\frac{j\pi}{N} + 40\}_{j=0}^{N}$ ($N = 161$) are shown in Figure.1.

The property of long-time tracking capability can be confirmed by Figure.2, when it is computed up to $t = 60$. The energy

$$\mathcal{H} = \int \left[\frac{\varepsilon}{6}u^3 - \frac{\mu}{2}u_x^2\right] dx,$$

is invariant with respect to time, since the initial value $u(x, 0)$ possesses a fast decay as $x \to \infty$. The relative errors in energy at some time points are shown in Table.2. The error is less than 9×10^{-4} after 60000 time steps, that means the algorithm nearly preserves the energy. The CPU time is less than $60s$ after nearly 60000 time steps, that means the method is effective.

Table.1 The accuracy of the cubic multiquadric quasi-interpolation method collocating with the uniform data points at $t = 1s$

N	h	RMS-error	rate	MAX-error	rate	CPU(s)
41	2.000	4.80×10^{-3}		8.80×10^{-3}		0.5
81	1.000	1.10×10^{-3}	2.1	3.70×10^{-3}	1.3	1.0
121	0.667	4.77×10^{-4}	2.1	1.50×10^{-3}	2.2	1.5
161	0.500	2.37×10^{-4}	2.4	7.20×10^{-4}	2.6	2.0

Table.2 Energy error of the cubic multiquadric quasi-interpolation method collocating with $N = 161$ uniform data points

t	the relative error in energy	CPU(s)
5	1.8×10^{-4}	3
10	2.7×10^{-4}	9
20	4.8×10^{-4}	18
40	6.7×10^{-4}	31
60	8.8×10^{-4}	45

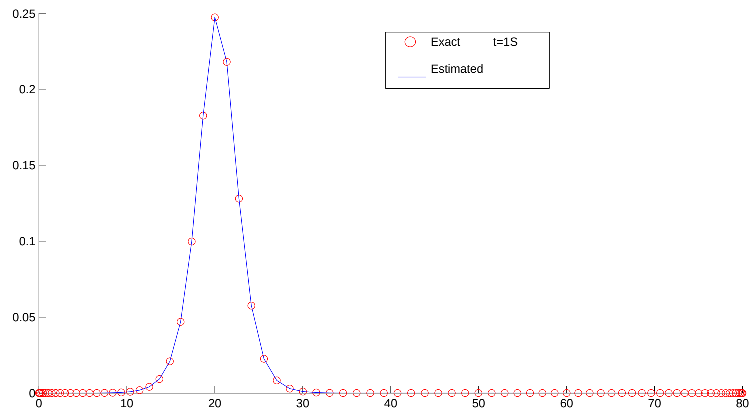

Figure.1 The graph of analytical and estimated solution at $t = 1s$ with the nonuniform data points, a single solitary wave

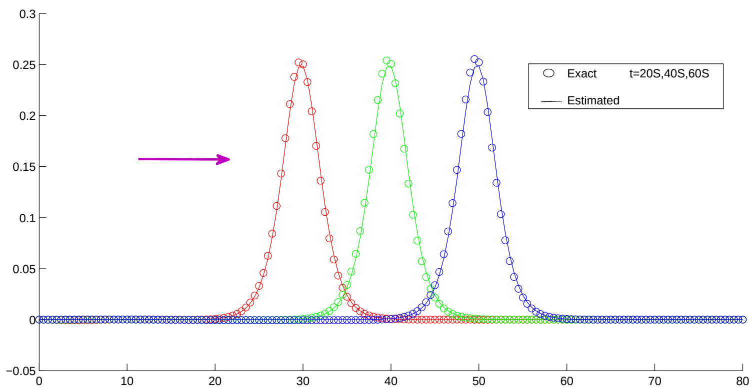

Figure.2 The graph of analytical and estimated solutions with the uniform points at t=20s,40s and 60s, a single solitary wave

Remark 5.2. *As Table.1 show, the errors decrease rapidly when N gets bigger, which verifies Theorem 4.1. As shown in Figure.1, multiquadric quasi-interpolation is easy to implement with the nonuniform knots. Figure.2 implies that the method has a good behavior in long-time simulation.*

Example 5.3. *Consider KdV equation* (1) *with the interaction of two solitary waves, where still* $\varepsilon = 6$ *and* $\mu = 1$. *The analytic solution is given in [6]*

$$u(x,t) = 5\left(\frac{4.5csch^2[1.5(x-9t+14.5)] + 2sech^2(x-4t+12)}{(3coth[1.5(x-9t+14.5)] - 2tanh(x-4t+12))^2}\right).$$

We can take the required initial and boundary functions from the exact solution. The graph of analytical and estimated solution at $t = 1s$ are shown in Figure.3, with $N = 101$ uniform points in $[-20, 0]$ and a shape parameter of $c = 0.5h$.

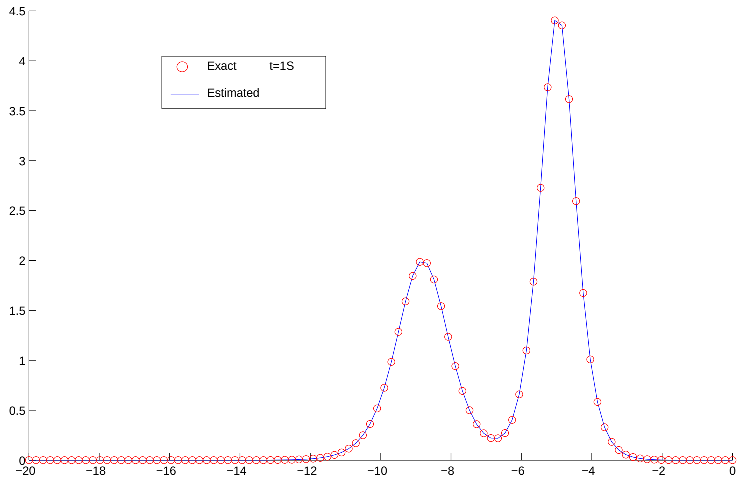

Figure.3 The graph of analytical and estimated solution at $t = 1s$ with the uniform data points, the interaction of two solitary waves

Remark 5.4. *From Example 5.1 and Example 5.3, when deal with no matter a single solitary wave and the interaction of two solitary waves, multiquadric quasi-interpolations method is effective in applications.*

6. Conclusions

Based on multiquadric quasi-interpolations, this paper develops a meshless symplectic algorithm for KdV equation. The method is based on a combination of multiquadric quasi-interpolations method and symplectic integrators. The paper also provides a systematic theoretical framework to show the conservativeness and convergence of the proposed method. Numerical results confirm that the proposed scheme is easy to implement with the nonuniform knots, high-order accurate, computationally efficient and possesses a long-time tracking capability for solving time-dependent KdV equation.

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References

[1] R. Beatson, N. Dyn, Multiquadric B-Splines, Journal of Approximation Theory 87 (1996) 1-24.
[2] R. Beatson, M. Powell, Univariate Multiquadric approximation: Quasi-interpolation to Scattered data, Constr Approx 27 (1992) 275-288.
[3] T. Bridge, S. Reich, Numerical methods for Hamiltonian PDEs, J Phys A: Math Gen 39 (2006) 5287-5320.
[4] A. Cheng, M. Golberg, E. Kansa, G. Zammito, Exponential Convergence and H-c Multiquadric Collocation Method for Partial Differential Equations, Numerical Methods for Partial Differential Equations 19 (2003) 571-594.
[5] G. Das, J. Sarma, A new mathematical approach for finding the solitary waves in dusty plasma, Phys. Plasmas 6 (1999) 4394-4397.
[6] M. Dehghan, A. Shokri, A numerical method for KdV equation using collocation and radial basis functions, Nonlinear Dyn 50 (2007) 111-120.
[7] D. Duncan, Symplectic finite difference approximations of the nonlinear Klein-Gordon equation, SIAM Journal on numerical analysis 34 (1997) 1742-1760.
[8] K. Feng, M. Qin, The symplectic methods for the computation of Hamiltonian equations, Lecture Notes in Mathematics 1297, ed. Zhu You-lan and Gu Ben-yu, Springer Berlin 1987
[9] R. Franke, Scattered Data Interpolation: Test of Some Methods, Math Comp 38 (1982) 181-200.
[10] B. Guo, J. Shen, On spectral approximations using modified Legendre rational functions: Application to the Korteweg-de Vries equation on the half line (Special Issue), Indiana Univ. Math J 50(2001) 181-204.
[11] R. Hardy, Multiquadric equations of topography and other irregular surfaces, Journal of Geophysical Research 76 (1971) 1905-1915.

[12] W. Hereman, P. Banerjee, A. Korpel, J. Assnto, A.V. Immerzeele, A. Meerpoel, Exact solitary wave solutions of nonlinear evolution and wave equations using a direct algebraic method, J.Phys.A (1986) 607-628.

[13] W. Huang, D. Sloan, The pseudospectral method for third-order differential equations, SIAM J. Numer.Anal 29(6) (1992) 1626-1647.

[14] Y. Lei, Z. Fajianj, W. Yinghai, The homogenous balance method, Lax pair, Hirota transformation and a general fifth-order KdV equation, Chaos Solitons Fractals 13 (2002) 337-340.

[15] A. Ludu, J. Draayer, Nonlinear modes of liquid drops as solitary waves, Phys.Rev.Lett 80 (1998) 2125-2128.

[16] H. Ma, W. Sun, A Legendre-Petrov-Galerkin and Chebyshev collocation method for third-order differential equations, SIAM J. Numer.Anal 38(5)(2000) 1425-1438.

[17] H. Ma, W. Sun, Optimal error estimates of the Legendre-Petrov-Galerkin method for the Korteweg-de Vries equation, SIAM J. Numer.Anal 39(4)(2001) 1380-1394.

[18] L. Ma, Z. Wu, Approximation to the $k-th$ derivatives by multiquadric quasi-interpolation method, Journal of Computational and Applied Mathematics 231 (2009) 925-932.

[19] R. McLachlan, Symplectic Integration of Hamiltonian Wave Equations, Numer Math 66 (1994) 465-492.

[20] A. Osborne, The inverse scattering transform: Tools for the nonlinear fourier analysis and filtering of ocean surface waves, Choas Solitons Fractals 5(1995) 2623-2637.

[21] L. Ostrovsky, Y. Stepanyants, Do internal solitons exist in the ocean? Rev.Geophys 27(3)(1989) 293-310.

[22] L. Reatto, D. Galli, What is a ROTON? Int.J.Mod.Phys.B 13(1999) 607-616.

[23] A. Refik Bahadır, Exponential finite-difference method applied to Kortewegñ-de Vries for small times, Appl. Math. Comput 160(3)(2005) 675-682.

[24] S. Rippa, An algorithm for selecting a good value for the parameter c in radial basis function interpolatio, Advances in Computional Mathematics (11) 1999 193-210.

[25] W. Strauss, Nonlinear wave equations, AMS Regional Conference Series no 73, 1989.

[26] A. Wazwaz, Two reliable methods for solving variants of the KdV equation with compact and noncompact structures, Chaos Solitions Fractals 28 (2006) 457-462.

[27] Z. Wu, R. Schaback, Shape preserving properties and convergence of univariate multiquadric quasi-interpolation, Acta Mathematicae Applicatae Sinica 10(1994) 441-446.

[28] W. Zhang, Z. Wu, Shape-preserving MQ-B-Splines Quasi-interpolation. GMAP.2004.1290030 Conference: Geometric Modeling and Processing, 2004. Proceedings

[29] L. Zhen, Y. Bai, Li, K. Wu, Symplectic and multi-symplectic schemes with finite element method, Phys Lett A 314 (2003) 443-455.

[30] H. Zhu, L. Tang, S. Song, Y. Tang, D. Wang, Symplectic wavelet collocation method for Hamiltonian wave equations, Journal of Computational Physics 229(2010) 2550-2572.