Filomat 32:14 (2018), 5123–130
https://doi.org/10.2298/FIL1814123M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Matrix Transformations and Application to Perturbed Problems of Some Sequence Spaces Equations with Operators

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Abstract. Given any sequence $z = (z_n)_{n\geq 1}$ of positive real numbers and any set E of complex sequences, we write E_z for the set of all sequences $y = (y_n)_{n\geq 1}$ such that $y/z = (y_n/z_n)_{n\geq 1} \in E$; in particular, $c_z = s_z^{(c)}$ denotes the set of all sequences y such that y/z converges. Starting with the equation $F_x = F_b$ we deal with some perturbed equation of the form $\mathcal{E} + F_x = F_b$, where $\mathcal{E}$ is a linear space of sequences. In this way we solve the previous equation where $\mathcal{E} = (E_a)_T$ and $(E, F) \in \{(\ell_\infty, c), (c_0, \ell_\infty), (c_0, c), (\ell^p, c), (\ell^p, \ell_\infty), (w_0, \ell_\infty)\}$ with $p \geq 1$, and T is a triangle.

1. Introduction

We write ω for the set of all complex sequences $y = (y_n)_{n\geq 1}$, ℓ_∞, c and c_0 for the sets of all bounded, convergent and null sequences, respectively, also, for $1 \leq p < \infty$,

$$\ell^p = \left\{ y \in \omega : \sum_{n=1}^{\infty} \left| y_n \right|^p < \infty \right\}.$$

If y, $z \in \omega$, then we write $yz = (y_n z_n)_{n\geq 1}$. Let $U = \{y \in \omega : y_n \neq 0\}$ and $U^+ = \{y \in \omega : y_n > 0\}$. We write $z/u = (z_n/u_n)_{n\geq 1}$ for all $z \in \omega$ and all $u \in U$, in particular $1/u = e/u$, where $e = \mathbf{1}$ is the sequence with $e_n = 1$ for all n. Finally, if $a \in U^+$ and E is any subset of ω, then we put

$$E_a = (1/a)^{-1} * E = \{y \in \omega : y/a \in E\}.$$

Let E and F be subsets of ω. In [2], the sets s_a, s_a^0 and $s_a^{(c)}$ were defined for positive sequences a by $(1/a)^{-1} * E$ and $E = \ell_\infty, c_0, c$, respectively. In [3] the sum $E_a + F_b$ and the product $E_a * F_b$ were defined where E, F are any of the symbols s, s^0, or $s^{(c)}$. Then in [6] the solvability was determined of sequences spaces equations inclusion $G_b \subset E_a + F_b$ where E, F, $G \in \left\{s^0, s^{(c)}, s\right\}$ and some applications were given to sequence spaces inclusions with operators. Recall that the spaces w_∞ and w_0 of strongly bounded and summable sequences

2010 *Mathematics Subject Classification*. Primary 40H05; Secondary 46A45
Keywords. BK space, matrix transformations, multiplier of sequence spaces, sequence spaces equations with operator
Received: 16 june 2018; Accepted: 20 September 2018
Communicated by Eberhard Malkowsky
Research supported by the Lebanese University
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are the sets of all y such that $\left(n^{-1}\sum_{k=1}^{n}|y_k|\right)_n$ is bounded and tends to zero respectively. These spaces were studied by Maddox [22] and Malkowsky, Rakočević [21]. In [9, 14] were given some properties of well known operators defined by the sets $W_a = (1/a)^{-1} * w_\infty$ and $W_a^0 = (1/a)^{-1} * w_0$. We are interested in solving special *sequence spaces inclusion equations (SSIE), (resp. sequence spaces equations (SSE)),* which are determined by an inclusion, (resp. identity), for which each term is a *sum* or *a sum of products of sets of the form* $(E_a)_T$ *and* $\left(E_{f(x)}\right)_T$ where f maps U^+ to itself, E is any linear space of sequences and T is a triangle. Some results on (SSE) and (SSIE) were stated in [4–8, 12, 15, 16, 18, 19]. In [6] we dealt with the (SSIE) with operators $E_a + (F_x)_\Delta \subset s_x^{(c)}$ where E and F are any of the sets c_0, c, or s_1. In [15] we determined the set of all positive sequences x for which the *(SSIE)* $\left(s_x^{(c)}\right)_{B(r,s)} \subset \left(s_x^{(c)}\right)_{B(r',s')}$ holds, where r, r', s' and s are real numbers, and $B(r,s)$ is the generalized operator of the first difference defined by $(B(r,s)y)_n = ry_n + sy_{n-1}$ for all $n \geq 2$ and $(B(r,s)y)_1 = ry_1$. In this way we determined the set of all positive sequences x for which $(ry_n + sy_{n-1})/x_n \to l$ implies $(r'y_n + s'y_{n-1})/x_n \to l'$ $(n \to \infty)$ for all y and for some scalars l and l'. In the paper [8] we used the sets of analytic and entire sequences denoted by $\mathbf{\Lambda}$ and $\mathbf{\Gamma}$ and defined by $\sup_{n\geq 1}\left(|y_n|^{1/n}\right) < \infty$ and $\lim_{n\to\infty}\left(|y_n|^{1/n}\right) = 0$, respectively. Then we dealt with a class of (SSE) with operators of the form $E_T + F_x = F_b$, where T is either Δ or Σ and E is any of the sets c_0, c, ℓ_∞, ℓ_p, $(p \geq 1)$, w_0, $\mathbf{\Gamma}$, or $\mathbf{\Lambda}$ and $F = c$, ℓ_∞ or $\mathbf{\Lambda}$. In [11] we solved the (SSE) defined by $(E_a)_\Delta + s_x^{(c)} = s_b^{(c)}$ where E is either c_0, or ℓ^p, and the (SSE) $(E_a)_\Delta + s_x^0 = s_b^0$ where E is either c, or ℓ_∞. In [10, 13] we dealt with the sequence spaces inclusion equations (SSIE) defined by $F_b \subset E_a + F'_x$ where a and b are positive sequences and E, F and F' are linear subspaces of ω and we solved the (SSE) defined by $E_a + F_x = F_b$ when $e \notin F$. In this paper we extend some of the results stated in [8] and solve a new class of sequence spaces equations of the form $(E_a)_T + F_x = F_b$ where (E,F) is any of the class (ℓ_∞, c), (ℓ^p, c), (c_0, c), (c_0, ℓ_∞), (ℓ^p, ℓ_∞), or (w_0, ℓ_∞) with $p \geq 1$ and T is a triangle whose the inverse has finite columns.

This paper is organized as follows. In Section 2 we recall some results on some sequence spaces and matrix transformations. In Section 3 we recall some results on matrix transformations and we define the set W_a^0. In Section 4 we deal with the solvability of the (SSE) $E_{\mathcal{T}} + F_x = F_b$ with $e \in F$ for some triangle $\mathcal{T}$. In Section 5 we deal with some *perturbed equation* of the form $\mathcal{E} + F_x = F_b$, where $\mathcal{E}$ is a linear space of sequences. In this way we solve such equations where $\mathcal{E} = (E_a)_T$ and $(E,F) \in \{(\ell_\infty, c), (c_0, \ell_\infty), (c_0, c), (\ell^p, c), (\ell^p, \ell_\infty), (w_0, \ell_\infty)\}$ with $p \geq 1$, and T is a triangle.

2. Preliminaries and notations

A *BK space* is a Banach space of sequences that is, an *FK space*. A BK space E *is said to have AK if for every sequence* $y = (y_k)_{k\geq 1} \in E$, *then* $y = \lim_{p\to\infty}\sum_{k=1}^{p} y_k e^{(k)}$, *where* $e^{(k)} = (0, ..., 0, 1, 0, ...)$, 1 *being in the k-th position.*

Let $\mathbb{R}$ be the set of all real numbers. For any given infinite matrix $A = (\mathbf{a}_{nk})_{n,k\geq 1}$ we define the operators $A_n = (\mathbf{a}_{nk})_{k\geq 1}$ for any integer $n \geq 1$, by $A_n y = \sum_{k=1}^{\infty} \mathbf{a}_{nk} y_k$, where $y = (y_k)_{k\geq 1}$, and the series are assumed convergent for all n. So we are led to the study of the operator A defined by $Ay = (A_n y)_{n\geq 1}$ mapping between sequence spaces. When *A maps E into F, where E and F are subsets of ω, we write $A \in (E,F)$*, (cf. [22, 23]). It is well known that if E has AK, then the set $\mathcal{B}(E)$ of all *bounded linear operators L* mapping in E, with norm $\|L\| = \sup_{y\neq 0}\left(\|L(y)\|_E / \|y\|_E\right)$ satisfies the identity $\mathcal{B}(E) = (E,E)$. We denote by ω, c_0, c and ℓ_∞ the sets of all sequences, the sets of null, convergent and bounded sequences. We write ℓ^p for the set of all p-absolutely convergent series, with $p \geq 1$, that is,

$$\ell^p = \left\{ y \in \omega : \|y\|_{\ell^p} = \sum_{k=1}^{\infty} |y_k|^p < \infty \right\}.$$

For any subset F of ω, we write $F_A = \{y \in \omega : Ay \in F\}$ for the matrix domain of A in F. Then for any given sequence $u = (u_n)_{n\geq 1} \in \omega$ we define the diagonal matrix D_u by $[D_u]_{nn} = u_n$ for all n. It is interesting to rewrite the set E_u using a diagonal matrix. Let E be any subset of ω and $u \in U^+$ we have $E_u = D_u * E = \{y = (y_n)_n \in \omega : y/u \in E\}$. We use the sets s_a^0, $s_a^{(c)}$, s_a and ℓ_a^p defined as follows (cf. [2]). For given $a \in U^+$ and $p \geq 1$ we put $D_a * c_0 = s_a^0$, $D_a * c = s_a^{(c)}$, $D_a * \ell_\infty = s_a$, and $D_a * \ell^p = \ell_a^p$. We frequently write c_a instead of $s_a^{(c)}$ to simplify. Each of the spaces $D_a * E$, where $E \in \{c_0, c, \ell_\infty\}$ is a *BK space normed* by $\|y\|_{s_a} = \sup_n \left(|y_n|/a_n\right)$ and s_a^0 *has AK*. The set ℓ^p, $(p \geq 1)$ normed by $\|y\|_{\ell^p}$ is a BK space with AK. If $a = (R^n)_{n\geq 1}$ with $R > 0$, we write s_R, s_R^0, $s_R^{(c)}$, (or c_R) and ℓ_R^p for the sets s_a, s_a^0, $s_a^{(c)}$ and ℓ_a^p, respectively. We also write D_R for $D_{(R^n)_{n\geq 1}}$. When $R = 1$, we obtain $s_1 = \ell_\infty$, $s_1^0 = c_0$ and $s_1^{(c)} = c$. Recall that $S_1 = (s_1, s_1)$ is a Banach algebra and $(c_0, s_1) = (c, \ell_\infty) = (s_1, s_1) = S_1$. We have $A \in S_1$ if and only if

$$\sup_n \left(\sum_{k=1}^{\infty} |\mathbf{a}_{nk}|\right) < \infty. \tag{1}$$

We are led to recall some well-known results on matrix transformations.

3. Some results on matrix transformations

3.1. The classes (c_0, c_0), (c_0, c), (c, c_0), (c, c), (ℓ_∞, c), (ℓ_∞, c_0) and (ℓ^p, F) where $F = c_0$, c, or ℓ_∞.

We recall the next well-known results.

Lemma 3.1. *[[21], Theorem 1.36, p. 160], [22]*
Let $A = (\mathbf{a}_{nk})_{n,k\geq 1}$ be an infinite matrix. Then we have

i) $A \in (c_0, c_0)$ if and only if (1) holds and

$$\lim_{n\to\infty} \mathbf{a}_{nk} = 0 \text{ for all } k. \tag{2}$$

ii) $A \in (c_0, c)$ if and only if (1) holds and

$$\lim_{n\to\infty} \mathbf{a}_{nk} = l_k \text{ for all } k \text{ and for some scalar } l_k. \tag{3}$$

iii) $A \in (c, c_0)$ if and only if the conditions in (1) and (2) hold and $\lim_{n\to\infty} \sum_{k=1}^{\infty} \mathbf{a}_{nk} = 0$.

iv) $A \in (c, c)$ if and only if (1), (3) hold and $\lim_{n\to\infty} \sum_{k=1}^{\infty} \mathbf{a}_{nk} = l$ for some scalar l.

v) $A \in (\ell_\infty, c)$ if and only if (3) holds and $\lim_{n\to\infty} \sum_{k=1}^{\infty} |\mathbf{a}_{nk}| = \sum_{k=1}^{\infty} |l_k|$.

vi) $A \in (\ell_\infty, c_0)$ if and only if $\lim_{n\to\infty} \sum_{k=1}^{\infty} |\mathbf{a}_{nk}| = 0$.

Characterization of (ℓ^p, F) where $F = c_0$, c, or ℓ_∞. For this, we let $q = p/(p-1)$ for $p > 1$ and we let $\mathcal{M}(\ell^p, \ell_\infty) = \sup_{n,k\geq 1} |\mathbf{a}_{nk}|$ if $p = 1$, and $\mathcal{M}(\ell^p, \ell_\infty) = \sup_{n\geq 1} \left(\sum_{k=1}^{\infty} |\mathbf{a}_{nk}|^q\right)$, if $p > 1$.

Lemma 3.2. *[[21], Theorem 1.37, p. 161]*
Let $p \geq 1$ and $A = (\mathbf{a}_{nk})_{n,k\geq 1}$ be an infinite matrix. Then we have

i) *$A \in (\ell^p, \ell_\infty)$ if and only if*

$$\mathcal{M}(\ell^p, \ell_\infty) < \infty. \tag{4}$$

ii) *$A \in (\ell^p, c_0)$ if and only if the conditions in (4) and (2) hold.*

iii) *$A \in (\ell^p, c)$ if and only if the conditions in (4) and (3) hold.*

We also use the well known property, stated as follows.

Lemma 3.3. *Let $a, b \in U^+$ and let E, $F \subset \omega$ be any linear spaces. Let $A = (\mathbf{a}_{nk})_{n,k\geq 1}$ be an infinite matrix. We have $A \in (E_a, F_b)$ if and only if $D_{1/b}AD_a \in (E, F)$, where $(D_{1/b}AD_a)_{nk} = b_n^{-1}\mathbf{a}_{nk}a_k$ for all n, $k \geq 1$.*

Lemma 3.4. *[[4], Lemma 9, p. 45]*
Let T' and T'' be any given triangles and let E, $F \subset \omega$. Then for any given operator T represented by a triangle we have $T \in (E_{T'}, F_{T''})$ if and only if $T''TT'^{-1} \in (E, F)$.

3.2. On the triangles $C(\lambda)$ and $\Delta(\lambda)$ and the sets W_a and W_a^0.

To solve the next equations we recall some definitions and results. The infinite matrix $T = (t_{nk})_{n,k\geq 1}$ is said to be a triangle if $t_{nk} = 0$ for $k > n$ and $t_{nn} \neq 0$ for all n. The infinite matrix $C(\lambda)$ with $\lambda = (\lambda_n)_n \in U$ is the triangle defined by $[C(\lambda)]_{nk} = 1/\lambda_n$ for $k \leq n$. It can be shown that the triangle $\Delta(\lambda)$ whose the nonzero entries are defined by $[\Delta(\lambda)]_{nn} = \lambda_n$, and $[\Delta(\lambda)]_{n,n-1} = -\lambda_{n-1}$ for all $n \geq 2$ is the inverse of $C(\lambda)$, that is, $C(\lambda)(\Delta(\lambda)y) = \Delta(\lambda)(C(\lambda)y)$ for all $y \in \omega$. If $\lambda = e = (1, ..., 1, ...)$ we obtain the well known operator of the first difference represented by $\Delta(e) = \Delta$. We then have $\Delta_n y = y_n - y_{n-1}$ for all $y \in \omega$ and for all $n \geq 1$, with the convention $y_0 = 0$. It is usually written $\Sigma = C(e)$ and then we may write $C(\lambda) = D_{1/\lambda}\Sigma$. Note that $\Delta = \Sigma^{-1}$. The Cesàro operator is defined By $C_1 = C((n)_{n\geq 1})$. We use the set of sequences that are *a–strongly convergent to zero* defined for $a \in U^+$ by

$$W_a^0 = (w_0)_a = \left\{ y \in \omega : \lim_{n\to\infty} \left(n^{-1} \sum_{k=1}^{n} \left| y_k \right| / a_k \right) = 0 \right\},$$

(cf. [9, 14, 17]). It can easily be seen that $W_a^0 = \left\{ y \in \omega : C_1 D_{1/a} \left| y \right| \in c_0 \right\}$. If $a = (r^n)_{n\geq 1}$ the set W_a^0 is denoted by W_r^0. For $r = 1$ we obtain the well known set

$$w_0 = \left\{ y \in \omega : \lim_{n\to\infty} \left(n^{-1} \sum_{k=1}^{n} \left| y_k \right| \right) = 0 \right\}$$

called the *space of sequences that are strongly summable to zero by the Cesàro method* (cf. [20]).

3.3. Characterization of (w_0, ℓ_∞) and (w_0, c_0).

Here we recall some results that are direct consequence of [1], Theorem 2.4], where we let $\sigma = (\sigma_n)_n$ with

$$\sigma_n = \sigma_n(A) = \sum_{\nu=0}^{\infty} 2^\nu \max_{2^\nu \leq k \leq 2^{\nu+1}-1} |\mathbf{a}_{nk}|, \tag{5}$$

for $A = (\mathbf{a}_{nk})_{n,k\geq 1}$. From [1] we obtain the following.

Lemma 3.5.

i) *We have $A \in (w_0, \ell_\infty)$ if and only if*

$$\sigma \in \ell_\infty. \tag{6}$$

ii) *$A \in (w_0, c_0)$ if and only if (6) and (2) hold.*

4. On the solvability of the (SSE) $E_{\mathcal{T}} + F_x = F_b$ where $\mathcal{T}$ is a triangle and $e \in F$

4.1. *On the multipliers of some sets.*

First we need to recall some well known results. Let y and z be sequences and let E and F be two subsets of ω, we then write

$$M(E,F) = \{y \in \omega : \ yz \in F \text{ for all } z \in E\},$$

the set $M(E,F)$ is called the *multiplier space of* E *and* F. Recall the next well known results.

Lemma 4.1. *Let* E, $\widetilde{E}$, F *and* $\widetilde{F}$ *be arbitrary subsets of* ω. *Then*

i) $M(E,F) \subset M\left(\widetilde{E},F\right)$ *whenever* $\widetilde{E} \subset E$,

ii) $M(E,F) \subset M\left(E,\widetilde{F}\right)$ *whenever* $F \subset \widetilde{F}$.

4.2. *On the sequence spaces equations.*

For $b \in U^+$ and for any subset F of ω, we denote by $cl^F(b)$ the equivalence class for the equivalence relation R_F defined by xR_Fy if $F_x = F_y$ for $x, y \in U^+$. It can easily be seen that $cl^F(b)$ is the set of all $x \in U^+$ such that $x/b \in M(F,F)$ and $b/x \in M(F,F)$, (cf. [18]). We then have $cl^F(b) = cl^{M(F,F)}(b)$. For instance $cl^c(b)$ is the set of all $x \in U^+$ such that $D_xc = D_bc$, that is, $s_x^{(c)} = s_b^{(c)}$. This is the set of all sequences $x \in U^+$ such that $x_n \sim Cb_n$ $(n \to \infty)$ for some $C > 0$. In [18] we denote by $cl^\infty(b)$ the class $cl^{\ell_\infty}(b)$. Recall that $cl^\infty(b)$ is the set of all $x \in U^+$, such that $K_1 \le x_n/b_n \le K_2$ for all n and for some $K_1, K_2 > 0$. For $a, b \in U^+$, we define the set $S_{a,b}(E,F) = \{x \in U^+ : E_a + F_x = F_b\}$ for $E, F \subset \omega$.

As we have just seen, for any given $b \in U^+$ the solutions of the (SSE) $F_x = F_b$ are determined by $x \in cl^F(b)$. Then the new (SSE) $\mathcal{E} + F_x = F_b$ where $\mathcal{E}$ is a linear space of sequences and $a \in U^+$can be considered as a *perturbed equation*. The question is: what are the conditions on $\mathcal{E}$ under which the (SSE) $F_x = F_b$ and the perturbed equation have the same solutions ?

Now we study some perturbed equations involving an operator represented by a triangle.

4.3. *On the (SSE) with operators represented by a triangle.*

Let $b \in U^+$, and E, F be two subsets of ω. We deal with the set $S_b(E_{\mathcal{T}}, F)$ of all the positive sequences that satisfy the (SSE) with operator

$$E_{\mathcal{T}} + F_x = F_b, \tag{7}$$

where $\mathcal{T}$ is a triangle and $x \in U^+$ is the unknown. The equation in (7) means for every $y \in \omega$, we have $y/b \in F$ if and only if there are $u, v \in \omega$ with $y = u + v$ such that $\mathcal{T}u \in E$ and $v/x \in F$. We assume $e \in F$. In the following we use the next two properties,

$$F \subset M(F,F), \tag{8}$$

and

$$F \subset F_{1/z} \text{ for all } z \in U^+ \text{ that satisfy } z_n \to 1 \ (n \to \infty). \tag{9}$$

Definition 4.2. *Let* $b \in U^+$ *and let* E, F *be linear spaces of sequences. We say that the (SSE) defined in (7), or the set* $S_b(E_{\mathcal{T}}, F)$ *is regular if*

$$S_b(E_{\mathcal{T}}, F) = \begin{cases} cl^F(b), & \text{if } D_{1/b}\mathcal{T}^{-1} \in (E,F), \\ \varnothing, & \text{if } D_{1/b}\mathcal{T}^{-1} \notin (E,F). \end{cases} \tag{10}$$

We recall the next result where we use the equivalence of $D_{1/b}\mathcal{T}^{-1} \in (E,F)$ and $1/b \in M(E_{\mathcal{T}}, F)$.

Lemma 4.3. *[[14], Proposition 6.1, p. 94]*
Let $b \in U^+$ and $\mathcal{T}$ be a triangle, let E, F be linear spaces of sequences with $e \in F$. Assume the space F satisfies the conditions in (8) and (9) and

$$M(E_{\mathcal{T}}, F) \subset M(E_{\mathcal{T}}, c_0). \tag{11}$$

Then the (SSE) defined in (7) is regular, that is,

$$S_b(E_{\mathcal{T}}, F) = \begin{cases} cl^F(b), & \text{if } 1/b \in M(E_{\mathcal{T}}, F), \\ \varnothing, & \text{if } 1/b \notin M(E_{\mathcal{T}}, F). \end{cases} \tag{12}$$

For any $b \in U^+$, if the perturbed equation $E_{\mathcal{T}} + F_x = F_b$ is regular, then it is equivalent to the (SSE) $F_x = F_b$. We may adapt the previous result using the notations of matrix transformations instead of the multiplier of sequence spaces. The proof of the next result follows from the equivalence of $z \in M(E_{\mathcal{T}}, \mathcal{F})$ and $D_z\mathcal{T}^{-1} \in (E, \mathcal{F})$ for any given set $\mathcal{F}$ of sequences. So we obtain the following result which is a direct consequence of Lemma 4.3.

Lemma 4.4. *[[14], Corollary 6.1, p. 94]*
Let $b \in U^+$ and $\mathcal{T}$ be a triangle and let E, F be linear spaces of sequences with $e \in F$. Assume the space F satisfies the conditions in (8) and (9) and

$$D_z\mathcal{T}^{-1} \in (E,F) \text{ implies } D_z\mathcal{T}^{-1} \in (E, c_0) \text{ for all } z \in U^+. \tag{13}$$

Then the (SSE) defined in (7) is regular.

5. Application to the solvability of the (SSE) of the form $(E_a)_T + F_x = F_b$ where $F \in \{c, \ell_\infty\}$

Let T be a triangle and let

$$\Theta = \{(\ell_\infty, c), (c_0, \ell_\infty), (c_0, c), (\ell^p, c), (\ell^p, \ell_\infty), (w_0, \ell_\infty)\}$$

with $p \geq 1$. Let a, b be positive sequences and consider the (SSE)

$$(E_a)_T + F_x = F_b, \tag{14}$$

where $(E,F) \in \Theta$. In the following we write $S_E^F(T) = S_b((E_a)_T, F)$ where E, $F \subset \omega$, and more precisely we write $S_\infty^c(T) = S_{\ell_\infty}^c(T)$, $S_0^\infty(T) = S_{c_0}^{\ell_\infty}(T)$, $S_0^c(T) = S_{c_0}^c(T)$, $S_p^c(T) = S_{\ell^p}^c(T)$, $S_p^\infty(T) = S_{\ell^p}^{\ell_\infty}(T)$ for $p \geq 1$, and $S_{w_0}^\infty(T) = S_{w_0}^{\ell_\infty}(T)$. So $S_\infty^c(T)$, $S_0^F(T)$, $S_p^F(T)$ and $S_{w_0}^\infty(T)$ are the sets of all positive sequences that satisfy the (SSE) $(s_a)_T + s_x^{(c)} = s_b^{(c)}$, $\left(s_a^0\right)_T + F_x = F_b$, $\left(\ell_a^p\right)_T + F_x = F_b$ where $F = c$, or ℓ_∞ with $p \geq 1$ and $\left(W_a^0\right)_T + s_x = s_b$, respectively. From Lemma 4.4 we obtain the next result.

Theorem 5.1. *Let a, $b \in U^+$, let T be a triangle and let $(E,F) \in \Theta$. We write $\mathbf{S}_T$ for the set of all positive sequences x that satisfy the (SSE) in (14). Assume for every positive integer k there is an integer $i_k > k$ such that*

$$\left(T^{-1}\right)_{nk} = 0 \text{ for all } n \geq i_k. \tag{15}$$

Then the set $\mathbf{S}_T$ is determined by (10), that is,

$$\mathbf{S}_T = \begin{cases} cl^F(b), & \text{if } D_{1/b}T^{-1}D_a \in (E,F), \\ \varnothing, & \text{if } D_{1/b}T^{-1}D_a \notin (E,F). \end{cases}$$

Proof. Let $\mathcal{T} = D_{1/a}T$, then $(E_a)_T = E_{\mathcal{T}}$ and therefore the (SSE) in (14) is equivalent to the (SSE) $E_{\mathcal{T}} + F_x = F_b$. From the characterizations of the classes $(E, F) \in \Theta$, and under the condition in (15), the condition $D_z T^{-1} D_a \in (E, F)$ holds if and only if $D_z T^{-1} D_a \in (E, c_0)$ for all $z \in U^+$. Since the conditions in (8) and (9) hold for $F = c$, or ℓ_∞ we conclude by Lemma 4.4 with $\mathcal{T} = D_{1/a}T$, that $\mathbf{S}_T$ is regular. More precisely we consider the case $(E, F) = (\ell_\infty, c)$. By v) in Lemma 3.1 where $l_k = 0$ for all k and using (15) we have $D_z T^{-1} D_a \in (\ell_\infty, c)$ if and only if $\lim_{n\to\infty} \left(D_z T^{-1} D_a\right)_{nk} = 0$ for all k, and $\lim_{n\to\infty} \sum_{k=1}^{n} \left|\left(D_z T^{-1} D_a\right)_{nk}\right| = 0$. So the condition $D_z T^{-1} D_a \in (\ell_\infty, c)$ implies $D_z T^{-1} D_a \in (\ell_\infty, c_0)$, for all $z \in U^+$, and Lemma 4.4 can be applied with $\mathcal{T} = D_{1/a}T$. The other cases can be shown in a similar way. □

To state the next results we use the sequence σ defined in Lemma 3.5 where A is a triangle L, so we obtain

$$\sigma_n = \sigma_n(L) = \sum_{\nu=0}^{\nu_n - 1} 2^\nu \max_{2^\nu \le k \le 2^{\nu+1}-1} |L_{nk}| + 2^{\nu_n} \max_{2^{\nu_n} \le k \le n} |L_{nk}| \tag{16}$$

where for every n, ν_n is an integer uniquely defined by $2^{\nu_n} \le n \le 2^{\nu_n + 1} - 1$. In the following we use the next conditions where T is a triangle

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{k=1}^{n} \left|T^{-1}_{nk}\right| a_k = 0, \tag{17}$$

$$\sup_n \left(\frac{1}{b_n} \sum_{k=1}^{n} \left|T^{-1}_{nk}\right| a_k \right) < \infty, \tag{18}$$

$$\sup_n \left(\frac{1}{b_n^q} \sum_{k=1}^{n} \left|T^{-1}_{nk}\right|^q a_k^q \right) < \infty \text{ with } q = p/(p-1), (p > 1), \tag{19}$$

and

$$\sup_{(n,k)\in\chi} \left(\frac{1}{b_n} \left|T^{-1}_{nk}\right| a_k \right) < \infty, \tag{20}$$

where we let $\chi = \{(n, k) : k \le n \text{ and } n \ge 1\}$. By Theorem 5.1 and using the characterization of each of the sets $(E, F) \in \Theta$ recalled in Lemma 3.1, Lemma 3.2 and Lemma 3.5 we obtain the next corollary.

Corollary 5.2. *Let $a, b \in U^+$ and let T be a triangle. Assume the condition in (15) holds. Then we have:*

i) *a) The solutions of the equation $(s_a)_T + s_x^{(c)} = s_b^{(c)}$ are determined by*

$$S_\infty^c(T) = \begin{cases} cl^c(b), & \text{if (17) holds,} \\ \emptyset, & \text{otherwise.} \end{cases}$$

b) The solutions of the equation $\left(s_a^0\right)_T + F_x = F_b$ with $F = c$, or ℓ_∞ are determined by

$$S_0^F(T) = \begin{cases} cl^F(b), & \text{if (18) holds,} \\ \emptyset, & \text{otherwise.} \end{cases}$$

ii) Let $F = c$, or ℓ_∞. Then the solutions of the equation $\left(\ell_a^p\right)_T + F_x = F_b$ with $p \ge 1$ are determined in the following way.

a) If $p > 1$, *then*

$$S_p^F(T) = \begin{cases} cl^F(b), & \text{if (19) holds,} \\ \varnothing, & \text{otherwise.} \end{cases}$$

b) If $p = 1$, *then*

$$S_1^F(T) = \begin{cases} cl^F(b), & \text{if (20) holds,} \\ \varnothing, & \text{otherwise.} \end{cases}$$

iii) The solutions of the equation $\left(W_a^0\right)_T + s_x = s_b$ *are determined by*

$$S_{w_0}^{\infty}(T) = \begin{cases} cl^{\infty}(b), & \text{if } \sup_n \left\{\sigma_n\left(D_{1/b}T^{-1}D_a\right)\right\} < \infty, \\ \varnothing, & \text{otherwise.} \end{cases}$$

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