Filomat 32:14 (2018), 4981–4991
https://doi.org/10.2298/FIL1814981V

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence of Nonoscillatory Solutions to Third Order Nonlinear Neutral Difference Equations

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Abstract. The authors consider the third order neutral delay difference equation with positive and negative coefficients

$$\Delta(a_n\Delta(b_n\Delta(x_n + px_{n-m}))) + p_n f(x_{n-k}) - q_n g(x_{n-l}) = 0,\ n \geq n_0,$$

and give some new sufficient conditions for the existence of nonoscillatory solutions. Banach's fixed point theorem plays a major role in the proofs. Examples are provided to illustrate their main results.

1. Introduction

Consider the third order neutral delay difference equation with positive and negative coefficients

$$\Delta(a_n\Delta(b_n\Delta(x_n + px_{n-m}))) + p_n f(x_{n-k}) - q_n g(x_{n-l}) = 0,\ n \geq n_0, \tag{1}$$

where n_0 is a nonnegative integer, subject to the following conditions:

(H_1) p is a real number, and m, k, and l are positive integers;
(H_2) $\{a_n\}$, $\{b_n\}$, $\{p_n\}$, and $\{q_n\}$ are positive real sequences for all $n \geq n_0$;
(H_3) f and g are continuous functions with $xf(x) > 0$ and $xg(x) > 0$ for $x \neq 0$;
(H_4) f and g satisfy local Lipschitz conditions, and the Lipschitz constants are denoted by $L_f(A)$ and $L_g(A)$, where A is a closed subset of the domain of f and g.

Let $\theta = \max\{m, k, l\}$. By a *solution* of equation (1), we mean a real sequence $\{x_n\}$ defined for all $n \geq n_0 - \theta$, and which satisfies equation (1) for all $n \geq n_0$. A solution of equation (1) is said to be *nonoscillatory* if it is either eventually positive or eventually negative, and is *oscillatory* otherwise.

Recently there has been an increasing interest in investigating the oscillatory and nonoscillatory behavior of various classes of third and higher order difference equations; see for example, the monograph [1], papers [2–4, 6–11, 13–17], and the references cited therein.

2010 *Mathematics Subject Classification*. Primary 39A10
Keywords. Third order, nonoscillation, neutral delay difference equation, positive and negative coefficients
Received: 23 February 2018; Accepted: 07 June 2018
Communicated by Jelena Manojlović
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In [6], the authors studied the existence of nonoscillatory solutions of the higher-order nonlinear neutral difference equation

$$\Delta^m(x(n) + p(n)x(\tau(n))) + f_1(n, x(\sigma_1(n))) - f_2(n, x(\sigma_2(n))) = 0, \tag{2}$$

where f_1 and f_2 are continuous functions satisfying local Lipschitz condition with $xf_i(n, x) > 0$ for $i = 1, 2$ and $x \neq 0$. Using the Banach contraction principle, the authors obtained some sufficient conditions for the existence of nonoscillatory solutions to equation (2).

In [11], the authors investigated the existence of nonoscillatory solutions to the third order neutral difference equation

$$\Delta^3(x_n + px_{n-k}) + q_n f(x_{n-l}) = h_n,\ n \geq n_0, \tag{3}$$

where $p \in \mathbb{R}$, k, l, $n_0 \in \mathbb{N}$, h_n, $q_n \in \mathbb{R}$, and $f \in C(\mathbb{R}, \mathbb{R})$ satisfies a local Lipschitz condition with $xf(x) > 0$ for $x \neq 0$. Some other special cases of equation (3) were considered in [7, 9, 10].

There appears to be few results available for third order nonlinear difference equations with positive and negative coefficients. This most likely is due to the technical difficulties arising in their analysis. Motivated by the above observations, in this paper we obtain some new sufficient conditions for the existence of nonoscillatory solutions to equation (1) for $p \neq -1$. Our method of proof involves defining appropriate subsets of a Banach space and then using Banach's fixed point theorem. Examples are provided to illustrate our main results.

2. Existence Theorems

In this section, we present nonoscillation results for equation (1) for different ranges of values of p. We begin with the following theorem.

Theorem 2.1. *Assume that* $p = 1$,

$$\sum_{n=n_0}^{\infty} \frac{R_n}{a_n} \sum_{s=n_0}^{n} p_s < \infty, \quad \textit{and} \quad \sum_{n=n_0}^{\infty} \frac{R_n}{a_n} \sum_{s=n_0}^{n} q_s < \infty, \tag{4}$$

where $R_n = \sum_{s=n_0}^{n} \frac{1}{b_s}$. *Then equation* (1) *has a bounded nonoscillatory solution.*

Proof. Let $B(n_0)$ be the Banach space of all bounded real sequences $x = \{x_n\}$ with the norm $\|x\| = \sup_{n\geq n_0} |x_n|$. In view of conditions (H_4) and (4), we can choose an integer $n_1 \geq n_0 + \theta$ sufficiently large such that, for all $n \geq n_1$,

$$\begin{aligned} \sum_{s=n}^{\infty} \Big(\frac{R_s - R_{n-1}}{a_s}\Big) \sum_{t=n_1}^{s} p_t &\leq \frac{1}{\alpha}, \\ \sum_{s=n}^{\infty} \Big(\frac{R_s - R_{n-1}}{a_s}\Big) \sum_{t=n_1}^{s} q_t &\leq \frac{1}{\beta}, \\ \sum_{s=n}^{\infty} \Big(\frac{R_s - R_{n-1}}{a_s}\Big) \sum_{t=n_1}^{s} (p_t + q_t) &< \min\left\{\frac{1}{L}, \frac{1}{\alpha} + \frac{1}{\beta}\right\}, \end{aligned} \tag{5}$$

where $\alpha = \max_{1\leq x\leq 3}\{f(x)\}$, $\beta = \max_{1\leq x\leq 3}\{g(x)\}$, and $L = \max\{L_f([1, 3]), L_g([1, 3])\}$. Define the closed, bounded, and convex subset S of $B(n_0)$ by

$$S = \{x = \{x_n\} \in B(n_0) : 1 \leq x_n \leq 3,\ n \geq n_0\}$$

and the operator $T : S \to B(n_0)$ by

$$(Tx)_n = \begin{cases} 2 - \sum_{j=1}^{\infty} \sum_{s=n+(2j-1)m}^{n+2jm} \frac{1}{b_s} \sum_{t=s}^{\infty} \frac{1}{a_t} \sum_{u=n_1}^{t-1} (p_u f(x_{u-k}) - q_u g(x_{u-l})), & n \ge n_1, \\ (Tx)_{n_1}, & n_0 \le n \le n_1. \end{cases}$$

Clearly, T is a continuous mapping on S. For every $x = \{x_n\} \in S$ and $n \ge n_1$, we have

$$\begin{aligned} (Tx)_n &\le 2 + \sum_{j=1}^{\infty} \left[\sum_{s=n+(2j-1)m}^{n+2jm} \frac{1}{b_s} \sum_{t=s}^{\infty} \frac{1}{a_t} \sum_{u=n_1}^{t-1} q_u g(x_{u-l}) \right. \\ &\quad \left. + \sum_{s=n+(2j-2)m}^{n+(2j-1)m} \frac{1}{b_s} \sum_{t=s}^{\infty} \frac{1}{a_t} \sum_{u=n_1}^{t-1} q_u g(x_{u-l}) \right] \\ &= 2 + \sum_{s=n}^{\infty} \frac{1}{b_s} \sum_{t=s}^{\infty} \frac{1}{a_t} \sum_{u=n_1}^{t-1} q_u g(x_{u-l}) \\ &= 2 + \sum_{t=n}^{\infty} \sum_{s=n}^{t} \frac{1}{b_s} \frac{1}{a_t} \sum_{u=n_1}^{t-1} q_u g(x_{u-l}) \\ &= 2 + \sum_{t=n}^{\infty} \left(\frac{R_t - R_{n-1}}{a_t} \right) \sum_{u=n_1}^{t-1} q_u g(x_{u-l}) \\ &\le 2 + \beta \sum_{t=n}^{\infty} \left(\frac{R_t - R_{n-1}}{a_t} \right) \sum_{u=n_1}^{t} q_u \le 3, \end{aligned}$$

and

$$\begin{aligned} (Tx)_n &\ge 2 - \sum_{j=1}^{\infty} \left[\sum_{s=n+(2j-1)m}^{n+2jm} \frac{1}{b_s} \sum_{t=s}^{\infty} \frac{1}{a_t} \sum_{u=n_1}^{t-1} p_u f(x_{u-k}) \right. \\ &\quad \left. + \sum_{s=n+(2j-2)m}^{n+(2j-1)m} \frac{1}{b_s} \sum_{t=s}^{\infty} \frac{1}{a_t} \sum_{u=n_1}^{t-1} p_u f(x_{u-k}) \right] \\ &= 2 - \sum_{s=n}^{\infty} \frac{1}{b_s} \sum_{t=s}^{\infty} \frac{1}{a_t} \sum_{u=n_1}^{t-1} p_u f(x_{u-k}) \\ &= 2 - \sum_{t=n}^{\infty} \sum_{s=n}^{t} \frac{1}{b_s} \frac{1}{a_t} \sum_{u=n_1}^{t-1} p_u f(x_{u-k}) \\ &= 2 - \sum_{t=n}^{\infty} \left(\frac{R_t - R_{n-1}}{a_t} \right) \sum_{u=n_1}^{t-1} p_u f(x_{u-k}) \\ &\ge 2 - \alpha \sum_{t=n}^{\infty} \left(\frac{R_t - R_{n-1}}{a_t} \right) \sum_{u=n_1}^{t} p_u \ge 1. \end{aligned}$$

Thus, $TS \subseteq S$.

To show that T is a contraction mapping on S, let $x, y \in S$. Then for $n \geq n_1$, we have

$$\begin{aligned}
|Tx_n - Ty_n| &\leq \sum_{j=1}^{\infty} \sum_{s=n+(2j-1)m}^{n+2jm} \frac{1}{b_s} \sum_{t=s}^{\infty} \frac{1}{a_t} \sum_{u=n_1}^{t-1} (p_u|f(x_{u-k}) - f(y_{u-k})| + q_u|g(x_{u-l}) - g(y_{u-l})|) \\
&\leq L\|x-y\| \sum_{s=n}^{\infty} \frac{1}{b_s} \sum_{t=s}^{\infty} \frac{1}{a_t} \sum_{u=n_1}^{t-1} (p_u + q_u) \\
&\leq L\|x-y\| \sum_{t=n}^{\infty} \sum_{s=n}^{t} \frac{1}{b_s} \frac{1}{a_t} \sum_{u=n_1}^{t-1} (p_u + q_u) \\
&\leq L\|x-y\| \sum_{t=n}^{\infty} \left(\frac{R_t - R_{n-1}}{a_t} \right) \sum_{u=n_1}^{t} (p_u + q_u).
\end{aligned}$$

This implies that

$$\|Tx - Ty\| \leq C_0 \|x - y\|$$

where $C_0 = L \sum_{t=n}^{\infty} \left(\frac{R_t - R_{n-1}}{a_t} \right) \sum_{u=n_1}^{t} (p_u + q_u)$. In view of (5), we see that $C_0 < 1$, and so T is a contraction mapping. Hence, T has a unique fixed point $x = \{x_n\}$. That is,

$$x_n = \begin{cases} 2 - \sum_{j=1}^{\infty} \sum_{s=n+(2j-1)m}^{n+2jm} \frac{1}{b_s} \sum_{t=s}^{\infty} \frac{1}{a_t} \sum_{u=n_1}^{t-1} \left[p_u f(x_{u-k}) - q_u g(x_{u-l}) \right], & n \geq n_1, \\ (Tx)_{n_1}, & n_0 \leq n \leq n_1. \end{cases}$$

Furthermore, we have

$$\begin{aligned}
x_n + x_{n-m} &= 4 - \sum_{j=1}^{m} \left[\sum_{s=n+(2j-1)m}^{n+2jm} + \sum_{s=n+(2j-2)m}^{n+(2j-1)m} \right] \frac{1}{b_s} \sum_{t=s}^{\infty} \frac{1}{a_t} \sum_{u=n_1}^{t-1} \left[p_u f(x_{u-k}) - q_u g(x_{u-l}) \right] \\
&= 4 - \sum_{s=n}^{\infty} \frac{1}{b_s} \sum_{t=s}^{\infty} \frac{1}{a_t} \sum_{u=n_1}^{t-1} \left[p_u f(x_{u-k}) - q_u g(x_{u-l}) \right].
\end{aligned}$$

Therefore,

$$\Delta(a_n \Delta(b_n \Delta(x_n + x_{n-m}))) + p_n f(x_{n-k}) - q_n g(x_{n-l}) = 0,$$

and $\{x_n\}$ is clearly a positive solution of equation (1). This completes the proof of the theorem. □

Our next result is for the case $0 \leq p < 1$.

Theorem 2.2. *Assume that $0 \leq p < 1$ and condition (4) holds. Then equation (1) has a bounded nonoscillatory solution.*

Proof. Let $B(n_0)$ be the Banach space defined in the proof of Theorem 2.1. By conditions (H_4) and (4), we can choose $n_3 \geq n_0 + \theta$ sufficiently large such that

$$\begin{aligned}
\sum_{s=n}^{\infty} \frac{R_s}{a_s} \sum_{t=n_3}^{s} p_t &\leq \frac{p - (1 - N_1)}{\alpha_1}, \\
\sum_{s=n}^{\infty} \frac{R_s}{a_s} \sum_{t=n_3}^{s} q_t &\leq \frac{1 - p - pN_1 - M_1}{\beta_1}, \\
\sum_{s=n}^{\infty} \frac{R_s}{a_s} \sum_{t=n_3}^{s} (p_t + q_t) &< \frac{1 - p}{L_1}
\end{aligned} \tag{6}$$

hold for all $n \geq n_3$, where $N_1 \geq M_1 > 0$, $1 - N_1 < p < \frac{(1-M_1)}{(1+N_1)}$, $\alpha_1 = \max_{M_1 \leq x \leq N_1}\{f(x)\}$, $\beta_1 = \max_{M_1 \leq x \leq N_1}\{g(x)\}$ and $L_1 = \max\{L_f([M_1, N_1]), L_g([M_1, N_1])\}$. Set

$$S_1 = \{x = \{x_n\} \in B(n_0) : M_1 \leq x_n \leq N_1,\ n \geq n_0\};$$

then S_1 is a closed, bounded, and convex subset of $B(n_0)$. Define the operator $T : S_1 \to B(n_0)$ by

$$(Tx)_n = \begin{cases} 1 - p - px_{n-m} + R_{n-1} \displaystyle\sum_{s=n-1}^{\infty} \frac{1}{a_s} \sum_{t=n_3}^{s-1} (p_t f(x_{t-k}) - q_t g(x_{t-l})) & \\ \quad + \displaystyle\sum_{s=n_3}^{n-2} \frac{R_s}{a_s} \sum_{t=n_3}^{s-1} (p_t f(x_{t-k}) - q_t g(x_{t-l})), & n \geq n_3, \\ (Tx)_{n_3}, & n_0 \leq n \leq n_3. \end{cases}$$

Clearly T is a continuous mapping on S_1. For every $x \in S_1$ and $n \geq n_3$, we have

$$\begin{aligned} (Tx)_n &\leq 1 - p + \alpha_1 R_{n-1} \sum_{s=n-1}^{\infty} \frac{1}{a_s} \sum_{t=n_3}^{s-1} p_t + \alpha_1 \sum_{s=n_3}^{n-2} \frac{R_s}{a_s} \sum_{t=n_3}^{s-1} p_t \\ &\leq 1 - p + \alpha_1 \sum_{s=n_3}^{\infty} \frac{R_s}{a_s} \sum_{t=n_3}^{s} p_t \leq N_1, \end{aligned}$$

and

$$\begin{aligned} (Tx)_n &\geq 1 - p - pN_1 - \beta_1 R_{n-1} \sum_{s=n-1}^{\infty} \frac{1}{a_s} \sum_{t=n_3}^{s-1} q_t - \beta_1 \sum_{s=n_3}^{n-2} \frac{R_s}{a_s} \sum_{t=n_3}^{s-1} q_t \\ &\geq 1 - p - pN_1 - \beta_1 \sum_{s=n_3}^{\infty} \frac{R_s}{a_s} \sum_{t=n_3}^{s} q_t \geq M_1. \end{aligned}$$

Thus, $TS_1 \subseteq S_1$. Now for $x, y \in S_1$ and $n \geq n_3$, we obtain

$$\begin{aligned} |Tx_n - Ty_n| &\leq p|x_{n-m} - y_{n-m}| + R_{n-1} \sum_{s=n-1}^{\infty} \frac{1}{a_s} \sum_{t=n_3}^{s-1} p_t |f(x_{t-k}) - f(y_{t-k})| \\ &\quad + \sum_{s=n_3}^{n-2} \frac{R_s}{a_s} \sum_{t=n_3}^{s-1} p_t |f(x_{t-k}) - f(y_{t-k})| \\ &\quad + R_{n-1} \sum_{s=n-1}^{\infty} \frac{1}{a_s} \sum_{t=n_3}^{s-1} q_t |g(x_{t-l}) - g(y_{t-l})| \\ &\quad + \sum_{s=n_3}^{n-2} \frac{R_s}{a_s} \sum_{t=n_3}^{s-1} q_t |g(x_{t-l}) - g(y_{t-l})| \\ &\leq p\|x - y\| + L_1 \|x - y\| \sum_{n=n_3}^{\infty} \frac{R_n}{a_n} \sum_{s=n_3}^{n} (p_s + q_s) \\ &= C_1 \|x - y\|, \end{aligned}$$

where $C_1 = p + L_1 \sum_{n=n_3}^{\infty} \frac{R_n}{a_n} \sum_{s=n_3}^{n} (p_s + q_s) < 1$ in view of (6). This implies

$$\|Tx - Ty\| \leq C_1 \|x - y\|,$$

and so T is a contraction mapping. Hence, by the Banach contraction mapping theorem, T has a unique fixed point that in turn is a positive solution of equation (1). This completes the proof. □

Theorem 2.3. *Assume that* $1 < p < \infty$ *and* (4) *holds. Then equation* (1) *has a bounded nonoscillatory solution.*

Proof. Let $B(n_0)$ be as in the proof of Theorem 2.1. By conditions (H_4) and (4), we can choose an integer $n_2 \geq n_0 + \theta$ such that

$$\begin{aligned} \sum_{s=n}^{\infty} \frac{R_s}{a_s} \sum_{t=n_2}^{s} p_t &\leq \frac{1 - p(1 - N_2)}{\alpha_2}, \\ \sum_{s=n}^{\infty} \frac{R_s}{a_s} \sum_{t=n_2}^{s} q_t &\leq \frac{(1 - M_2)p - (1 + N_2)}{\beta_2}, \\ \sum_{s=n}^{\infty} \frac{R_s}{a_s} \sum_{t=n_2}^{s} (p_t + q_t) &< \frac{p-1}{L_2} \end{aligned} \tag{7}$$

for all $n \geq n_2$, where $N_2 \geq M_2 > 0$, $(1 - M_2)p > 1 + N_2$, $p(1 - N_2) < 1$, $\alpha_2 = \max_{M_2 \leq x \leq N_2}\{f(x)\}$, $\beta_2 = \max_{M_2 \leq x \leq N_2}\{g(x)\}$, and $L_2 = \max\{L_f([M_2, N_2]), L_g([M_2, N_2])\}$. Let

$$S_2 = \{x = \{x_n\} \in B(n_0) : M_2 \leq x_n \leq N_2,\ n \geq n_0\}.$$

Define the operator $T : S_2 \to B(n_0)$ by

$$(Tx)_n = \begin{cases} 1 - \dfrac{1}{p} - \dfrac{1}{p} x_{n+m} + \dfrac{1}{p} R_{n+m-1} \displaystyle\sum_{s=n+m-1}^{\infty} \frac{1}{a_s} \sum_{t=n_2}^{s-1} (p_t f(x_{t-k}) - q_t g(x_{t-l})) & \\ \quad + \dfrac{1}{p} \displaystyle\sum_{s=n_2}^{n+m-2} \frac{R_s}{a_s} \sum_{t=n_2}^{s-1} (p_t f(x_{t-k}) - q_t g(x_{t-l})), & n \geq n_2, \\ (Tx)_{n_2}, & n_0 \leq n \leq n_2. \end{cases}$$

Clearly T is continuous on S_2. For every $x \in S_2$ and $n \geq n_2$, we have

$$\begin{aligned} (Tx)_n &\leq 1 - \frac{1}{p} + \frac{1}{p}\alpha_2 \sum_{s=n+m-1}^{\infty} \frac{R_s}{a_s} \sum_{t=n_2}^{s-1} p_t + \frac{1}{p}\alpha_2 \sum_{s=n_2}^{n+m-2} \frac{R_s}{a_s} \sum_{t=n_2}^{s-1} p_t \\ &\leq 1 - \frac{1}{p} + \frac{1}{p}\alpha_2 \sum_{s=n_2}^{\infty} \frac{R_s}{a_s} \sum_{t=n_2}^{s} p_t \leq N_2, \end{aligned}$$

and

$$\begin{aligned} (Tx)_n &\geq 1 - \frac{1}{p} - \frac{1}{p}N_2 - \frac{1}{p}\beta_2 \sum_{s=n+m-1}^{\infty} \frac{R_s}{a_s} \sum_{t=n_2}^{s-1} q_t - \frac{1}{p}\beta_2 \sum_{s=n_2}^{n+m-2} \frac{R_s}{a_s} \sum_{t=n_2}^{s-1} q_t \\ &\geq 1 - \frac{1}{p} - \frac{1}{p}N_2 - \frac{1}{p}\beta_2 \sum_{s=n_2}^{\infty} \frac{R_s}{a_s} \sum_{t=n_2}^{s} p_t \geq M_2. \end{aligned}$$

Thus, $TS_2 \subseteq S_2$. Since S_2 is a bounded, closed, and convex subset of $B(n_0)$, we need to prove that T is a contraction in order to apply the contraction mapping principle. Now for $x, y \in S_2$ and $n \geq n_2$, we see that

$$\begin{aligned}
|Tx_n - Ty_n| &\leq \frac{1}{p}|x_{n+m} - y_{n+m}| + \frac{1}{p}R_{n+m-1}\sum_{s=n+m-1}^{\infty}\frac{1}{a_s}\sum_{t=n_2}^{s-1}p_t|f(x_{t-k}) - f(y_{t-k})| \\
&\quad + \frac{1}{p}\sum_{s=n_2}^{n+m-2}\frac{R_s}{a_s}\sum_{t=n_2}^{s-1}p_t|f(x_{t-k}) - f(y_{t-k})| \\
&\quad + \frac{1}{p}R_{n+m-1}\sum_{s=n+m-1}^{\infty}\frac{1}{a_s}\sum_{t=n_2}^{s-1}q_t|g(x_{t-l}) - g(y_{t-l})| \\
&\quad + \frac{1}{p}\sum_{s=n_2}^{n+m-2}\frac{R_s}{a_s}\sum_{t=n_2}^{s-1}q_t|g(x_{t-l}) - g(y_{t-l})| \\
&\leq \frac{1}{p}\|x - y\| + \frac{1}{p}L_2\|x - y\|\sum_{s=n_2}^{\infty}\frac{R_s}{a_s}\sum_{t=n_2}^{s}(p_t + q_t) \\
&= C_2\|x - y\|,
\end{aligned}$$

which implies

$$\|Tx - Ty\| \leq C_2\|x - y\|.$$

From (7), we have $C_2 = \frac{1}{p}\left(1 + L_2\sum_{s=n_2}^{\infty}\frac{R_s}{a_s}\sum_{t=n_2}^{s}(p_t + q_t)\right) < 1$, and therefore T is a contraction mapping. Hence, T has a unique fixed that is a positive solution of equation (1). This proves the theorem. □

Our next two theorems are for cases where $p < 0$.

Theorem 2.4. *Assume that* $-1 < p < 0$ *and condition* (4) *holds. Then equation* (1) *has a bounded nonoscillatory solution.*

Proof. Let $B(n_0)$ be as in Theorem 2.1. By conditions (H_4) and (4), we can choose $n_4 \geq n_0 + \theta$ such that

$$\begin{aligned}
&\sum_{s=n}^{\infty}\frac{R_s}{a_s}\sum_{t=n_4}^{s}p_t \leq \frac{(1+p)N_3 - (1+p)}{\alpha_3}, \\
&\sum_{s=n}^{\infty}\frac{R_s}{a_s}\sum_{t=n_4}^{s}q_t \leq \frac{1 + p - M_3(1+p)}{\beta_3}, \\
&\sum_{s=n}^{\infty}\frac{R_s}{a_s}\sum_{t=n_4}^{s}(p_t + q_t) < \frac{1+p}{L_3}
\end{aligned} \tag{8}$$

hold for $n \geq n_4$, where M_3 and N_3 are positive constants satisfying $0 < M_3 < 1 < N_3$, $\alpha_3 = \max_{M_3 \leq x \leq N_3}\{f(x)\}$, $\beta_3 = \max_{M_3 \leq x \leq N_3}\{g(x)\}$, and $L_3 = \max\{L_f([M_3, N_3]), L_g([M_3, N_3])\}$. Set

$$S_3 = \{x = \{x_n\} \in B(n_0) : M_3 \leq x_n \leq N_3,\ n \geq n_0\}.$$

Then S_3 is a bounded, closed, and convex subset of $B(n_0)$. Define the operator $T : S_3 \to B(n_0)$ by

$$(Tx)_n = \begin{cases}
1 + p - px_{n-m} + R_{n-1}\displaystyle\sum_{s=n-1}^{\infty}\frac{1}{a_s}\sum_{t=n_4}^{s-1}(p_t f(x_{t-k}) - q_t g(x_{t-l})) & \\
\quad + \displaystyle\sum_{s=n_4}^{n-2}\frac{R_s}{a_s}\sum_{t=n_4}^{s-1}(p_t f(x_{t-k}) - q_t g(x_{t-l})), & n \geq n_4, \\
(Tx)_{n_4}, & n_0 \leq n \leq n_4.
\end{cases}$$

Then T is a continuous mapping on S_3 and for every $x \in S_3$ and $n \geq n_4$,

$$\begin{aligned}(Tx)_n &\leq 1+p-pN_3+\alpha_3\sum_{s=n-1}^{\infty}\frac{R_s}{a_s}\sum_{t=n_4}^{s-1}p_t+\alpha_3\sum_{s=n_4}^{n-2}\frac{R_s}{a_s}\sum_{t=n_4}^{s-1}p_t\\ &\leq 1+p-pN_3+\alpha_3\sum_{s=n_4}^{\infty}\frac{R_s}{a_s}\sum_{t=n_4}^{s}p_t\leq N_3.\end{aligned}$$

Similarly,

$$(Tx)_n \geq 1+p-pM_3-\beta_3\sum_{s=n_4}^{\infty}\frac{R_s}{a_s}\sum_{t=s_4}^{s}q_t\geq M_3,$$

and so $TS_3 \subseteq S_3$.

To prove that T is a contraction mapping on S_3, take $x, y \in S_3$, Then for $n \geq n_4$, we have

$$\begin{aligned}|Tx_n-Ty_n| &\leq -p|x_{n-m}-y_{n-m}|+R_{n-1}\sum_{s=n-1}^{\infty}\frac{1}{a_s}\sum_{t=n_4}^{s-1}p_t|f(x_{t-k})-f(y_{t-k})|\\ &\quad+\sum_{s=n_4}^{n-2}\frac{R_s}{a_s}\sum_{t=n_4}^{s-1}p_t|f(x_{t-k})-f(y_{t-k})|\\ &\quad+R_{n-1}\sum_{s=n-1}^{\infty}\frac{1}{a_s}\sum_{t=n_4}^{s-1}q_t|g(x_{t-l})-g(y_{t-l})|\\ &\quad+\sum_{s=n_4}^{n-2}\frac{R_s}{a_s}\sum_{t=n_4}^{s-1}q_t|g(x_{t-l})-g(y_{t-l})|\\ &\leq -p\|x-y\|+L_3\|x-y\|\sum_{s=n_4}^{\infty}\frac{R_s}{a_s}\sum_{t=n_4}^{s}(p_t+q_t)\\ &= C_3\|x-y\|.\end{aligned}$$

Hence,

$$\|Tx-Ty\|\leq C_3\|x-y\|$$

where $C_3=-p+L_3\sum_{n=n_4}^{\infty}\frac{R_n}{a_n}\sum_{s=n_4}^{n}(p_s+q_s)<1$ by (8). Hence, T is a contraction mapping, and therefore T has a unique fixed point that is a positive solution of equation (1). This completes the proof of the theorem. □

Theorem 2.5. *Assume that $-\infty < p < -1$ and condition* (4) *holds. Then equation* (1) *has a bounded nonoscillatory solution.*

Proof. Again let $B(n_0)$ be as in Theorem 2.1. In view of conditions (H_4) and (4), we can choose an integer $n_5 \geq n_0+\theta$large such that

$$\begin{aligned}&\sum_{s=n}^{\infty}\frac{R_s}{a_s}\sum_{t=n_5}^{s}p_t\leq\frac{-(p+1)(N_4-1)}{\beta_4},\\ &\sum_{s=n}^{\infty}\frac{R_s}{a_s}\sum_{t=n_5}^{s}q_t\leq\frac{-(1+p)(1-M_4)}{\alpha_4},\\ &\sum_{s=n}^{\infty}\frac{R_s}{a_s}\sum_{t=n_5}^{s}(p_t+q_t)<\frac{-(p+1)}{L_4}\end{aligned}\tag{9}$$

for $n \geq n_5$, where M_4 and N_4 are positive constants satisfying $0 < M_4 < 1 < N_4$, $\alpha_4 = \max_{M_4 \leq x_n \leq N_4}\{f(x)\}$, $\beta_4 = \max_{M_4 \leq x \leq N_4}\{g(x)\}$, and $L_4 = \max\{L_f([M_4, N_4]), L_g([M_4, N_4])\}$. Set

$$S_4 = \{x = \{x_n\} \in B(n_0) : M_4 \leq x_n \leq N_4,\ n \geq n_0\}$$

which we see is a closed, bounded, and convex subset of $B(n_0)$. Define the operator $T : S_4 \to B(n_0)$

$$(Tx)_n = \begin{cases} 1 + \dfrac{1}{p} - \dfrac{1}{p}x_{n+m} + \dfrac{1}{p}R_{n+m-1} \displaystyle\sum_{s=n+m-1}^{\infty} \frac{1}{a_s} \sum_{t=n_5}^{s-1} (p_t f(x_{t-k}) - q_t g(x_{t-l})) & \\ \quad + \dfrac{1}{p} \displaystyle\sum_{s=n_5}^{n+m-2} \frac{R_s}{a_s} \sum_{t=n_5}^{s-1} (p_t f(x_{t-k}) - q_t g(x_{t-l})), & n \geq n_5, \\ (Tx)_{n_5}, & n_0 \leq n \leq n_5 \end{cases}$$

Now T is continuous on S_4, and for every $x \in S_4$ and $n \geq n_5$, we have

$$\begin{aligned} (Tx)_n &\leq 1 + \frac{1}{p} - \frac{1}{p}N_4 - \frac{1}{p}\beta_4 \sum_{s=n+m-1}^{\infty} \frac{R_s}{a_s} \sum_{t=n_5}^{s-1} q_t - \frac{1}{p}\beta_4 \sum_{s=n_5}^{n+m-2} \frac{R_s}{a_s} \sum_{t=n_5}^{s-1} q_t \\ &\leq 1 + \frac{1}{p} - \frac{1}{p}N_4 - \frac{1}{p}\beta_4 \sum_{s=n_5}^{\infty} \frac{R_s}{a_s} \sum_{t=n_5}^{s} q_t \leq N_4, \end{aligned}$$

and

$$\begin{aligned} (Tx)_n &\geq 1 + \frac{1}{p} - \frac{1}{p}M_4 + \frac{1}{p}\alpha_4 \sum_{s=n+m-1}^{\infty} \frac{R_s}{a_s} \sum_{t=n_5}^{s-1} p_t + \frac{1}{p}\alpha_4 \sum_{s=n_5}^{n+m-2} \frac{R_s}{a_s} \sum_{t=n_5}^{s-1} p_t \\ &\geq 1 + \frac{1}{p} - \frac{1}{p}M_4 + \alpha_4 \sum_{s=n_5}^{\infty} \frac{R_s}{a_s} \sum_{t=n_5}^{s} p_t \geq M_4. \end{aligned}$$

Thus, $TS_4 \subseteq S_4$.

To prove that T is a contraction, let $x, y \in S_4$. Then for $n \geq n_5$, we have

$$\begin{aligned} |Tx_n - Ty_n| \leq\ & -\frac{1}{p}|x_{n+m} - y_{n+m}| - \frac{1}{p}R_{n+m-1} \sum_{s=n+m-1}^{\infty} \frac{1}{a_s} \sum_{t=n_5}^{s-1} p_t |f(x_{t-k}) - f(y_{t-k})| \\ & -\frac{1}{p}R_{n+m-1} \sum_{s=n+m-1}^{\infty} \frac{1}{a_s} \sum_{t=n_5}^{s-1} q_t |g(x_{t-l}) - g(y_{t-l})| \\ & -\frac{1}{p} \sum_{s=n_5}^{n+m-2} \frac{R_s}{a_s} \sum_{t=n_5}^{s-1} p_t |f(x_{t-k}) - f(y_{t-k})| \\ & -\frac{1}{p} \sum_{s=n_5}^{n+m-2} \frac{R_s}{a_s} \sum_{t=n_5}^{s-1} q_t |g(x_{t-l}) - g(y_{t-l})| \end{aligned}$$

$$\begin{aligned} &\leq -\frac{1}{p}\|x - y\| - \frac{1}{p}L_4 \|x - y\| \sum_{s=n_5}^{\infty} \frac{R_s}{a_s} \sum_{t=n_5}^{s} (p_t + q_t) \\ &= C_4 \|x - y\|, \end{aligned}$$

which implies that

$$\|Tx - Ty\| \leq C_4 \|x - y\|.$$

In view of (9), $C_4 = \frac{1}{p}(-1 - L_4 \sum_{s=n_5}^{\infty} \frac{R_s}{a_s} \sum_{t=n_5}^{s} (p_t + q_t)) < 1$, and this implies T is a contraction mapping. Hence, T has a unique fixed point that a positive solution of equation (1). This proves the theorem. □

Remark 2.6. *It is easy to see that Theorems 4–8 include the results in [11] as a special case. They also include the results in [6] for* $m = 3$. *The results obtained in this paper are new and extend or complement those in [6, 7, 11, 16, 17].*

3. Examples

In this section, we provide some examples to illustrate our results.

Example 3.1. *Consider the third order neutral difference equation*

$$\Delta\left(n(n-1)\Delta\left(n(n+1)\Delta(x_n + x_{n-1})\right)\right) + \frac{8(n-2)}{n^4(2n-3)}x_{n-2} - \frac{8(n-2)^3}{n^4(2n-3)^3}x_{n-2}^3 = 0,\ n \geq 2. \tag{10}$$

Here, $p = 1$, $a_n = n(n-1)$, $b_n = n(n+1)$, $f(x) = x$, $g(x) = x^3$, $p_n = \frac{8(n-2)}{n^4(2n-3)}$, *and* $q_n = \frac{8(n-2)^3}{n^4(2n-3)^3}$. *Simple calculations show that*

$$R_n = \sum_{s=2}^{n} \frac{1}{s(s+1)} = \frac{n-1}{2(n+1)},$$

$$\sum_{n=2}^{\infty} \frac{R_n}{a_n} \sum_{s=2}^{n} p_s = \sum_{n=2}^{\infty} \frac{1}{2n(n+1)} \sum_{s=2}^{n} \frac{8(s-2)}{s^4(2s-3)} < \infty,$$

and

$$\sum_{n=2}^{\infty} \frac{R_n}{a_n} \sum_{s=2}^{n} q_s = \sum_{n=2}^{\infty} \frac{1}{2n(n+1)} \sum_{s=2}^{n} \frac{8(s-2)^3}{s^4(2s-3)^3} < \infty,$$

Hence condition (4) *is satisfied. By Theorem 2.1, the equation* (10) *has a bounded nonoscillatory solution. In fact, the sequence* $\{x_n\} = \{2 + \frac{1}{n}\}$ *is such a solution of equation* (10).

Example 3.2. *Consider the equation*

$$\Delta\left(2^n\Delta\left(2^n\Delta\left(x_n + \frac{1}{2}x_{n-1}\right)\right)\right) + \frac{1}{4^n(2^n+16)}x_{n-2} - \frac{16}{(2^n+16)^3}x_{n-2}^3 = 0,\ n \geq 1 \tag{11}$$

We have $a_n = 2^n$, $b_n = 2^n$, $p = \frac{1}{2}$, $f(x) = x$, $g(x) = x^3$, $p_n = \frac{1}{4^n(2^n+16)}$, *and* $q_n = \frac{16}{(2^n+16)^3}$. *Simple calculations give*

$$R_n = \sum_{s=1}^{n} \frac{1}{b_s} = \sum_{s=1}^{n} \frac{1}{2^s} = \left(1 - \frac{1}{2^n}\right),$$

and

$$\sum_{n=1}^{\infty} \frac{R_n}{a_n} \sum_{s=1}^{n} p_s = \sum_{n=1}^{\infty} \frac{1}{2^n}\left(1 - \frac{1}{2^n}\right) \sum_{s=1}^{n} \frac{1}{4^s(2^s+16)} < \infty,$$

$$\sum_{n=1}^{\infty} \frac{R_n}{a_n} \sum_{s=1}^{n} q_s = \sum_{n=1}^{\infty} \frac{1}{2^n}\left(1 - \frac{1}{2^n}\right) \sum_{s=1}^{n} \frac{16}{(2^s+16)^3} < \infty.$$

Hence, condition (4) *holds. By Theorem 2.2, the equation* (11) *has a bounded nonoscillatory solution in* $[M_1, N_1]$. *In fact by taking* $M_1 = \frac{1}{16}$ *and* $N_1 = \frac{3}{4}$, *we see that the sequence* $\{x_n\} = \{\frac{1}{4} + \frac{1}{2^n}\}$ *is such a solution of equation* (11) *in* $[\frac{1}{16}, \frac{3}{4}]$.

Example 3.3. *Consider the equation*

$$\Delta\left(3^n\Delta\left(3^n\Delta\left(x_n - \frac{1}{3}x_{n-1}\right)\right)\right) + \frac{1}{(3^n+9)^3}x_{n-2}^3 - \frac{1}{9^n(3^n+3)}x_{n-1} = 0,\ n \geq 1 \tag{12}$$

In this case $a_n = 3^n$, $b_n = 3^n$, $p = -\frac{1}{3}$, $f(x) = x^3$, $g(x) = x$, $p_n = \frac{1}{(3^n+9)^3}$, *and* $q_n = \frac{1}{9^n(3^n+3)}$. *We see that*

$$R_n = \sum_{s=1}^{n} \frac{1}{b_s} = \sum_{s=1}^{n} \frac{1}{3^s} = \frac{1}{2}\left(\frac{3^n - 1}{3^n}\right),$$

$$\sum_{n=1}^{\infty} \frac{R_n}{a_n} \sum_{s=1}^{n} p_s = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{3^n} - \frac{1}{9^n}\right) \sum_{s=1}^{n} \frac{1}{(3^n + 9)^3} < \infty,$$

and

$$\sum_{n=1}^{\infty} \frac{R_n}{a_n} \sum_{s=1}^{n} q_s = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{3^n} - \frac{1}{9^n}\right) \sum_{s=1}^{n} \frac{1}{9^n(3^n + 3)} < \infty.$$

By Theorem 2.4, equation (3.3) *has a bounded nonoscillatory solution in* $[M_3, N_3]$. *Taking* $M_3 = \frac{1}{2}$ *and* $N_3 = 3$, *we see that the sequence* $\{x_n\} = \{1 + \frac{1}{3^n}\}$ *is a solution of equation* (3.3) *in* $[\frac{1}{2}, 3]$.

Remark 3.4. *In conclusion, we have shown that for any value of* $p \neq -1$, *condition* (4) *implies that equation* (1) *has a bounded nonoscillatory solution. It is well known that for neutral equations, the value* $p = -1$ *behaves as a bifurcation point in the behavior of solutions, so it is no surprise that it is eliminated from consideration here (see [5, 12]).*

References

[1] R. P. Agarwal, M. Bohner, S. R. Grace, and D. O'Regan, Discrete Oscillation Theory, Hindawi, New York, 2005.

[2] R. P. Agarwal, S. R. Grace, and D. O'Regan, On the oscillation of certain third order difference equations, Advances in Difference Equations 2005 (2005), 345–367.

[3] M. Bohner, C. Dharuman, R. Srinivasan, and E. Thandapani, Oscillation criteria for third order nonlinear functional difference equations with damping, Applied Mathematics & Information Sciences 11 (2017), 1–8.

[4] S. R. Grace, R. P. Agarwal, and J. R. Graef, Oscillation criteria for certain third order nonlinear difference equations, Applicable Analysis and Discrete Mathematics 3 (2009), 27–38.

[5] J. R. Graef and P. W. Spikes, Some asymptotic properties of solutions of a neutral delay equation with an oscillatory coefficient, Canadian Mathematical Bulletin 36 (1993), 263–272.

[6] Q. Li, H. Liang, W. Dong, and Z. Zhang, Existence of nonoscillatory solutions of higher order difference equations with positive and negative coefficients, Bulletin of the Korean Mathematical Society 45 (2008), 23–31.

[7] Z. Liu, H. Wu, and S. M. Kang, Bounded nonoscillatory solutions for a third order difference equations, International Journal of Mathematics and Analysis 8 (2014), 89–99.

[8] S. H. Saker, J. O. Alzabut, and A. Mukheimar, On the oscillatory behavior of a certain class of third order nonlinear delay difference equations, Electronic Journal of Qualitative Theory of Differential Equations 67 (2010), 1–16.

[9] B. Smith, Oscillation and nonoscillation theorems for third order quasi-adjoint difference equations, Portugaliae Mathematica 45 (1986), 229–243.

[10] B. Smith and W. E. Taylor, Jr., Oscillation and nonoscillation in nonlinear third order difference equations, International Journal of Mathematics and Mathematical Sciences 13 (1990), 281–286.

[11] E. Thandapani, R. Karunakaran, and I. M. Arockiasamy, Existence results for nonoscillatory solutions of third order nonlinear neutral difference equations, Sarajevo Journal of Mathematics 5 (2009), 73–87.

[12] E. Thandapani, P. Sundaram, J. R. Graef, and P. W. Spikes, Asymptotic behavior and oscillation of solutions of neutral delay difference equations of arbitrary order, Mathematica Slovaca 47 (1997), 539–551.

[13] E. Thandapani, M. Vijaya, and T. Li, On the oscillation of third order half-linear neutral type difference equations, Electronic Journal of Qualitative Theory of Differential Equation 76 (2011), 1–13.

[14] K. S. Vidhyaa, C. Dharuman, and E. Thandapani, Oscillation of third order nonlinear difference equations with positive and negative terms, Kyungpook Mathematical Journal, to appear.

[15] X. Wang and L. Huang, Oscillation for an odd order delay difference equations with several delays, International Journal of Qualitative Theory of Differential Equations and Applications 2 (2016), 15–23.

[16] Y. Zhou, Existence of nonoscillatory solutions of higher order neutral difference equations with general coefficients, Applied Mathematics Letters 15 (2002), 785–791.

[17] Y. Zhou and Y. Q. Huang, Existence for nonoscillatory solutions of higher order nonlinear neutral difference equations, Journal of Mathematical Analysis and Applications 280 (2003), 63–76.