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On Lower Density Type Operators and Topologies Generated by Them

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Abstract. In the paper we concentrate on lower, almost-lower and semi-lower density operators on measurable spaces. The existence of maximal element in the families of such operators is investigated. Moreover, we consider topologies generated by the above operators. Among others the existence of the greatest of such topologies (with respect to the inclusion) is studied.

1. Introduction and Preliminaries

In the paper we will concentrate on measurable spaces and density type operators defined on some families of subsets of this space. By a measurable space we will mean a triple $\langle X, \mathcal{S}, \mathcal{J}\rangle$, where X is a non-empty set, $\mathcal{S}$ is an algebra of subsets of X and $\mathcal{J} \subset \mathcal{S}$ is a proper ideal of sets. Moreover, from now on we will assume that $\bigcup \mathcal{J} = X$ i.e. $\mathcal{J}$ contains all singletons. One can see at once that $\mathcal{S}_0 = \{A \in \mathcal{S} : A \in \mathcal{J} \vee X \setminus A \in \mathcal{J}\}$ is the smallest algebra containing the family $\mathcal{J}$.

We will use the symbols $\mathcal{L}$ and $\mathbb{L}$ to denote the σ-algebra of Lebesgue measurable sets and the σ-ideal of Lebesgue measure zero sets in $\mathbb{R}$, respectively. If we consider $\mathbb{R}$ with natural topology τ_{nat} then we will use the symbol $\mathcal{B}a$ to denote the σ-algebra of set with the Baire property and the symbol $\mathbb{K}$ to denote the σ-ideal of the first category sets in $(\mathbb{R}, \tau_{nat})$.

Let X be a non-empty set. The family of all subsets of X will be denoted by 2^X. For any $A, B \in 2^X$ the symbol $A \triangle B$ will stand for the set $(A \setminus B) \cup (B \setminus A)$. Moreover, for any measurable space $\langle X, \mathcal{S}, \mathcal{J}\rangle$ and $A, B \subset X$ the symbol $A \sim B$ will mean that $A \triangle B \in \mathcal{J}$. We will write $A \underset{\sim}{\subset} B$ to state that $A \setminus B \in \mathcal{J}$. If $\{\mathcal{T}_w\}_{w \in W}$ is a family of topologies on X then the smallest topology generated by $\bigcup_{w \in W} \mathcal{T}_w$ will be denoted by $\sigma(\bigcup_{w \in W} \mathcal{T}_w)$.

We shall say that a measurable space $\langle X, \mathcal{S}, \mathcal{J}\rangle$ has the hull property if for any set $A \subset X$ there is a set $V \in \mathcal{S}$ such that $A \subset V$ and for any $Z \in \mathcal{S}$ if $Z \subset V \setminus A$ then $Z \in \mathcal{J}$.

As mentioned at the beginning of this section we will consider particular operators, so now we briefly recall their definitions. We will start with a lower density operator has played a special role in many considerations (e.g. [2, 12]).

Definition 1.1. Let $\langle X, \mathcal{S}, \mathcal{J}\rangle$ be a measurable space. We shall say that $\Phi : \mathcal{S} \to 2^X$ is a lower density operator on $\langle X, \mathcal{S}, \mathcal{J}\rangle$ if

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[I] $\Phi(\emptyset) = \emptyset$ and $\Phi(X) = X$;
[II] $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$ for any $A, B \in \mathcal{S}$;
[III] for any $A, B \in \mathcal{S}$ if $A \triangle B \in \mathcal{J}$ then $\Phi(A) = \Phi(B)$;
[IV] $\Phi(A) \triangle A \in \mathcal{J}$ for any $A \in \mathcal{S}$.

If Φ satisfies the additional condition:

[V] $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$ for any $A, B \in \mathcal{S}$

we shall call it a lifting. If the operator Φ satisfies conditions [I]–[III] then it is called a semi-lower density operator on $\langle X, \mathcal{S}, \mathcal{J} \rangle$ (see [5]). If we replace condition [IV] by the following one:

[IV*] $\Phi(A) \setminus A \in \mathcal{J}$ for any $A \in \mathcal{S}$,

we obtain an almost-lower density operator on $\langle X, \mathcal{S}, \mathcal{J} \rangle$ (see [3]).

For any measurable space $\langle X, \mathcal{S}, \mathcal{J} \rangle$ the family of all lower density operators on $\langle X, \mathcal{S}, \mathcal{J} \rangle$ will be denoted by $\mathcal{LDO}(\langle X, \mathcal{S}, \mathcal{J} \rangle)$, the family of all almost-lower density operators by $\mathcal{ALDO}(\langle X, \mathcal{S}, \mathcal{J} \rangle)$ and the family of all semi-lower density operators by $\mathcal{SLDO}(\langle X, \mathcal{S}, \mathcal{J} \rangle)$. The family of all liftings on $\langle X, \mathcal{S}, \mathcal{J} \rangle$ will be denoted by $\mathcal{LLDO}(\langle X, \mathcal{S}, \mathcal{J} \rangle)$. We will write simply $\mathcal{LDO}$, $\mathcal{ALDO}$, $\mathcal{SLDO}$ and $\mathcal{LLDO}$ when no confusion can arise.

Obviously for any measurable space $\langle X, \mathcal{S}, \mathcal{J} \rangle$ we have $\mathcal{LLDO} \subset \mathcal{LDO} \subset \mathcal{ALDO} \subset \mathcal{SLDO}$. Moreover, one can see at once that if $\Phi \in \mathcal{LDO}$ then $\Phi(A) \in \mathcal{S}$ for any $A \in \mathcal{S}$, so $\Phi : \mathcal{S} \to \mathcal{S}$. It is worth adding that condition [V] can be replaced by the following one

[V*] $\Phi(A) \cup \Phi(X \setminus A) = X$ for any $A \in S$.

Let $\langle X, \mathcal{S}, \mathcal{J} \rangle$ be a measurable space and $\mathcal{P}$ be any family of operators included in $\mathcal{SLDO}$. Let $\Phi_1, \Phi_2 \in \mathcal{P}$. We shall say that Φ_2 is greater than or equal to Φ_1 ($\Phi_1 \preceq \Phi_2$) if and only if $\Phi_1(A) \subset \Phi_2(A)$ for every $A \in \mathcal{S}$. The relation $\preceq$ is of course a partial order in $\mathcal{P}$.

In this paper we will also focus on topologies generated by operators mentioned above. We will investigate, among others, the existence of the largest topology with respect to the inclusion. The existence of the smallest with respect to the relation $\subset$ topology generated by lower and almost-lower density operators was investigated in [6]. Moreover, some other relations in the family of all topologies generated by operators mentioned above were studied in [10].

Definition 1.2. Let $\langle X, \mathcal{S}, \mathcal{J} \rangle$ be a measurable space and Φ be an operator such that $\Phi : \mathcal{S} \to 2^X$. If the family

$$\mathcal{T}_\Phi = \{A \in \mathcal{S} : A \subset \Phi(A)\}$$

is a topology on X then we say that Φ generates topology $\mathcal{T}_\Phi$ on X.

The following definition is well known.

Definition 1.3. Let $\langle X, \mathcal{S}, \mathcal{J} \rangle$ be a measurable space. We shall say that a topology $\tau \subset 2^X$ is an abstract density topology on $\langle X, \mathcal{S}, \mathcal{J} \rangle$ if there exists $\Phi \in \mathcal{LDO}$ generating topology $\mathcal{T}_\Phi$ such that $\tau = \mathcal{T}_\Phi$. The family of all abstract density topologies on $\langle X, \mathcal{S}, \mathcal{J} \rangle$ will be denoted by $\mathfrak{T}_{\langle X, \mathcal{S}, \mathcal{J} \rangle}$ (or simply $\mathfrak{T}$ when no confusion can arise) .

Analogously, we can define an almost-abstract (a semi-abstract) density topology.

Definition 1.4. Let $\langle X, \mathcal{S}, \mathcal{J} \rangle$ be a measurable space. We shall say that a topology $\tau \subset 2^X$ is an almost-abstract (a semi-abstract) density topology on $\langle X, \mathcal{S}, \mathcal{J} \rangle$ if there exists $\Phi \in \mathcal{ALDO}$ ($\Phi \in \mathcal{SLDO}$) generating topology $\mathcal{T}_\Phi$ such that $\tau = \mathcal{T}_\Phi$. The family of all almost-abstract (semi-abstract) density topologies on $\langle X, \mathcal{S}, \mathcal{J} \rangle$ will be denoted by $\mathfrak{T}^a_{\langle X, \mathcal{S}, \mathcal{J} \rangle}$ ($\mathfrak{T}^s_{\langle X, \mathcal{S}, \mathcal{J} \rangle}$) or simply $\mathfrak{T}^a$ ($\mathfrak{T}^s$) when no confusion can arise.

2. The Families $\mathcal{LDO}$ and $\mathfrak{T}$.

There exists a close connection between the hull property and the existence of a topology generated by operator from $\mathcal{LDO}$ (see [8, 11]).

Theorem 2.1. *Let $\langle X, \mathcal{S}, \mathcal{J}\rangle$ be a measurable space and $\Phi \in \mathcal{LDO}$. The space $\langle X, \mathcal{S}, \mathcal{J}\rangle$ has the hull property if and only if the operator Φ generates the topology $\mathcal{T}_\Phi$ on X.*

Clearly, the above theorem implies that if $\langle X, \mathcal{S}, \mathcal{J}\rangle$ is a measurable space having the hull property and $\mathcal{LDO} \neq \emptyset$ than $\mathfrak{T} \neq \emptyset$. Moreover, there is a connection between the partial order in the family $\mathcal{LDO}$ and the partial order in $\mathfrak{T}$ connected with the relation $\subset$. It is obvious that

Property 2.2. *If $\mathcal{T}_{\Phi_1}$ and $\mathcal{T}_{\Phi_1}$ are abstract density topologies on $\langle X, \mathcal{S}, \mathcal{J}\rangle$ generated by $\Phi_1, \Phi_2 \in \mathcal{LDO}$ then $\mathcal{T}_{\Phi_1} \subset \mathcal{T}_{\Phi_2}$ if and only if $\Phi_1 \leq \Phi_2$.*

Taking into account the above property we see that examination of the existence of the smallest or the largest topology in the family $\mathfrak{T}$ ordered by the relation $\subset$ is equivalent to the study of existence of the least or the greatest element in the family of all lower density operators ordered by the relation $\leq$.

Topologies from the families $\mathfrak{T}_{\langle \mathbb{R}, \mathcal{L}, \mathbb{L}\rangle}$ and $\mathfrak{T}_{\langle \mathbb{R}, \mathcal{B}a, \mathbb{K}\rangle}$ are investigated in many papers (e.g. [9, 12, 13]). It turns out that in none of these families there is the smallest and the largest topology with respect to the relation $\subset$. In the case of the smallest topology, the appropriate justification could be found in [6]. Now, we would like to concentrate on the smallest topology containing the union of all abstract density topologies on $\langle \mathbb{R}, \mathcal{L}, \mathbb{L}\rangle$.

Let us start with a reminder of information about $\mathcal{J}$-density operator introduced in [9].

We shall say that a sequence $\{J_n\}_{n\in\mathbb{N}}$ of non-degenerate closed intervals tends to 0 if $\lim_{n\to\infty} \operatorname{diam}(J_n \cup \{0\}) = 0$, where $\operatorname{diam}(J_n \cup \{0\})$ is the diameter of the set $J_n \cup \{0\}$. Let $\mathfrak{J}$ denote the family of all sequences of non-degenerate closed intervals tending to 0. To shorten notation we will write J instead of a sequence $\{J_n\}_{n\in\mathbb{N}}$ from the family $\mathfrak{J}$. Moreover, we will denote by $\mathfrak{J}_\alpha$ the set of all sequences $J \in \mathfrak{J}$ such that

$$\limsup_{n\to\infty} \frac{\operatorname{diam}(J_n \cup \{0\})}{|J_n|} < \infty.$$

Let $J \in \mathfrak{J}$ and $A \in \mathcal{L}$. Putting

$$\Phi_J(A) = \left\{ x \in \mathbb{R} : \lim_{n\to\infty} \frac{A \cap (J_n + x)}{|J_n|} = 1 \right\}$$

we obtain an almost lower density operator on $(\mathbb{R}, \mathcal{L}, \mathbb{L})$ called a J-density operator. Moreover, we have that the family $\mathcal{T}_{\Phi_J} = \{A \in \mathcal{L} : \Phi_J(A) \subset A\}$ is a topology on $\mathbb{R}$ (see [9]). If $J \in \mathfrak{J}_\alpha$, then Φ_J is a lower density operator.

Theorem 2.3. *The family $\bigcup_{J\in\mathfrak{J}_\alpha} \mathcal{T}_{\Phi_J}$ is not a topology. The smallest topology containing the family $\bigcup_{J\in\mathfrak{J}_\alpha} \mathcal{T}_{\Phi_J}$ is equal to $2^{\mathbb{R}}$.*

Proof. Let $J = \{J_n\}_{n\in\mathbb{N}}$ and $K = \{K_n\}_{n\in\mathbb{N}}$ be sequences from $\mathfrak{J}_\alpha$ such that $J_n \subset [0, \infty)$ and $K_n \subset (-\infty, 0]$ for any $n \in \mathbb{N}$. It is easy to see that for any $\alpha < 0$ and any $\beta > 0$ we get $(\alpha, 0] \in \mathcal{T}_{\Phi_K} \subset \bigcup_{J\in\mathfrak{J}_\alpha} \mathcal{T}_{\Phi_J}$ and $[0, \beta) \in \mathcal{T}_{\Phi_J} \subset \bigcup_{J\in\mathfrak{J}_\alpha} \mathcal{T}_{\Phi_J}$. Clearly, $(\alpha, 0] \cap [0, \beta) = \{0\} \notin \bigcup_{J\in\mathfrak{J}_\alpha} \mathcal{T}_{\Phi_J}$, so $\bigcup_{J\in\mathfrak{J}_\alpha} \mathcal{T}_{\Phi_J}$ is not a topology.

Let $\mathcal{T}^*$ be a topology containing the family $\bigcup_{J\in\mathfrak{J}_\alpha} \mathcal{T}_{\Phi_J}$. Obviously, $(\alpha, 0] \cap [0, \beta) = \{0\} \in \mathcal{T}^*$. Since for every $J \in \mathfrak{J}_\alpha$ the topology $\mathcal{T}_{\Phi_J}$ is invariant under translation (see [7]), it follows that $\{x\} \in \mathcal{T}^*$ for every $x \in \mathbb{R}$. Therefore $\mathcal{T}^* = 2^{\mathbb{R}}$. $\square$

If we consider J-density operators connected with category (see [15]) we can prove the analogue of the above theorem. From these facts it follows that the smallest topology containing the union of all abstract density topologies on $\langle \mathbb{R}, \mathcal{L}, \mathbb{L}\rangle$ and $\langle \mathbb{R}, \mathcal{B}a, \mathbb{K}\rangle$ is equal to $2^{\mathbb{R}}$. Clearly, if for $\langle X, \mathcal{S}, \mathcal{J}\rangle$ we have $\sigma(\bigcup_{\Phi\in\mathcal{LDO}} \mathcal{T}_\Phi) = 2^X$, then there is no largest topology in $\mathfrak{T}$ with respect to the relation $\subset$. Therefore there is no largest abstract density topology in $\langle \mathbb{R}, \mathcal{L}, \mathbb{L}\rangle$ and $\langle \mathbb{R}, \mathcal{B}a, \mathbb{K}\rangle$ with respect to the relation $\subset$. One can ask about a general case.

In [6] one can find that in a measurable space $\langle X, \mathcal{S}, \mathcal{J}\rangle$ such that $\bigcup \mathcal{J} = X$, the existence of the smallest topology in the family $\mathfrak{T}$ ordered by relation $\subset$ is equivalent to the equality $\mathcal{S} = \mathcal{S}_0$. The existence of the largest topology in the family $\mathfrak{T}$ ordered by the relation $\subset$ is also connected with the analogous condition. In fact we have

Theorem 2.4. *Let $\langle X, \mathcal{S}, \mathcal{J}\rangle$ be a measurable space such that $\mathfrak{T} \neq \emptyset$. Then $\sigma(\bigcup \mathfrak{T}) = 2^X$ if and only if $\mathcal{S} \neq \mathcal{S}_0$.*

Proof. If $\mathcal{S} = \mathcal{S}_0$ then the family $\mathfrak{T}$ consists of the one topology $\mathcal{T} = \{A \subset X : A = \emptyset \land X \setminus A \in \mathcal{J}\}$ which is generated by a lower density operator

$$\Phi_0(A) = \begin{cases} \emptyset & \text{if } A \in \mathcal{J}, \\ X & \text{if } X \setminus A \in \mathcal{J}, \end{cases}$$

so the smallest topology generated by the family $\bigcup \mathfrak{T}$ is equal to $\mathcal{T}$ and, in consequence, it is not equal to 2^X.

Assume that $\mathcal{S} \neq \mathcal{S}_0$. Since $\mathfrak{T} \neq \emptyset$, Theorem 2.1 implies that $\langle X, \mathcal{S}, \mathcal{J}\rangle$ has the hull property. Moreover, we have $\mathcal{LDO} \neq \emptyset$ and, in consequence, there is lifting Ψ on $\langle X, \mathcal{S}, \mathcal{J}\rangle$ (see [14]). Obviously, by Theorem 2.1, Ψ generates topology $\mathcal{T}_\Psi$.

Let $A \subset X$ be such that $A \in \mathcal{S} \setminus \mathcal{S}_0$. Clearly, $X \setminus A \in \mathcal{S} \setminus \mathcal{S}_0$. Fix $x_1 \in \Psi(A)$ and $x_2 \in \Psi(X \setminus A)$. Let $x \in \Psi(A)$. Putting

$$\Phi_x(B) = \begin{cases} \Psi(B) \cup \{x\} & \text{if } x_2 \in \Psi(B), \\ \Psi(B) \setminus \{x\} & \text{if } x_2 \notin \Psi(B) \end{cases}$$

for any $B \in \mathcal{S}$ we obtain a lower density operator on $\langle X, \mathcal{S}, \mathcal{J}\rangle$. Theorem 2.1 gives that Φ_x generates topology $\mathcal{T}_{\Phi_x}$. We check at once that $\Phi_x(X \setminus A) \in \mathcal{T}_{\Phi_x} \in \mathfrak{T}$ and $\Psi(A) \in \mathcal{T}_\Psi \in \mathfrak{T}$. Therefore, $\{x\} = \Psi(A) \cap \Phi_x(X \setminus A) \in \sigma(\bigcup \mathfrak{T})$.

The same conclusion can be drawn for $x \in \Psi(X \setminus A)$. In this case, it suffices to consider the operator defined as follows:

$$\Phi_x(B) = \begin{cases} \Psi(B) \cup \{x\} & \text{if } x_1 \in \Psi(B), \\ \Psi(B) \setminus \{x\} & \text{if } x_1 \notin \Psi(B) \end{cases}$$

for any $B \in \mathcal{S}$.

Since Ψ is a lifting, $X = \Psi(A) \cup \Psi(X \setminus A)$. Therefore, for any $x \in X$ we obtain $\{x\} \in \sigma(\bigcup \mathfrak{T})$, which means that $\sigma(\bigcup \mathfrak{T}) = 2^X$. $\square$

From the above we immediately obtain the following theorem

Theorem 2.5. *Let $\langle X, \mathcal{S}, \mathcal{J}\rangle$ be a measurable space such that $\mathfrak{T} \neq \emptyset$. There is the largest abstract density topology in $\mathfrak{T}$ ordered by relation $\subset$ if and only if $\mathcal{S} = \mathcal{S}_0$.*

Taking into account the above theorem and Property 2.2 we obtain immediately

Theorem 2.6. *Let $\langle X, \mathcal{S}, \mathcal{J}\rangle$ be a measurable space such that $\mathcal{LDO} \neq \emptyset$. The following statements are equivalent:*

(i) $\mathcal{S} = \mathcal{S}_0$;

(ii) *there exists the greatest (with respect to $\leq$) element in the family $\mathcal{LDO}$.*

The considerations associated with maximal elements in the family $\mathcal{LDO}$ ordered by $\preceq$ will end this section. It turns out that operators from the family $\mathcal{LLDO}$ play a special role in the considerations. As we mentioned earlier in [14] one can find that for any measurable space $\langle X, \mathcal{S}, \mathcal{J}\rangle$ if $\mathcal{LDO} \neq \emptyset$ then $\mathcal{LLDO} \neq \emptyset$. We start with the following lemma useful in the next part of the paper.

Lemma 2.7. *Let $\langle X, \mathcal{S}, \mathcal{J}\rangle$ be a measurable space and $\Phi \in \mathcal{ALDO}$. If there are a set $C \in \mathcal{S}$ and $x_0 \in X$ such that $x_0 \notin \Phi(C) \cup \Phi(X \setminus C)$ then there exists an operator $\Phi_* \in \mathcal{ALDO}$ which is greater than Φ with respect to $\preceq$. Moreover, if we additionally assume that $\Phi \in \mathcal{LDO}$ then $\Phi_* \in \mathcal{LDO}$.*

Proof. Assume, first that $\Phi \in \mathcal{ALDO}$. For any $A \in \mathcal{S}$ we put

$$\Phi_*(A) = \begin{cases} \Phi(A) \cup \{x_0\} & \text{if there exists } K \in \mathcal{S} \text{ such that } x_0 \in \Phi(K) \text{ and } (K \cap C) \underset{\sim}{\subset} A; \\ \Phi(A) & \text{otherwise.} \end{cases}$$

Obviously, $\Phi_*(X) = X$. Moreover, $\Phi_*(\emptyset) = \emptyset$, because if it existed $K \in \mathcal{S}$ such that $(K \cap C) \underset{\sim}{\subset} \emptyset$ and $x_0 \in \Phi(K)$, then we would have $x_0 \in \Phi(K) \subset \Phi(X \setminus C)$, which is impossible. Conditions [III] and [IV*] are easy to see. To prove the condition [II], first observe that for any $D, E \in \mathcal{S}$ if $x_0 \in \Phi_2(D)$ and $D \subset E$ then $x_0 \in \Phi_*(E)$. Therefore, we get that $\Phi_*(A \cap B) \subset \Phi_*(A) \cap \Phi_*(B)$. To prove the converse inclusion let us assume that $x_0 \in \Phi_*(A) \cap \Phi_*(B)$. There are three cases:

1. if $x_0 \in \Phi(A) \cap \Phi(B)$ then $x_0 \in \Phi(A \cap B) \subset \Phi_*(A \cap B)$.
2. $x_0 \in \Phi(A)$ but $x_0 \notin \Phi(B)$. Then there is $K \in \mathcal{S}$ such that $(K \cap C) \underset{\sim}{\subset} B$ and $x_0 \in \Phi(K)$. Thus $x_0 \in \Phi(A \cap K)$ and $((A \cap K) \cap C) \underset{\sim}{\subset} A \cap B$, so $x_0 \in \Phi_*(A \cap B)$.
3. $x_0 \notin \Phi(A)$ and $x_0 \notin \Phi(B)$. Then there are $K, L \in \mathcal{S}$ such that $(K \cap C) \underset{\sim}{\subset} A$, $(L \cap C) \underset{\sim}{\subset} B$ and $x_0 \in \Phi(K \cap L)$. This gives $((K \cap L) \cap C) \underset{\sim}{\subset} A \cap B$ and, in consequence, $x_0 \in \Phi_*(A \cap B)$.

To end the proof it is sufficient to observe that $\Phi(P) \subset \Phi_*(P)$ for any P and $\Phi(C) \neq \Phi_*(C)$.

It is easy to check that if $\Phi \in \mathcal{LDO}$ then the operator Φ_* defined above satisfies condition [IV]. □

Theorem 2.8. *Let $\langle X, \mathcal{S}, \mathcal{J}\rangle$ be a measurable space such that $\mathcal{LDO} \neq \emptyset$. For any $\Phi \in \mathcal{LDO}$ the following conditions are equivalent:*

(i) *Φ is a maximal element in the family $\mathcal{LDO}$ ordered by the relation $\preceq$;*
(ii) *Φ is a lifting.*

Proof. (*i*) $\Rightarrow$ (*ii*) Let Φ be a maximal element in the family $\mathcal{LDO}$ ordered by the relation $\preceq$. Suppose that Φ is not a lifting. Observe that for any $C, B \in \mathcal{S}$ the following inclusion holds : $\Phi(C) \cup \Phi(B) \subset \Phi(C \cup B)$. Hence there exist $C, B \in \mathcal{S}$ and $x_0 \in X$ such that $x_0 \in \Phi(C \cup B) \setminus (\Phi(C) \cup \Phi(B))$.

At the beginning observe that $x_0 \notin \Phi(X \setminus C)$. Indeed, otherwise we would have

$$x_0 \in \Phi(X \setminus C) \cap \Phi(C \cup B) = \Phi((X \setminus C) \cap (C \cup B)) = \Phi(B \setminus C) \subset \Phi(B),$$

which is impossible.

Lemma 2.7 gives that there is an operator $\Phi_* \in \mathcal{LDO}$ such that $\Phi \preceq \Phi_*$ and $\Phi \neq \Phi_*$, contrary to our assumption. This contradiction ends the proof of this implication.

Now, we will prove the implication (*ii*) $\Rightarrow$ (*i*). Let us assume that Φ is a lifting and suppose that there is $\Phi_2 \in \mathcal{LDO}$ such that $\Phi \preceq \Phi_2$ and $\Phi \neq \Phi_2$. Then there exist $A \in \mathcal{S}$ and $a \in X$ such that $a \in \Phi_2(A) \setminus \Phi(A)$. Since Φ is a lifting, we have $a \in \Phi(X \setminus A) \subset \Phi_2(X \setminus A)$. Therefore $a \in \Phi_2(X \setminus A) \cap \Phi_2(A) = \Phi_2(\emptyset)$, a contradiction. □

3. The Families $\mathcal{ALDO}$ and $\mathfrak{T}^a$.

There exists a measurable space $\langle X, \mathcal{S}, \mathcal{J}\rangle$ having the hull property such that $\mathcal{LDO} = \emptyset$ and, in consequence, $\mathfrak{T} = \emptyset$ (see [1]). For the family $\mathcal{ALDO}$ the situation is different.

Property 3.1. *For any $\langle X, \mathcal{S}, \mathcal{J}\rangle$ we have $\mathcal{ALDO} \neq \emptyset$ and $\mathfrak{T}^a \neq \emptyset$.*

Proof. To see that $\mathcal{ALDO} \neq \emptyset$ it is enough to consider the following operator:

$$\Phi_0(A) = \begin{cases} \emptyset & \text{if } A \in \mathcal{J}, \\ X & \text{if } X \setminus A \in \mathcal{J}. \end{cases} \tag{1}$$

One can prove at once that Φ_0 is an almost-lower density operator. The fact that $\mathfrak{T}^a \neq \emptyset$ is obvious. □

As in the case of lower density operators we have that for $\Phi_1, \Phi_2 \in \mathcal{ALDO}$ generating topologies $\mathcal{T}_{\Phi_1}, \mathcal{T}_{\Phi_2}$, respectively, if $\Phi_1 \leq \Phi_2$ then $\mathcal{T}_{\Phi_1} \subset \mathcal{T}_{\Phi_2}$. However, for operators from family $\mathcal{ALDO}$ the converse theorem is not true. For these reasons, in the first part of this section we focus on the family $\mathcal{ALDO}$ ordered by the relation $\leq$ and in the next part we focus on the family $\mathfrak{T}^a$ ordered by the relation $\subset$.

We start with considerations connected with maximal element in the family $\mathcal{ALDO}$ ordered by $\leq$. Lemma 2.7 implies immediately the following fact

Lemma 3.2. *Let $\langle X, \mathcal{S}, \mathcal{J}\rangle$ be a measurable space and $\Phi \in \mathcal{ALDO}$. If there are a set $C \in \mathcal{S}$ and $x_0 \in X$ such that $x_0 \notin \Phi(C) \cup \Phi(X \setminus C)$, then Φ is not a maximal element in the family $\mathcal{ALDO}$ ordered by the relation $\leq$.*

Moreover, we have

Theorem 3.3. *Let $\langle X, \mathcal{S}, \mathcal{J}\rangle$ be a measurable space. If $\Phi \in \mathcal{ALDO}$ is a maximal element in the family $\mathcal{ALDO}$ ordered by the relation $\leq$ then $\Phi \in \mathcal{LDO}$.*

Proof. Suppose that $\Phi \notin \mathcal{LDO}$, so there is $A \in \mathcal{S}$ such that $A \setminus \Phi(A) \notin \mathcal{J}$. Clearly, $A \cap \Phi(X \setminus A) \in \mathcal{J}$. Hence one can find $x_0 \notin \Phi(A) \cup \Phi(X \setminus A)$. By virtue of Lemma 2.7 Φ is not a maximal element in the family $\mathcal{ALDO}$ ordered by the relation $\leq$, a contradiction. □

The next theorem shows, among others, the connection between the existence of a maximal element in the family $\mathcal{ALDO}$ in some space $\langle X, \mathcal{S}, \mathcal{J}\rangle$ and the existence of a lower density operator in $\langle X, \mathcal{S}, \mathcal{J}\rangle$.

Theorem 3.4. *Let $\langle X, \mathcal{S}, \mathcal{J}\rangle$ be a measurable space. The following conditions are equivalent:*

(i) *there is a maximal element in the family $\mathcal{ALDO}$ ordered by the relation $\leq$;*
(ii) *$\mathcal{LDO} \neq \emptyset$;*
(iii) *$\mathcal{LLDO} \neq \emptyset$.*

Proof. The implication $(i) \Rightarrow (ii)$ is a consequence of Theorem 3.3. In [14] one can find that for any measurable space $\langle X, \mathcal{S}, \mathcal{J}\rangle$ if $\mathcal{LDO} \neq \emptyset$ then $\mathcal{LLDO} \neq \emptyset$.

Now, assume that $\Phi \in \mathcal{LLDO}$. By Theorem 2.8 we get that Φ is a a maximal element in the family $\mathcal{LDO}$ ordered by the relation $\leq$. We see at once that Φ is a maximal element in the family $\mathcal{ALDO}$ ordered by the relation $\leq$. Indeed, suppose that there exists $\Phi_1 \in \mathcal{ALDO}$ such that $\Phi \leq \Phi_1$ and $\Phi \neq \Phi_1$. Thus $\Phi(A) \subset \Phi_1(A)$ for any $A \in \mathcal{S}$. Therefore $A \setminus \Phi_1(A) \subset A \setminus \Phi(A) \in \mathcal{J}$. This gives that $\Phi_1 \in \mathcal{LDO}$, which is impossible. □

As we mentioned earlier for any measurable space $\langle X, \mathcal{S}, \mathcal{J}\rangle$ such that $\mathcal{LDO} \neq \emptyset$ we have $\mathcal{LLDO} \neq \emptyset$. From this fact and Theorem 2.8 we obtain that if $\mathcal{LDO} \neq \emptyset$ then there exists a maximal element in the family $\mathcal{LDO}$ ordered by the relation $\leq$. It is worth adding that this property is not true for the family $\mathcal{ALDO}$. Indeed, Theorem 3.4 and the fact that there exists a measurable space $\langle X, \mathcal{S}, \mathcal{J}\rangle$ such that $\mathcal{LDO} = \emptyset$, imply that there is a measurable space $\langle X, \mathcal{S}, \mathcal{J}\rangle$ such that there is no maximal element in the family $\mathcal{ALDO}$. However, as in the case of the family $\mathcal{LDO}$, we have

Theorem 3.5. *Let $\langle X, \mathcal{S}, \mathcal{J}\rangle$ be a measurable space. There exists the greatest (with respect to $\leq$) element in the family $\mathcal{ALDO}$ if and only if $\mathcal{S} = \mathcal{S}_0$.*

Now, we focus on the family $\mathfrak{T}^a$ ordered by the relation $\subset$. The following lemma, useful in the next part of this paper, is easily seen.

Lemma 3.6. *Let $\langle X, \mathcal{S}, \mathcal{J}\rangle$ be a measurable space. If there is $A \subset X$ such that $A \in S \setminus \mathcal{J}$ and $X \setminus A \in S \setminus \mathcal{J}$ then the operator Φ given by the formula*

$$\Phi(B) = \begin{cases} X & \text{if } B \sim X; \\ A & \text{if } \neg(B \sim X) \wedge A \underset{\sim}{\subseteq} B; \\ \emptyset & \text{if } \neg(B \sim X) \wedge \neg(A \underset{\sim}{\subseteq} B) \end{cases}$$

for $B \in \mathcal{S}$, belongs to $\mathcal{ALDO}$.

As in the case of the family $\mathcal{LDO}$ we obtain

Theorem 3.7. *For any measurable space $\langle X, \mathcal{S}, \mathcal{J}\rangle$ the following conditions are equivalent:*

(i) $\mathcal{S} \neq \mathcal{S}_0$;

(ii) $\sigma(\bigcup \mathfrak{T}^a) = 2^X$.

Proof. Assume (i). Thus there is $A \subset X$ such that $A \in \mathcal{S} \setminus \mathcal{J}$ and $X \setminus A \in \mathcal{S} \setminus \mathcal{J}$. Let $x \in X$. There are two possibilities.

The first one: $x \in A$. Put for any $B \in \mathcal{S}$

$$\Phi_1(B) = \begin{cases} X & \text{if } B \sim X; \\ A & \text{if } \neg(B \sim X) \wedge A \underset{\sim}{\subseteq} B; \\ \emptyset & \text{if } \neg(B \sim X) \wedge \neg(A \underset{\sim}{\subseteq} B) \end{cases}$$

and

$$\Phi_2(B) = \begin{cases} X & \text{if } B \sim X; \\ X \setminus A \cup \{x\} & \text{if } \neg(B \sim X) \wedge (X \setminus A \cup \{x\}) \underset{\sim}{\subseteq} B; \\ \emptyset & \text{if } \neg(B \sim X) \wedge \neg((X \setminus A \cup \{x\}) \underset{\sim}{\subseteq} B). \end{cases}$$

Lemma 3.6 gives that $\Phi_1, \Phi_2 \in \mathcal{ALDO}$. Moreover, $A \in \mathcal{T}_{\Phi_1}$ and $X \setminus A \cup \{x\} \in \mathcal{T}_{\Phi_2}$. Therefore $\{x\} \in \sigma(\bigcup \mathfrak{T}^a)$.

The second one: $x \in X \setminus A$. We now apply the above argument again, with A replaced by $X \setminus A$, to obtain $\{x\} \in \sigma(\bigcup \mathfrak{T}^a)$.

Finally, we proved that $\{x\} \in \sigma(\bigcup \mathfrak{T}^a)$ for any $x \in X$, which gives (ii). The converse implication is obvious. □

Taking into account the above theorem, we can immediately prove the following property.

Theorem 3.8. *Let $\langle X, \mathcal{S}, \mathcal{J}\rangle$ be a measurable space. There is the largest almost-abstract density topology in $\mathfrak{T}^a$ ordered by the relation $\subset$ if and only if $\mathcal{S} = \mathcal{S}_0$.*

4. The Families $\mathcal{SLDO}$ and $\mathfrak{T}^s$.

Conversely to lower and almost-lower density operators, semi-lower density operators on $\langle X, \mathcal{S}, \mathcal{J}\rangle$ with the hull property do not have to generate topology.

Example 4.1. Let $A, B \in \mathcal{L} \setminus \mathbb{L}$ and $A \cap B = \emptyset$. Define Φ as follows:

$$\Phi(C) = \begin{cases} \mathbb{R} & \text{if } C \sim \mathbb{R}; \\ A \cup B & \text{if } A \underset{\sim}{\subseteq} C; \\ \emptyset & \text{if } \neg(B \sim \mathbb{R}) \wedge \neg(A \underset{\sim}{\subseteq} C) \end{cases}$$

for any $C \in \mathcal{L}$. It is easy to check that $\Phi \in \mathcal{SLDO}(\langle \mathbb{R}, \mathcal{L}, \mathbb{L}\rangle)$. On the other hand the family $\mathcal{T}_\Phi$ is not a topology. To see that it is sufficient to consider a set $D \subset B$ and $D \notin \mathcal{L}$. Then for any $d \in D$ we have $A \cup \{d\} \in \mathcal{T}_\Phi$ because $A \cup \{d\} \subset \Phi(A \cup \{d\}) = A \cup B$. Obviously, $\bigcup_{d \in D} (A \cup \{d\}) \notin \mathcal{L}$ and, in consequence, $\bigcup_{d \in D} (A \cup \{d\}) \notin \mathcal{T}_\Phi$, so $\mathcal{T}_\Phi$ is not a topology on $\mathbb{R}$.

It is worth adding that in general there exists an operator $\Phi \in \mathcal{SLDO}(\langle X, \mathcal{S}, \mathcal{J} \rangle)$ not generating a topology if and only if $\mathcal{S}_0 \neq \mathcal{S} \neq 2^X$ (see [4]).

Obviously, for any measurable space $\langle X, \mathcal{S}, \mathcal{J} \rangle$ we have $\mathfrak{T}^a \subset \mathfrak{T}^s$, so $\mathfrak{T}^s \neq \emptyset$. Moreover, analysis similar to that in the proof of Lemma 2.7 shows

Lemma 4.2. *Let $\langle X, \mathcal{S}, \mathcal{J} \rangle$ be a measurable space and $\Phi \in \mathcal{SLDO}$. If there are a set $C \in \mathcal{S}$ and $x_0 \in X$ such that $x_0 \notin \Phi(C) \cup \Phi(X \setminus C)$ then there exists an operator $\Phi_* \in \mathcal{SLDO}$ which is strictly greater than Φ with respect to $\preceq$.*

Using this lemma one can prove

Theorem 4.3. *Let $\langle X, \mathcal{S}, \mathcal{J} \rangle$ be a measurable space. An operator $\Phi \in \mathcal{SLDO}$ is a maximal element in the family $\mathcal{SLDO}$ ordered by the relation $\preceq$ if and only if*

$$\forall_{A \in \mathcal{S}} \Phi(A) \cup \Phi(X \setminus A) = X. \tag{2}$$

Proof. Let Φ be a maximal element in the family $\mathcal{SLDO}$ ordered by the relation $\preceq$. Suppose that $\Phi(A) \cup \Phi(X \setminus A) \neq X$ for some $A \in \mathcal{S}$. Thus there is $x_0 \in X \setminus (\Phi(A) \cup \Phi(X \setminus A))$. By Lemma 4.2 we get that there exists an operator $\Phi_* \in \mathcal{SLDO}$ such that $\Phi \preceq \Phi_*$ and $\Phi \neq \Phi_*$. This contradiction ends the proof of necessity.

Now assume that an operator $\Phi \in \mathcal{SLDO}$ satisfies condition (2). Suppose that Φ is not a maximal element in the family $\mathcal{SLDO}$ ordered by the relation $\preceq$. Therefore there exists $\Phi^* \in \mathcal{SLDO}$ such that $\Phi \preceq \Phi^*$ and $\Phi \neq \Phi^*$. Thus there are $A \in \mathcal{S}$ and $a \in X$ such that $a \in \Phi^*(A) \setminus \Phi(A)$. Condition (2) gives that $a \in \Phi(X \setminus A) \subset \Phi^*(X \setminus A)$. It implies that $a \in \Phi^*(X \setminus A) \cap \Phi^*(A) = \emptyset$, a contradiction. □

Evidently the operator Φ_0 defined in (1) is the smallest element in the family $\mathcal{SLDO}$ ordered by the relation $\preceq$. In the case of the greatest element we have the following property.

Theorem 4.4. *Let $\langle X, \mathcal{S}, \mathcal{J} \rangle$ be a measurable space. The following statements are equivalent:*

(i) $\mathcal{S} \neq \mathcal{S}_0$;
(ii) *there in no greatest element in the family $\mathcal{SLDO}$ ordered by the relation $\preceq$.*

Proof. Let us assume that $\mathcal{S} \neq \mathcal{S}_0$ and suppose that Φ is the greatest element in the family $\mathcal{SLDO}$ ordered by the relation $\preceq$. There exists $A \in \mathcal{S}$ such that $A \in \mathcal{S} \setminus \mathcal{J}$ and $X \setminus A \in \mathcal{S} \setminus \mathcal{J}$. Let us define Φ_* in the following way:

$$\Phi_*(B) = \begin{cases} X & \text{if } B \sim X; \\ A & \text{if } B \subsetneq A \wedge \neg(A \sim X); \\ X \setminus A & \text{if } B \subsetneq X \setminus A \wedge \neg((X \setminus A) \sim X); \\ \emptyset & \text{if } \neg(B \sim X) \wedge \neg(B \subsetneq A) \wedge \neg(B \subsetneq X \setminus A) \end{cases}$$

for $B \in \mathcal{S}$. One could check at once that $\Phi_* \in \mathcal{SLDO}$. By Theorem 4.3 we obtain $\Phi(A) \cup \Phi(X \setminus A) = X$. Therefore $\Phi(A) \neq \emptyset$ or $\Phi(X \setminus A) \neq \emptyset$. Without loss of generality we can assume that $\Phi(A) \neq \emptyset$. Let us fix $a \in \Phi(A)$. Since $\Phi_* \preceq \Phi$ we get $\Phi_*(X \setminus A) = X \setminus A \subset \Phi(X \setminus A)$ and, in consequence, $\Phi(X \setminus A) \neq \emptyset$. Let us fix $b \in \Phi(X \setminus A)$. Putting

$$\Phi_{**}(B) = \begin{cases} \Phi(B) \cup \{b\} & \text{if } a \in \Phi(B); \\ \Phi(B) \setminus \{b\} & \text{if } a \notin \Phi(B) \end{cases}$$

for every $B \in \mathcal{S}$ we obtain an operator Φ_{**} from $\mathcal{SLDO}$. Moreover, we get $\Phi(A) \subset \Phi_{**}(A)$ and $\Phi(A) \neq \Phi_{**}(A)$, which contradicts the fact that Φ is the greatest element in $\mathcal{SLDO}$ ordered by the relation $\preceq$.

The implication (ii)⇒(i) is obvious because if $\mathcal{S} = \mathcal{S}_0$ then $\mathcal{SLDO} = \{\Phi_0\}$, where Φ_0 is defined in (1). □

Theorem 4.5. *Let $\langle X, \mathcal{S}, \mathcal{J} \rangle$ be a measurable space. There exists a maximal element in the family $\mathcal{SLDO}$ ordered by the relation $\preceq$.*

Proof. To find a maximal element in $\mathcal{SLDO}$ we apply here a method described in the acknowledgment section of the paper [14]. Let $\Phi \in \mathcal{SLDO}$ and let $\mathcal{F}_x = \{A \in \mathcal{S} : x \in \Phi(A)\}$ for $x \in X$. Then $\mathcal{F}_x$ is a filter on $\langle X, \mathcal{S}, \mathcal{J} \rangle$ for any $x \in X$. Consider, for any $x \in \mathbb{R}$, a maximal filter $\mathfrak{F}_x$ containing the filter $\mathcal{F}_x$. Let Ψ be the operator defined on $\mathcal{S}$ in the following way:

$$\underset{A \in \mathcal{S}}{\forall} \Psi(A) = \{x \in X : A \in \mathfrak{F}_x\}.$$

Then $\Psi \in \mathcal{SLDO}$. Moreover, $\Psi(A) \cup \Psi(X \setminus A) = X$ for every $A \in \mathcal{S}$. Thus, by Theorem 4.3, Ψ is a maximal element in the family $\mathcal{SLDO}$ ordered by the relation $\preceq$. $\square$

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