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Resolvents of Functions of Operators with Hilbert-Schmidt Hermitian Components

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Abstract. Let $\mathcal{H}$ be a separable Hilbert space with the unit operator I. We derive a sharp norm estimate for the operator function $(\lambda I - f(A))^{-1}$ ($\lambda \in \mathbb{C}$), where A is a bounded linear operator in $\mathcal{H}$ whose Hermitian component $(A - A^*)/2i$ is a Hilbert-Schmidt operator and $f(z)$ is a function holomorphic on the convex hull of the spectrum of A. Here A^* is the operator adjoint to A. Applications of the obtained estimate to perturbations of operator equations, whose coefficients are operator functions and localization of spectra are also discussed.

1. Introduction and statement of the main result

Throughout the present paper $\mathcal{H}$ is a complex separable Hilbert space with a scalar product $(.,.)$, the norm $\|.\| = \sqrt{(.,.)}$ and unit operator I; $\mathcal{L}(\mathcal{H})$ denotes the space of bounded operators in $\mathcal{H}$. For an $A \in \mathcal{L}(\mathcal{H})$, $\|A\|$ is the operator norm, A^* is the adjoint operator, $\sigma(A)$ is the spectrum, $r_s(A)$ is the spectral radius; $R_\lambda(A) = (A - \lambda I)^{-1}$ ($\lambda \notin \sigma(A)$) is the resolvent; $co(A)$ is the convex hull of $\sigma(A)$; SN_2 is the Hilbert-Schmidt ideal with the norm $N_2(A) = (\text{Trace } A^*A)^{1/2}$.

It is assumed that

$$A \in \mathcal{L}(\mathcal{H}) \text{ and } A_I := (A - A^*)/2i \in SN_2. \tag{1.1}$$

Numerous integral operators satisfy this condition. Let f be a scalar-valued function, which is analytic on a neighborhood of $\sigma(A)$. Let contour L consist of a finite number of rectifiable Jordan curves, oriented in the positive sense customary in the theory of complex variables. Suppose that L is the boundary of an open set $M \supset \sigma(A)$ and $M \cup L$ is contained in the domain of analyticity of f. As usually, $f(A)$ is defined by the equality

$$f(A) = -\frac{1}{2\pi i} \int_L f(\lambda) R_\lambda(A) d\lambda. \tag{1.2}$$

In the present paper we derive a sharp norm estimate for the operator function $(\lambda I - f(A))^{-1}$ ($\lambda \notin \sigma(f(A))$. Recall that one of the first norm estimates for functions of non-normal matrices has been established by I.M. Gel'fand and G.E. Shilov [8] in connection with their investigations of partial differential equations.

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However that estimate is not sharp, it is not attained for any matrix. In the paper [9] the author has derived a sharp estimate for matrix-valued functions regular on $co(A)$. That estimate is attained for normal matrices. The results of the paper [9] were generalized to various operators, cf. [14]. Obviously, $(\lambda - f(z))^{-1}$ can be nonregular on $co(A)$. Our approach is based on a combined usage of the results from [9] with the estimate for the nilpotent part of $f(A)$ derived below.

We also discuss applications of the obtained estimate to solutions of operator equations whose coefficients are operator functions (function operator equations) and to localization of spectra.

Operator (in particular, matrix) equations naturally arise in various applications, in particular, in the theories of differential and difference equations, and control theory, cf. [5, 18, 20]. The theory of operator equations is well developed. About the classical results see the just cited books, the recent results can be found, in particular, in [1–3, 6, 17, 21, 24–26] and references given therein.

To formulate the results introduce the quantity

$$g_I(A) := \sqrt{2}\left[N_2^2(A_I) - \sum_{k=1}^{\infty}(\Im\ \lambda_k(A))^2\right]^{1/2},$$

where $\lambda_k(A)$ $(k = 1, 2, ...)$ are the eigenvalues of A taken with their multiplicities. Obviously, $g_I(A) \leq \sqrt{2}N_2(A_I)$. Assume that f is regular on a neighborhood of $co(A)$ and put

$$g_I(f, A) := \sum_{j=1}^{\infty} \sup_{\lambda\in co(A)} |f^{(j)}(\lambda)| \frac{g_I^j(A)}{j!\sqrt{(j-1)!}}. \tag{1.3}$$

For example, if $f(z) = z^\nu$ with a positive integer ν, then $g_I(f, A) = g_I(A^\nu)$. Here

$$g_I(A^\nu) = \sum_{j=1}^{\nu} r_s^{\nu-j}(A)\frac{\nu(\nu-1)...(\nu-j+1)g_I^j(A)}{j!\sqrt{(j-1)!}}. \tag{1.4}$$

If A is normal, then $g_I(A) = 0$ and therefore $g_I(f, A) = 0$. Now we are in a position to formulate our main result.

Theorem 1.1. *Let conditions (1.1) hold, f be regular on a neighborhood of $co(A)$ and*

$$\rho(f(A), \lambda) := \inf_{\mu\in\sigma(A)} |f(\mu) - \lambda| > 0.$$

Then

$$\left\|(f(A) - \lambda I)^{-1}\right\| \leq \sum_{k=0}^{\infty} \frac{g_I^k(f, A)}{\sqrt{k!}\rho^{k+1}(f(A), \lambda)}. \tag{1.5}$$

The proof of this theorem is divided into a series of lemmas which are presented in the next section. The theorem is sharp. Inequality (1.5) becomes the equality, if A is normal.

By the Schwarz inequality for constants $c \in (0, 1)$ and $a \geq 0$ we have

$$\sum_{k=0}^{\infty}\frac{a^k}{\sqrt{k!}} = \sum_{k=0}^{\infty}\frac{(ac)^k}{c^k\sqrt{k!}} \leq \left(\sum_{j=0}^{\infty}c^{2j}\sum_{k=0}^{\infty}\frac{a^{2k}}{c^{2k}k!}\right)^{1/2} = \frac{1}{(1-c^2)^{1/2}}e^{a^2/(2c^2)}.$$

Thus (1.5) implies

$$\|(f(A) - \lambda I)^{-1}\| \leq \frac{1}{(1-c^2)^{1/2}\rho(f(A), \lambda)}\exp\left[\frac{g_I^2(f, A)}{2c^2\rho^2(f(A), \lambda)}\right] \quad (\lambda \notin \sigma(f(A))).$$

In particular, taking $c^2 = 1/2$ we obtain

$$\|(f(A) - \lambda I)^{-1}\| \leq \frac{\sqrt{2}}{\rho(f(A), \lambda)}\exp\left[\frac{g_I^2(f, A)}{\rho^2(f(A), \lambda)}\right] \quad (\lambda \notin \sigma(f(A))). \tag{1.6}$$

2. Proof of Theorem 1.1

2.1. Preliminaries

In this subsection $\mathcal{H} = \mathbb{C}^n$-the n-dimensional Euclidean space ($n < \infty$) and A is an $n\times n$-matrix. Introduce the quantity (the departure from normality)

$$g(A) := [N_2^2(A) - \sum_{k=1}^{n} |\lambda_k(A)|^2]^{1/2}. \tag{2.1}$$

Recall that $\lambda_k(A)$ ($k = 1, 2, ..., n$) are the eigenvalues of A taken with their multiplicities. The following properties of $g(A)$ are checked in [14, Section 3.1]:

$$g(A) = g_I(A), \tag{2.2}$$

$g^2(A) \le N_2^2(A) - |\text{Trace } A^2|$, if A is normal, then $g(A) = 0$. In the n-dimensional case

$$g_I(A) = \sqrt{2}\left[N_2^2(A_I) - \sum_{k=1}^{n}(\Im\ \lambda_k(A))^2\right]^{1/2} \le \sqrt{2}N_2(A_I).$$

About other properties of $g(A)$ see [14, Section 3.1].

By Schur's theorem [7], there is an orthogonal normal (Schur's) basis $\{e_k\}_{k=1}^n$, in which A has the triangular representation

$$Ae_k = \sum_{j=1}^{k} a_{jk}e_j \text{ with } a_{jk} = (Ae_k, e_j) \ \ (k = 1, ..., n).$$

Schur's basis is not unique. We can write

$$A = D + V \ \ (\sigma(A) = \sigma(D)) \tag{2.3}$$

with a normal (diagonal) operator D defined by $De_j = \lambda_j(A)e_j$ ($j = 1, ..., n$) and a nilpotent operator V defined by

$$Ve_k = \sum_{j=1}^{k-1} a_{jk}e_j \ \ (k = 2, ..., n),\ Ve_1 = 0.$$

Equality (2.3) is called *the triangular representation of A; D and V are called the diagonal part and nilpotent part of A, respectively*. Put

$$P_j = \sum_{k=1}^{j}(., e_k)e_k \ \ (j = 1, ..., n),\ \ P_0 = 0.$$

So $0 = P_0\mathbb{C}^n \subset P_1\mathbb{C}^n \subset ... \subset P_n\mathbb{C}^n = \mathbb{C}^n$, $\dim\,(P_k - P_{k-1})\mathbb{C}^n = 1$,

$$AP_k = P_kAP_k;\ \ VP_k = P_{k-1}VP_k \quad (k = 1, ..., n) \text{ and } D = \sum_{k=1}^{n}\lambda_k(A)\Delta P_k,$$

where $\Delta P_k = P_k - P_{k-1}$ ($k = 1, ..., n$). Since $f(A)$ commutes with A, $f(A)$ and A have the joint Schur basis. Besides the diagonal part of $f(A)$ is $f(D) : f(D)e_j = f(\lambda_j(A))e_j$ ($j = 1, ..., n$) and its nilpotent part is $V_{f,A} = f(A) - f(D)$. In addition, $f(A)$, $V_{f,A}$ and $f(D)$ have joint invariant subspaces and

$$f(D) = \sum_{k=1}^{n} f(\lambda_k(A))\Delta P_k.$$

Moreover, according to [14, Lemma 3.1] $N_2(V) = g(A)$ and consequently,

$$N_2(V_{f,A}) = g(f(A)), \tag{2.4}$$

where

$$g(f(A)) := (N_2^2(f(A)) - N_2^2(f(D)))^{1/2} = (N_2^2(f(A)) - \sum_{k=1}^{n} |f(\lambda_k(A))|^2)^{1/2}.$$

Denote by $|V|_e$ the matrix whose entries in $\{e_k\}$ are the absolute values of the entries of V in the Schur basis. That is,

$$|V|_e = \sum_{k=2}^{n} \sum_{j=1}^{k-1} |a_{jk}|(., e_k)e_j,$$

where $a_{jk} = (Ae_k, e_j)$. Put

$$I_{j_1 \ldots j_{k+1}} = \frac{(-1)^{k+1}}{2\pi i} \int_L \frac{f(\lambda)d\lambda}{(\lambda_{j_1} - \lambda) \ldots (\lambda_{j_{k+1}} - \lambda)} \quad (\lambda_j = \lambda_j(A)).$$

Lemma 2.1. *Let $A \in \mathbb{C}^{n\times n}$ and f be holomorphic in a Jordan domain containing $\sigma(A)$. Then*

$$N_2(V_{f,A}) \le N_2(V) \sum_{k=1}^{n-1} J_k \| \, |V|_e^{k-1} \|,$$

where $J_k = \max\{|I_{j_1 \ldots j_{k+1}}| : 1 \le j_1 < \ldots < j_{k+1} \le n\}$.

Proof. Since $R_\lambda(D)V$ is a nilpotent matrix, $(R_\lambda(D)V)^n = 0$. By (2.3) we have

$$R_\lambda(A) = (D + V - \lambda)^{-1} = (I + (D - \lambda)^{-1}V)^{-1}(D - \lambda)^{-1} = \sum_{k=0}^{n-1} (-1)^k (R_\lambda(D)V)^k R_\lambda(D).$$

Hence according (1.2)

$$f(A) - f(D) = -\frac{1}{2\pi i} \int_L f(\lambda)(R_\lambda(A) - R_\lambda(D))d\lambda = \sum_{k=1}^{n-1} B_k, \tag{2.5}$$

where

$$B_k = (-1)^{k+1} \frac{1}{2\pi i} \int_L f(\lambda)(R_\lambda(D)V)^k R_\lambda(D)d\lambda.$$

Since D is a diagonal matrix with respect to $\{e_k\}$ and its diagonal entries are the eigenvalues of A, we can write

$$R_\lambda(D) = \sum_{j=1}^{n} \frac{\Delta P_j}{\lambda_j(A) - \lambda}.$$

Recall that $\Delta P_k = (., e_k)e_k$. Thus,

$$B_k = \sum_{j_{k+1}=1}^{n} \sum_{j_k=1}^{n} \ldots \sum_{j_2=1}^{n} \sum_{j_1=1}^{n} \Delta P_{j_1} V \Delta P_{j_2} V \Delta P_{j_3} \ldots \Delta P_{j_k} V \Delta P_{j_{k+1}} I_{j_1 j_2 \ldots j_{k+1}}.$$

In addition, $\Delta P_j V \Delta P_k = 0$ for $j \ge k$. Consequently,

$$B_k = \sum_{j_{k+1}=1}^{n} \sum_{j_k=1}^{j_{k+1}-1} \ldots \sum_{j_2=1}^{j_3-1} \sum_{j_1=1}^{j_2-1} \Delta P_{j_1} V \Delta P_{j_2} V \Delta P_{j_3} \ldots \Delta P_{j_k} V \Delta P_{j_{k+1}} I_{j_1 j_2 \ldots j_{k+1}}.$$

Let $|B_k|_e$ be the operator whose entries in basis $\{e_k\}$ are the absolute values of the entries of B_k in that basis. Then

$$|B_k|_e \le J_k \sum_{j_{k+1}=1}^{n} \sum_{j_k=1}^{j_{k+1}-1} \dots \sum_{j_2=1}^{j_3-1} \sum_{j_1=1}^{j_2-1} \Delta P_{j_1} |V|_e \Delta P_{j_2} |V|_e \dots |V|_e \Delta P_{j_{k+1}}$$
$$= J_k P_{n-k} |V|_e P_{n-k+1} |V|_e P_{n-k+2} \dots P_{n-1} |V|_e,$$

where the inequality is understood in the entry-wise sense. But

$$P_{n-k}|V|_e P_{n-k+1}|V|_e P_{n-k+2} \dots P_{n-1}|V|_e = |V|_e P_{n-k+1}|V|_e P_{n-k+2} \dots P_{n-1}|V|_e$$
$$= |V|_e^2 P_{n-k+2} \dots P_{n-1}|V|_e \le |V|_e^k.$$

Thus $|B_k|_e \le J_k |V|_e^k$. Since $N_2(|V|_e) = N_2(V)$ and $N_2(|B|_e) = N_2(B)$, we get

$$N_2(B_k) \le J_k N_2(V) \| |V|_e^{k-1} \|.$$

Now (2.5) implies the required result. □

Due to Lemma 3.4 [14]

$$\| |V|_e^k \| \le \frac{N_2^k(|V|_e)}{\sqrt{k!}} \quad (k = 1, 2, ..., n-1). \tag{2.6}$$

Since $N_2(|V|_e) = N_2(V) = g(A)$, by the previous lemma

$$N_2(V_{f,A}) \le \sum_{k=1}^{n-1} \frac{g^k(A) J_k}{\sqrt{(k-1)!}}. \tag{2.7}$$

If f is holomorphic on a neighborhood $co(A)$, then by [14, Lemma 3.8]

$$J_k \le \frac{1}{k!} \sup_{\lambda \in co(A)} |f^{(k)}(\lambda)|, k = 1, ..., n-1. \tag{2.8}$$

Now (2.7) implies

Lemma 2.2. *One has $g(f(A)) = N_2(V_{f,A}) \le g_{A,f}$, where*

$$g_{f,A} := \sum_{j=1}^{n-1} \sup_{\lambda \in co(A)} |f^{(j)}(\lambda)| \frac{g^j(A)}{j! \sqrt{(j-1)!}}.$$

Furthermore, due to [14, Theorem 3.2],

$$\|R_\lambda(A)\| \le \sum_{k=0}^{n-1} \frac{g^k(A)}{\sqrt{k!} \rho^{k+1}(A, \lambda)} \quad (\lambda \notin \sigma(A)), \tag{2.9}$$

where $\rho(A, \lambda) = \min_k |\lambda - \lambda_k(A)|$. From (2.9) it follows

$$\|(f(A) - \lambda I)^{-1}\| \le \sum_{k=0}^{n-1} \frac{g^k(f(A))}{\sqrt{k!} \rho^{k+1}(f(A), \lambda)} \quad (\lambda \notin \sigma(f(A))). \tag{2.10}$$

Making use of (2.10) and Lemma 2.2 we arrive at

Lemma 2.3. *Let $A \in \mathbb{C}^{n \times n}$ and f be regular on a neighborhood of $co(A)$. Then*

$$\left\|(f(A) - \lambda I)^{-1}\right\| \le \sum_{k=0}^{n-1} \frac{g_{f,A}^k}{\sqrt{k!} \rho^{k+1}(f(A), \lambda)} \quad (\lambda \notin \sigma(f(A))).$$

2.2. Proof of Theorem 1.1

Let A_n be n-dimensional operator in $\mathcal{H}$. Then

$$g_I(A_n) = \sqrt{2}\left[N_2^2(A_{In}) - \sum_{k=1}^{n}(\Im\ \lambda_k(A_n))^2\right]^{1/2} \quad (A_{In} = (A_n - A_n^*)/2i).$$

Due to (2.2) $g(A_n) = g_I(A_n)$ and therefore $g_{f,A_n} = g_I(f, A_n)$. Here

$$g_I(f, A_n) = \sum_{j=1}^{n-1} \sup_{\lambda\in co(A)} |f^{(j)}(\lambda)| \frac{g_I^j(A_n)}{j!\sqrt{(j-1)!}}.$$

Lemma 2.3 implies

$$\left\|(f(A_n) - \lambda I)^{-1}\right\| \le \sum_{k=0}^{n-1} \frac{g_I^k(f, A_n)}{\sqrt{k!}\rho^{k+1}(f(A_n), \lambda)}. \tag{2.11}$$

According to [14, Lemma 10.2] (see also [13]), under condition (1.1) there is a sequence of n-dimensional operators A_n strongly converging to A, such that $\sigma(A_n) \subseteq \sigma(A)$, $n < \infty$ and by [14, Corollary 10.2] $g_I(A_n) \to g_I(A)$ (see also [13]). Hence $g_I(f, A_n) \to g_I(f, A)$ as $n \to \infty$. Now the required result follows from (2.11). □

3. Applications to operator equations

In this section we derive solution estimates for the equation

$$Xf(A) + (f(A))^*X = C, \tag{3.1}$$

where $A, C \in \mathcal{L}(\mathcal{H})$ are given, and $X \in \mathcal{L}(\mathcal{H})$ should be found. It is assumed that

$$\beta(f(A)) := \inf \Re\ \sigma(f(A)) > 0. \tag{3.2}$$

To the best of our knowledge, bounds for solutions of function operator equation (3.1) have been established only for $f(z) = z$, cf. [14] and the references given therein. In particular in the case $f(z) = z^\nu$ with a positive integer ν we obtain conditions that provide localization of the spectrum of a perturbed matrix in a certain angle. Such conditions play an essential role, in the theories of periodic differential equations [5] and fractional differential and difference equations, [4, 15, 19, 22, 23].

Lemma 3.1. *Let condition (3.2) hold. Then equation (3.1) has a unique solution X representable in the form*

$$X = \int_0^\infty e^{-f^*(A)t}Ce^{-f(A)t}dt. \tag{3.3}$$

Proof. Consider the equation $YB + B^*Y = C$ where $B, C \in \mathcal{L}(\mathcal{H})$ are given, and $Y \in \mathcal{L}(\mathcal{H})$ should be found. If $Re\ \sigma\ (B) < 0$, then as it is well known $Y = -\int_0^\infty e^{B^*t}Ce^{Bt}dt$, cf. [5, Section 1.5] (see also [14, Theorem 2.4]). Hence (3.3) follows. □

By the Parseval equality, (3.3) implies

$$(Xy, y_1) = \int_0^\infty (Ce^{f(A)t}y, e^{f(A)t}y_1)dt = \frac{1}{2\pi}\int_{-\infty}^\infty (C(f(A) + i\omega I)^{-1}y, (f(A) + i\omega I)^{-1}y_1)d\omega$$

($y, y_1 \in \mathcal{H}$). Since $\|X\| = \sup_{y,y_1\in\mathcal{H};\ \|y\|=\|y_1\|=1} |(Xy, y_1)$, we have

$$\|X\| \le \|C\|\frac{1}{2\pi}\int_{-\infty}^\infty \|(f(A) + i\omega I)^{-1}\|^2 d\omega. \tag{3.4}$$

Hence, under the hypothesis of Theorem 1.1 we get

$$\|X\| \le \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\sum_{k=0}^{\infty}\frac{g_I^k(f,A)}{\sqrt{k!}\rho^{k+1}(f(A),-i\omega)}\right)^2 d\omega. \tag{3.5}$$

Put $\zeta(f(A)) := \sup_{\mu\in\sigma(A)}|\Im\, f(\mu)|$. Then $\rho(f(A),-i\omega) \ge w(f(A),\omega)$, where

$$w(f(A),\omega) = \begin{cases} \beta(f(A)) & \text{if } |\omega| \le \zeta(f(A)), \\ (\beta^2(f(A)) + (|\omega| - \zeta(f(A)))^2)^{1/2} & \text{if } |\omega| \ge \zeta(f(A)). \end{cases}$$

We thus arrive at.

Corollary 3.2. *Let conditions (1.1) and (3.2) hold and f be regular on a neighborhood of $co(A)$. Then $\|X\| \le \chi(f(A))\|C\|$, where*

$$\chi(f(A)) := \frac{1}{\pi}\int_0^{\infty}\left(\sum_{k=0}^{\infty}\frac{g_I^k(f,A)}{\sqrt{k!}w^{k+1}(f(A),\omega)}\right)^2 d\omega. \tag{3.6}$$

Similarly we can estimate the unique solution to the equation

$$Y_0 - f(A)Y_0 f^*(A) = C, \tag{3.7}$$

under the condition

$$r_s(f(A)) = \sup_{z\in\sigma(A)}|f(z)| < 1, \tag{3.8}$$

making use of the representation of the unique solution Y_0 of (3.7):

$$Y_0 = \frac{1}{2\pi}\int_0^{2\pi}(Ie^{-i\omega} - f(A))^{-1}C(e^{i\omega}I - f^*(A))^{-1}d\omega. \tag{3.9}$$

To obtain this representation it is enough to consider the equation $Y_1 - BY_1B^* = C$, whose unique solution Y_1 is representable as

$$Y_1 = \frac{1}{2\pi}\int_0^{2\pi}(Ie^{-i\omega} - B)^{-1}C(e^{i\omega}I - B^*)^{-1}d\omega,$$

provided $r_s(B) < 1$, cf. [14, Lemma 2.2] and references therein. Hence (3.9) follows. Now we can directly apply Theorem 1.1 to (3.9).

4. Spectrum perturbations

Let $A, \tilde{A} \in \mathcal{L}(\mathcal{H})$ and condition (3.2) hold. In this section we derive the conditions that provide the relation

$$\beta(f(\tilde{A})) = \inf \Re\sigma(f(\tilde{A})) > 0. \tag{4.1}$$

Furthermore, due to the Lyapunov theorem (see [5]) an operator B satisfies the inequality $\beta(B) > 0$ if and only if for some positive definite operator Z, the operator $ZB + B^*Z > 0$, i.e. positive definite. Take $B = f(\tilde{A})$. Then we have (4.1), provided

$$(f(\tilde{A}))^*X_L + X_L f(\tilde{A}) > 0, \tag{4.2}$$

where X_L is a solution to the equation

$$(f(A))^*X_L + X_L f(A) = 2I. \tag{4.3}$$

With $\Delta f = f(\tilde{A}) - f(A)$, (4.3) gives

$$(f(A) + \Delta f)^*X_L + X_L(f(A) + \Delta f) = 2I + \Delta f^*X_L + X_L\Delta f.$$

According to (4.2) we arrive at

Lemma 4.1. *Let f be holomorphic on a neighborhood of $\sigma(A) \cup \sigma(\tilde{A})$ and the conditions (3.2) and*

$$\|X_L\|\|f(\tilde{A}) - f(A)\| < 1 \tag{4.4}$$

hold. Then inequality (4.1) is valid.

Let condition (1.1) hold. Then by Corollary 3.2 $\|X_L\| \leq 2\chi(f(A))$, provided f is holomorphic a neighborhood of $co(A)$. Now Lemma 4.1 implies

Corollary 4.2. *Let f be holomorphic on a neighborhood of $co(A) \cup \sigma(\tilde{A})$ and the conditions (1.1), (3.2) and*

$$2\|f(A) - f(\tilde{A})\|\chi(f(A)) < 1 \tag{4.5}$$

hold. Then inequality (4.1) is valid.

For example, let $f(z) = z^\nu$ with an integer $\nu > 0$, and let the spectrum of A lie inside the angle

$$S_\nu = \{z \in \mathbb{C} : |\arg\ z| < \pi/(2\nu)\},$$

then we have $|\arg\ z^\nu| < \pi/2$, i.e. condition (3.2) holds in the form

$$\beta(A^\nu) = \inf\ \Re\sigma(A^\nu) > 0. \tag{4.6}$$

In addition, $\zeta(f(A)) = \zeta(A^\nu) = \sup_{\mu\in\sigma(A)} |\Im\mu^\nu(A)|$ and

$$w(f(A), \omega) = w(A^\nu, \omega) := \begin{cases} \beta(A^\nu) & \text{if } |\omega| \leq \zeta(A^\nu), \\ \beta^2(A^\nu) + (|\omega| - \zeta(A^\nu))^2)^{1/2} & \text{if } |\omega| \geq \zeta(A^\nu). \end{cases}$$

In this case $g_I(f, A)$ is defined by (1.4). So $\chi(f(A)) = \chi(A^\nu)$, where

$$\chi(A^\nu) = \frac{1}{\pi} \int_0^\infty \left(\sum_{k=0}^\infty \frac{g_I^k(A^\nu)}{\sqrt{k!}w^{k+1}(A^\nu, \omega)} \right)^2 d\omega$$

and condition (4.5) takes the form

$$2\|A^\nu - \tilde{A}^\nu\|\chi(A^\nu) < 1. \tag{4.7}$$

Since for any $\mu(A) \in \sigma(A)$ we have $\mu^\nu(A) = \mu(A^\nu) \in \sigma(A^\nu)$ and $\mu(A) = \sqrt[\nu]{\mu(A^\nu)}$, where *the principal branch of the root is taken*, making use of Corollary 4.2 we obtain

Corollary 4.3. *Let $\sigma(A) \subset S_\nu$ and conditions (1.1) and (4.7) hold. Then also $\sigma(\tilde{A}) \subset S_\nu$.*

Note that

$$A^\nu - \tilde{A}^\nu = \sum_{k=0}^{\nu-1} A^{\nu-k-1}(A - \tilde{A})\tilde{A}^k.$$

(see also [14, Corollary 13.1]). So

$$\|A^\nu - \tilde{A}^\nu\| \leq q \sum_{k=0}^{\nu-1} \|A^{\nu-k-1}\|\|\tilde{A}^k\|, \tag{4.8}$$

where $q := \|A - \tilde{A}\|$. About norm estimates for A^k and other operator functions see [14].

Note that to apply Theorem 3.1 we need estimates for the norm of $f(A) - f(\tilde{A})$. In particular recall that

$$e^A - e^{\tilde{A}} \leq \int_0^1 e^{As}(A - \tilde{A})e^{\tilde{A}(1-s)}ds.$$

Hence,

$$\|e^A - e^{\tilde{A}}\| \leq q \int_0^1 \|e^{As}\|\|e^{\tilde{A}(1-s)}\|ds.$$

Furthermore, for an $r_0 > \max(r_s(A), r_s(\tilde{A}))$, let

$$f(z) = \sum_{k=0}^{\infty} c_k z^k \ (z \in \mathbb{C} : |z| < r_0). \text{ Then } f(A) - f(\tilde{A}) = \sum_{k=0}^{\infty} c_k (A^k - \tilde{A}^k).$$

Suppose that

$$\|A^k\| \le a_0 r_0^k, \|\tilde{A}^k\| \le \tilde{a}_0 r_0^k \ \ (k = 1, 2, ...),$$

where a_0 and $\tilde{a}_0$ are constants. Then by (4.8) $\|A^n - \tilde{A}^n\| \le q a_0 \tilde{a}_0 \ n r_0^{n-1}$ $(n = 1, 2, ...)$, and

$$\|f(A) - f(\tilde{A})\| \le q a_0 \tilde{a}_0 \sum_{k=1}^{\infty} |c_k| k r_0^{k-1}$$

provided the series converges. Additional estimates for the norms of $f(A) - f(\tilde{A})$ with non-self-adjoint operators A and $\tilde{A}$ can be found in [14, Chapter 13].

5. Additional perturbation results

For two operators A and $\tilde{A}$, the spectral variation of $\tilde{A}$ with respect to A is defined by

$$sv_A(\tilde{A}) := \sup_{t \in \sigma(\tilde{A})} \inf_{s \in \sigma(A)} |t - s|.$$

Let f be holomorphic on a neighborhood of $\sigma(A) \cup \sigma(\tilde{A})$. Following the just pointed definition, the quantity

$$sv_{f(A)}(f(\tilde{A})) := \sup_{t \in \sigma(\tilde{A})} \inf_{s \in (A)} |f(t) - f(s)|.$$

will be called the spectral variation of $f(\tilde{A})$ with respect to $f(A)$.

Lemma 5.1. *Let f be holomorphic on a neighborhood of the convex hull $co(A, \tilde{A})$ of $\sigma(A) \cup \sigma(\tilde{A})$. Then*

$$sv_{f(A)}(f(\tilde{A})) \le \sup_{z \in co(A, \tilde{A})} |f'(z)| sv_A(\tilde{A})$$

Proof. We need the well-known relation $f(z_1) - f(z_2) = f'(c)(z_2 - z_1)$ $(z_1, z_2 \in \mathbb{C})$ for some c belonging to the segment connecting z_1 and z_2. Now take $t \in \sigma(\tilde{A})$ and $s \in \sigma(A)$ Then

$$f(t) - f(s) = f'(\theta)(t - s) \ \ (\theta \in co(A, \tilde{A})).$$

Hence

$$\sup_{t \in \sigma(\tilde{A})} \inf_{s \in (A)} |f(t) - f(s)| \le \sup_{\theta \in co(A, \tilde{A})} |f'(\theta)| \sup_{t \in \sigma(\tilde{A})} \inf_{s \in \sigma(A)} |t - s|.$$

This proves the lemma. □

Put

$$F(x) = \sum_{k=0}^{\infty} \frac{g_I^k(A)}{\sqrt{k!}} x^{k+1} \ \ (x \in \mathbb{R}).$$

Then taking in Theorem 1.1 $f(z) = z$, we have $\|(A - \lambda I)^{-1}\| \le F(1/\rho(A, \lambda))$ $(\lambda \notin \sigma(A))$. Hence by [14, Lemma 1.10] we have $sv_A(\tilde{A}) \le z(q, A)$, where $z(q, A)$ is the unique positive root of the equation $qF(1/z) = 1$. Now the previous lemma implies

Theorem 5.2. *Let condition (1.1) hold and f be holomorphic on a neighborhood of $co(A, \tilde{A})$. Then*

$$sv_{f(A)}(f(\tilde{A})) \le \sup_{z \in co(A, \tilde{A})} |f'(z)| z(q, A).$$

If A is normal, then $g_I(A) = 0$, $z(q, A) = q$ and $sv_{f(A)}(f(\tilde{A})) \le q \sup_{z \in co(A, \tilde{A})} |f'(z)|$.

6. Integral operators

Throughout this section $\tilde{A}$ is an operator in $L^2(0,1)$ defined by

$$(\tilde{A}h)(x) = \phi(x)h(x) + \int_0^1 k(x,s)h(s)ds \ (h \in L^2, x \in [0,1]), \tag{6.1}$$

where ϕ is a positive bounded measurable function and $k(x,s)$ is a Hilbert-Schmidt kernel defined on $[0,1]^2$:

$$\int_0^1 \int_0^1 |k(x,s)|^2 ds\, dx < \infty. \tag{6.2}$$

Introduce the operators S, $\hat{V}$ and W by $(Sh)(x) = \phi(x)h(x)$,

$$(\hat{V}h)(x) = \int_0^x k(x,s)h(s)ds \text{ and } (Wh)(x) = \int_x^1 k(x,s)h(s)ds \ (h \in L^2, x \in [0,1]).$$

Take $A = S + \hat{V}$ and $f(z) = z^2$. Simple calculations show that $\sigma(A) = \sigma(S)$ is real, $A_I = (\hat{V} - \hat{V}^*)/2i$ and

$$g_I^2(A) = 2N_2^2(A_I) = 2N_2^2(\hat{V} - \hat{V}^*)/4 = \frac{1}{2}|\text{Trace } (\hat{V} - \hat{V}^*)^2| = |\text{Trace } (\hat{V}\hat{V}^*) - \frac{1}{2}\text{Trace } (\hat{V}^2 + (\hat{V}^*)^2)| = N_2^2(\hat{V}).$$

According to (1.4) $g_I(f,A) = g_I(A^2) = 2r_s(A)N_2(\hat{V}) + N_2^2(\hat{V})$. Besides, $r_s(A) = \sup_x \phi(x)$, $\sigma(A^2)$ is real, $\zeta(A^2) = 0$ and $\beta(A^2) = b(\phi)$, where $b(\phi) := \inf_x \phi^2(x)$. Therefore, $w(A^2,\omega) = (b^2(\phi) + \omega^2)^{1/2}$, and

$$\chi(A^2) = \frac{1}{\pi}\int_0^\infty \left(\sum_{k=0}^\infty \frac{g_I^k(A^2)}{\sqrt{k!}(b^2(\phi) + \omega^2)^{(k+1)/2}}\right)^2 d\omega.$$

Hence, according to (1.6)

$$\chi(A^2) \le \frac{2}{\pi}\int_0^\infty \frac{1}{b^2(\phi) + \omega^2} \exp\,[\frac{2g_I^2(A^2)}{b^2(\phi) + \omega^2}]d\omega \le \exp\,[\frac{2g_I^2(A^2)}{b^2(\phi)}]\frac{2}{\pi}\int_0^\infty \frac{d\omega}{b^2(\phi) + \omega^2}$$

$$= \exp\,[\frac{2g_I^2(A^2)}{b^2(\phi)}].$$

In addition, $\|A^2 - \tilde{A}^2\| = \|W^2 + AW + WA\|$ and condition (4.5) is provided by the inequality

$$\|W^2 + AW + WA\| \exp\,[\frac{2g_I^2(A^2)}{b^2(\phi)}] < 1. \tag{6.3}$$

We thus arrive at

Corollary 6.1. *Let conditions (6.2) and (6.3) hold. Then the spectrum of the operator $\tilde{A}$ defined by (6.1) lies in the angle $\{z \in \mathbb{C} : |\arg\ z| < \pi/4\}$.*

About other bounds for the spectrum of integral operators see [10, 11] and references therein.

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