Filomat 32:14 (2018), 4865–4873
https://doi.org/10.2298/FIL1814865K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A New Characterization of Browder's Theorem

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Abstract. We give a new characterization of Browder's theorem using spectra originated from Drazin-Fredholm theory.

1. Introduction and Preliminaries

Throughout, X denotes a complex Banach space, $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X, let I be the identity operator, and for $T \in \mathcal{B}(X)$ we denote by T^*, $N(T)$, $R(T)$, $R^\infty(T) = \bigcap_{n\geq 0} R(T^n)$, $\rho(T)$, $\sigma(T)$ respectively the adjoint, the null space, the range, the hyper-range, the resolvent set and the spectrum of T.

Let E be a subset of X. E is said T-invariant if $T(E) \subseteq E$. We say that T is completely reduced by the pair (E, F) and we denote $(E, F) \in Red(T)$ if E and F are two closed T-invariant subspaces of X such that $X = E \oplus F$. In this case we write $T = T_{\restriction E} \oplus T_{\restriction F}$ and we say that T is the direct sum of $T_{\restriction E}$ and $T_{\restriction F}$. An operator $T \in \mathcal{B}(X)$ is said to be semi-regular, if $R(T)$ is closed and $N(T) \subseteq R^\infty(T)$ ([1]).

In the other hand, recall that an operator $T \in \mathcal{B}(X)$ admits a generalized Kato decomposition, (GKD for short), if there exists $(X_1, X_2) \in Red(T)$ such that $T_{\restriction X_1}$ is semi-regular and $T_{\restriction X_2}$ is quasi-nilpotent, in this case T is said a pseudo Fredholm operator. If we assume in the definition above that $T_{\restriction X_2}$ is nilpotent, then T is said to be of Kato type. Clearly, every semi-regular operator is of Kato type and a quasi-nilpotent operator has a GKD, see [17, 20] for more information about generalized Kato decomposition.

A bounded linear operator is called an upper semi-Fredholm (resp, lower semi Fredholm) if $dimN(T) < \infty$ and $R(T)$ is closed (resp, $codimR(T) < \infty$). T is semi-Fredholm if it is a lower or upper semi-Fredholm operator. The index of a semi-Fredholm operator T is defined by $ind(T) := dimN(T) - codimR(T)$. Also, T is a Fredholm operator if it is a lower and upper semi-Fredholm operator, and T is called a Weyl operator if it is a Fredholm of index zero.
The essential and Weyl spectra of T are closed and defined by :

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a Fredholm operator}\};$$

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a Weyl operator}\}.$$

2010 *Mathematics Subject Classification*. Primary 47A53; Secondary 47A10
Keywords. Generalized Kato-Riesz decomposition, generalized Drazin-Riesz invertible, Browder's theorem, Riesz operator.
Received: 09 February 2018; Accepted: 25 July 2018
Communicated by Snežana Č Živković-Zlatanović
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Recall that an operator $R \in \mathcal{B}(X)$ is said to be Riesz if $R - \mu I$ is Fredholm for every non-zero complex number μ ([1]). Of course compact and quasi-nilpotent operators are particular cases of Riesz operators.

In [26], Živković-Zlatanović SČ and M D. Cvetković introduced and studied a new concept of Kato decomposition to extend the Mbektha concept to "generalized Kato-Riesz decomposition". In fact, an operator $T \in \mathcal{B}(X)$ admits a generalized Kato-Riesz decomposition, (GKRD for short), if there exists $(X_1, X_2) \in Red(T)$ such that $T_{|X_1}$ is semi-regular and $T_{|X_2}$ is Riesz. The generalized Kato-Riesz spectrum is defined by

$$\sigma_{gKR}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ does not admit a generalized Kato-Riesz decomposition}\}.$$

Let $T \in \mathcal{B}(X)$, the ascent of T is defined by $a(T) = min\{p \in \mathbb{N} : N(T^p) = N(T^{p+1})\}$, if such p does not exist we let $a(T) = \infty$. Analogously the descent of T is $d(T) = min\{q \in \mathbb{N} : R(T^q) = R(T^{q+1})\}$, if such q does not exist we let $d(T) = \infty$ [19]. It is well known that if both $a(T)$ and $d(T)$ are finite then $a(T) = d(T)$ and we have the decomposition $X = R(T^p) \oplus N(T^p)$ where $p = a(T) = d(T)$.
An operator $T \in \mathcal{B}(X)$ is upper semi-Browder if T is upper semi-Fredholm and $a(T) < \infty$. If $T \in \mathcal{B}(X)$ is lower semi-Fredholm and $d(T) < \infty$ then T is lower semi-Browder. T is called Browder operator if it is a lower and an upper Browder operator.

An operator $T \in \mathcal{B}(X)$ is said to be B-Fredholm, if for some integer $n \geq 0$ the range $R(T^n)$ is closed and T_n, the restriction of T to $R(T^n)$ is a Fredholm operator. This class of operators, introduced and studied by Berkani et al. in a series of papers extends the class of semi-Fredholm operators ([11], [12]). T is said to be a B-Weyl operator if T_n is a Fredholm operator of index zero. The B-Fredholm and B-Weyl spectra are defined by

$$\sigma_{BF}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not B-Fredholm}\};$$

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not B-Weyl}\}.$$

Note that T is a B-Fredholm operator if there exists $(X_1, X_2) \in Red(T)$ such that $T_{|X_1}$ is Fredholm and $T_{|X_2}$ is nilpotent, see [11, Theorem 2.7]. Also, T is a B-Weyl operator if and only if $T_{|X_1}$ is a Weyl operator and $T_{|X_2}$ is a nilpotent operator.

More recently, B-Fredholm and B-Weyl operators were generalized to pseudo B-Fredholm and pseudo B-Weyl, see [13] [22][23] [25], precisely, T is a pseudo B-Fredholm operator, if there exists $(X_1, X_2) \in Red(T)$ such that $T_{|X_1}$ is a Fredholm operator and $T_{|X_2}$ is a quasi-nilpotent operator. T is said to be pseudo B-Weyl operator if there exists $(X_1, X_2) \in Red(T)$ such that $T_{|X_1}$ is a Weyl operator and $T_{|X_2}$ is a quasi-nilpotent operator. The pseudo B-Fredholm and pseudo B-Weyl spectra are defined by:

$$\sigma_{pBF}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not pseudo B-Fredholm}\};$$

$$\sigma_{pBW}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not pseudo B-Weyl}\}.$$

Let $T \in \mathcal{B}(X)$, T is said to be Drazin invertible if there exist a positive integer k and an operator $S \in \mathcal{B}(X)$ such that

$$ST = TS, \ \ T^{k+1}S = T^k \ \ and \ S^2T = S.$$

Which is also equivalent to the fact that $T = T_1 \oplus T_2$; where T_1 is invertible and T_2 is nilpotent. The Drazin spectrum is defined by

$$\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible }\}.$$

The concept of Drazin invertible operators has been generalized by Koliha [16]. In fact, $T \in \mathcal{B}(X)$ is generalized Drazin invertible if and only if $0 \notin acc(\sigma(T))$, where $acc(\sigma(T))$ is the set of accumulation points of $\sigma(T)$. This is also equivalent to the fact that there exists $(X_1, X_2) \in Red(T)$ such that $T_{|X_1}$ is invertible and $T_{|X_2}$ is quasi-nilpotent. The generalized Drazin spectrum is defined by

$$\sigma_{gD}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not generalized Drazin invertible }\}.$$

The concept of analytical core for an operator has been introduced by Vrbova in [24] and study by Mbekhta [20, 21], that is the following set:

$$K(T) = \{x \in X \ : \ \exists (x_n)_{n\geq 0} \subset X \text{ and } \delta > 0 \text{ such that } x_0 = x, \ Tx_n = x_{n-1} \ \forall n \geq 1 \text{ and } \|x_n\| \leq \delta^n \|x\|\}$$

The quasi-nilpotent part of T, $H_0(T)$ is given by :

$$H_0(T) := \{x \in X; r_T(x) = 0\} \text{ where } r_T(x) = \lim_{n\to+\infty} \|T^n x\|^{\frac{1}{n}}.$$

In [14], M D. Cvetković and SČ. Živković-Zlatanović introduced and studied a new concept of generalized Drazin invertibility of bounded operators as a generalization of generalized Drazin invertible operators. In fact, an operator $T \in \mathcal{B}(X)$ is said to be generalized Drazin bounded below if $H_0(T)$ is closed and complemented with a subspace M in X such that $(M, H_0(T)) \in Red(T)$ and $T(M)$ is closed which is equivalent to there exists $(M, N) \in Red(T)$ such that $T_{\shortmid M}$ is bounded below and $T_{\shortmid N}$ is quasi-nilpotent, see [14, Theorem 3.6]. An operator $T \in \mathcal{B}(X)$ is said to be generalized Drazin surjective if $K(T)$ is closed and complemented with a subspace N in X such that $N \subseteq H_0(T)$ and $(K(T), N) \in Red(T)$ which is equivalent to there exists $(M, N) \in Red(T)$ such that $T_{\shortmid M}$ is surjective and $T_{\shortmid N}$ is quasi-nilpotent, see [14, Theorem 3.7]. The generalized Drazin bounded below and surjective spectra of $T \in \mathcal{B}(X)$ are defined respectively by:

$$\sigma_{gD\mathcal{M}}(T) = \{\lambda \in \mathbb{C}, \ T - \lambda I \text{ is not generalized Drazin bounded below}\};$$

$$\sigma_{gD\mathcal{Q}}(T) = \{\lambda \in \mathbb{C}, \ T - \lambda I \text{ is not generalized Drazin surjective}\}.$$

From [14], we have:

$$\sigma_{gD}(T) = \sigma_{gD\mathcal{M}}(T) \cup \sigma_{gD\mathcal{Q}}(T).$$

Recently, Živković-Zlatanović SČ and M D. Cvetković [26] introduced and studied a new concept of pseudo-inverse to extend the Koliha concept, generalized Drazin bounded below, and generalized Drazin surjective to ″generalized Drazin-Riesz invertible″, ″generalized Drazin-Riesz bounded below″ and ″generalized Drazin-Riesz surjective″ respectively. In fact, an operator $T \in \mathcal{B}(X)$ is said to be generalized Drazin-Riesz invertible, if there exists $S \in \mathcal{B}(X)$ such that

$$TS = ST, \quad STS = S \quad \text{and } TST - T \text{ is Riesz}$$

Živković-Zlatanović SČ and M D. Cvetković also showed that T is generalized Drazin-Riesz invertible iff it has a direct sum decomposition $T = T_1 \oplus T_0$ with T_1 is invertible and T_0 is Riesz. If we assume in the characterization above that T_1 is bounded below (surjective), then T is said to be generalized Drazin-Riesz bounded below(generalized Drazin-Riesz surjective). The generalized Drazin-Riesz, generalized Drazin-Riesz bounded below and generalized Drazin-Riesz surjective spectra of $T \in \mathcal{B}(X)$ are defined respectively by:

$$\sigma_{gDR}(T) = \{\lambda \in \mathbb{C}, \quad T - \lambda I \text{ is not generalized Drazin-Riesz invertible}\}$$

$$\sigma_{gDR\mathcal{M}}(T) = \{\lambda \in \mathbb{C}, \quad T - \lambda I \text{ is not generalized Drazin-Riesz bounded below }\}$$

$$\sigma_{gDR\mathcal{Q}}(T) = \{\lambda \in \mathbb{C}, \quad T - \lambda I \text{ is not generalized Drazin-Riesz surjective}\}$$

Also , they introduced the definition of operators which are a direct sum of a Riesz operator and a Fredholm (Weyl, upper (lower) semi-Fredholm, upper (lower) semi-Weyl) operator([26]). These operators generalize the class of generalized Drazin invertible operators and also the class of generalized Drazin-Riesz invertible operators and hence, we shall call them generalized Drazin-Riesz Fredholm (generalized Drazin-Riesz Weyl, generalized Drazin-Riesz upper (lower) semi-Fredholm, generalized Drazin-Riesz (lower) semi-Weyl, ..) operators, and we shall use the following notations:

$$gDRR_*(X) = \{T \in \mathcal{B}(X) : T = T_1 \oplus T_2, \quad T_1 \in R_*, \quad T_2 \text{ is Riesz }\}$$

where $R_*(X)$ run the Fredholm class $\Phi(X)$, upper (lower) semi-Fredholm class $\Phi_+(X)$ $(\Phi_-(X))$, Weyl class $\mathcal{W}(X)$, upper (lower) semi-Weyl class $\mathcal{W}_+(X)$ $(\mathcal{W}_-(X))$.

These classes of operators motivate the definition of several spectra. The generalized Drazin-Riesz lower(upper) semi-Weyl and generalized Drazin-Riesz Weyl spectra of $T \in \mathcal{B}(X)$ are defined respectively by:

$$\sigma_{gDR\mathcal{W}_-}(T) = \{\lambda \in \mathbb{C},\ T - \lambda I \text{ is not generalized Drazin-Riesz lower semi-Weyl}\};$$

$$\sigma_{gDR\mathcal{W}_+}(T) = \{\lambda \in \mathbb{C},\ T - \lambda I \text{ is not generalized Drazin-Riesz upper semi-Weyl}\}.$$

$$\sigma_{gDR\mathcal{W}}(T) = \{\lambda \in \mathbb{C},\ T - \lambda I \text{ is not generalized Drazin-Riesz Weyl}\}.$$

From [26], we have:

$$\sigma_{gDR\mathcal{W}}(T) = \sigma_{gDR\mathcal{W}_+}(T) \cup \sigma_{gDR\mathcal{W}_-}(T);$$

The generalized Drazin-Riesz upper (lower) semi-Fredholm and generalized Drazin-Riesz Fredholm spectra of $T \in \mathcal{B}(X)$ are defined respectively by:

$$\sigma_{gDR\Phi_+}(T) = \{\lambda \in \mathbb{C},\ T - \lambda I \text{ is not generalized Drazin-Riesz upper semi-Fredholm }\};$$

$$\sigma_{gDR\Phi_-}(T) = \{\lambda \in \mathbb{C},\ T - \lambda I \text{ is not generalized Drazin-Riesz lower semi-Fredholm}\}.$$

$$\sigma_{gDR\Phi}(T) = \{\lambda \in \mathbb{C},\ T - \lambda I \text{ is not generalized Drazin-Riesz Fredholm}\}.$$

Also, from [26], we have:

$$\sigma_{gDR\Phi}(T) = \sigma_{gDR\Phi_+}(T) \cup \sigma_{gDR\Phi_-}(T).$$

$$\sigma_{gKR}(T) \subset \sigma_{gDR\Phi+}(T) \subset \sigma_{gDR\mathcal{W}+}(T) \subset \sigma_{gDR\mathcal{M}}(T)$$

$$\sigma_{gKR}(T) \subset \sigma_{gDR\Phi-}(T) \subset \sigma_{gDR\mathcal{W}-}(T) \subset \sigma_{gDR\mathcal{Q}}(T)$$

$$\sigma_{gKR}(T) \subset \sigma_{gDR\Phi}(T) \subset \sigma_{gDR\mathcal{W}}(T) \subset \sigma_{gDR}(T)$$

A Banach space operator satisfies "Browder's theorem" if the Browder spectrum coincides with the Weyl spectrum. Browder's theorem has been studied by several authors (see [4], [3], [5], [6]). In this paper we shall give some characterizations of operators satisfying Browder's theorem. In particular, we shall see that Browder's theorem for a bounded linear operator is equivalent to the equality between the generalized Drazin-Riesz Weyl spectrum and generalized Drazin-Riesz spectrum. Also, we will give serval necessary and sufficient conditions for T to have equality between the spectra originated from Drazin-Fredholm theory.

2. Main Results

Recall that $T \in \mathcal{B}(X)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (SVEP for short) if for every open neighbourhood $U \subseteq \mathbb{C}$ of λ_0, the only analytic function $f : U \longrightarrow X$ which satisfies the equation $(T - zI)f(z) = 0$ for all $z \in U$ is the function $f \equiv 0$. An operator T is said to have the SVEP if T has the SVEP for every $\lambda \in \mathbb{C}$. Obviously, every operator $T \in \mathcal{B}(X)$ has the SVEP at every $\lambda \in \rho(T) = \mathbb{C} \setminus \sigma(T)$, hence T and T^* have the SVEP at every point of the boundary $\partial(\sigma(T))$ of the spectrum. Also, we have the implication

$$a(T) < \infty \Longrightarrow T \text{ has SVEP at } 0.$$

$$d(T) < \infty \Longrightarrow T^* \text{ has SVEP at } 0.$$

In [26], the authors gave some examples showing that $\sigma_{gDR\mathcal{W}+}(T) \subset \sigma_{gDR\mathcal{M}}(T)$, $\sigma_{gDR\mathcal{W}-}(T) \subset \sigma_{gDR\mathcal{Q}}(T)$ and $\sigma_{gDR\mathcal{W}}(T) \subset \sigma_{gDR}(T)$ can be proper. In the following results we give serval necessary and sufficient conditions for T to have equality.

Proposition 2.1. *Let $T \in \mathcal{B}(X)$, then $\sigma_{gDR\mathcal{M}}(T) = \sigma_{gDR\mathcal{W}+}(T)$ if and only if T has SVEP at every $\lambda \notin \sigma_{gDR\mathcal{W}+}(T)$*

Proof. Assume that T has SVEP at every $\lambda \notin \sigma_{gDR\mathcal{W}_+}(T)$. If $\lambda \notin \sigma_{gDR\mathcal{W}_+}(T)$, then $T-\lambda I$ is generalized Drazin Riesz upper semi-Weyl, then there exists $(M,N) \in Red(T-\lambda I)$ such that $(T-\lambda I)_{|M}$ is semi-regular and $(T-\lambda I)_{|N}$ is Riesz. T has SVEP at every $\lambda \notin \sigma_{gDR\mathcal{W}_+}(T)$, it follows that $(T-\lambda I)_{|M}$ has the SVEP at 0, then $(T-\lambda I)_{|M}$ is bounded below, see [18, Corollary 3.1.7]. Hence $T-\lambda I$ is generalized Drazin Riesz bounded below, $\lambda \notin \sigma_{gDR\mathcal{M}}(T)$, and since the reverse implication holds for every operator we conclude that $\sigma_{gDR\mathcal{M}}(T) = \sigma_{gDR\mathcal{W}_+}(T)$. Conversely, suppose that $\sigma_{gDR\mathcal{M}}(T) = \sigma_{gDR\mathcal{W}_+}(T)$. If $\lambda \notin \sigma_{gDR\mathcal{W}_+}(T)$ then $T-\lambda I$ is generalized Drazin Riesz bounded below so, T has SVEP at λ, by [26, Theorem 2.4]. □

We denote by $\sigma_{lb}(T)$ and $\sigma_{lw}(T)$ respectively the lower Browder and lower Weyl spectra. In the same way we have the following result.

Proposition 2.2. *Let $T \in \mathcal{B}(X)$, then $\sigma_{gDRQ}(T) = \sigma_{gDR\mathcal{W}_-}(T)$ if and only if T^* has SVEP at every $\lambda \notin \sigma_{gDR\mathcal{W}_-}(T)$*

Proof. Suppose that T^* has SVEP at every $\lambda \notin \sigma_{gDR\mathcal{W}_-}(T)$. If $\lambda \notin \sigma_{gDR\mathcal{W}_-}(T)$, then by [26, Theorem 2.6], $T-\lambda I$ admits GKRD and $\lambda \notin acc\sigma_{lw}(T)$. T^* has SVEP at every $\lambda \notin \sigma_{gDR\mathcal{W}_-}(T)$, then T^* has SVEP at every $\lambda \notin \sigma_{lw}(T)$, and so $\sigma_{lb}(T) = \sigma_{lw}(T)$. Then $\lambda \notin acc\sigma_{lb}(T)$. Therefore, $T-\lambda I$ is generalized Drazin Riesz surjective according to [26, Theorem 2.5], $\lambda \notin \sigma_{gDRQ}(T)$ and since the reverse implication holds for every operator we conclude that $\sigma_{gDRQ}(T) = \sigma_{gDR\mathcal{W}_-}(T)$. Conversely, suppose that $\sigma_{gDRQ}(T) = \sigma_{gDR\mathcal{W}_-}(T)$. If $\lambda \notin \sigma_{gDR\mathcal{W}_-}(T)$, then $T-\lambda I$ is generalized Riesz Drazin surjective so, T has SVEP at λ, by [26, Theorem 2.5]. □

As a consequence of the two previous results we have the following corollary.

Corollary 2.3. *Let $T \in \mathcal{B}(X)$, then $\sigma_{gDR}(T) = \sigma_{gDR\mathcal{W}}(T)$ if and only if T and T^* have the SVEP at every $\lambda \notin \sigma_{gDR\mathcal{W}}(T)$*

Proof. Suppose that $\sigma_{gDR}(T) = \sigma_{gDR\mathcal{W}}(T)$. If $\lambda \notin \sigma_{gDR\mathcal{W}}(T)$, then $T-\lambda I$ is generalized Riesz Drazin invertible so, T and T^* have SVEP at λ, by [26, Theorem 2.3]. The "if" is an immediate consequence of Proposition 2.1 and Proposition 2.2.
□

Moreover, we have the following result.

Proposition 2.4. *Let $T \in \mathcal{B}(X)$, the following statements are equivalent :*
1) $\sigma_{gDR}(T) = \sigma_{gDR\mathcal{W}}(T)$;
2) T or T^* has SVEP at every $\lambda \notin \sigma_{gDR\mathcal{W}}(T)$.

Proof. -If T has SVEP at every $\lambda \notin \sigma_{gDR\mathcal{W}}(T)$. If $\lambda \notin \sigma_{gDR\mathcal{W}}(T)$, then by [26, Theorem 2.6], $T-\lambda I$ admits GKRD and $\lambda \notin acc\sigma_w(T)$. T has SVEP at every $\lambda \notin \sigma_{gDR\mathcal{W}}(T)$, then T has SVEP at every $\lambda \notin \sigma_w(T)$, and so $\sigma_b(T) = \sigma_w(T)$ [1, Theorem 4.23]. Thus $\lambda \notin acc\sigma_b(T)$. Therefore, $T-\lambda I$ is generalized Drazin Riesz invertible by [26, Theorem 2.3], $\lambda \notin \sigma_{gDR}(T)$ and since the reverse implication holds for every operator we conclude that $\sigma_{gDR}(T) = \sigma_{gDR\mathcal{W}}(T)$.

-If T^* has SVEP at every $\lambda \notin \sigma_{gDR\mathcal{W}}(T)$. Since $\sigma_b(T) = \sigma_b(T^*)$ and $\sigma_w(T) = \sigma_w(T^*)$, we have $\sigma_{gDR}(T) = \sigma_{gDR\mathcal{W}}(T)$.

Conversely, suppose that $\sigma_{gDR}(T) = \sigma_{gDR\mathcal{W}}(T)$. If $\lambda \notin \sigma_{gDR\mathcal{W}}(T)$, then $T-\lambda I$ is generalized Riesz Drazin invertible so, T and T^* have SVEP at λ, by [26, Theorem 2.3]. □

We shall say that T satisfies Browder's theorem if $\sigma_w(T) = \sigma_b(T)$, or equivalently $acc\sigma(T) \subseteq \sigma_w(T)$, where $\sigma_b(T)$ is the Browder spectrum of T ([15]).
It is known from [2] that a-Browder's theorem holds for T if $\sigma_{uw}(T) = \sigma_{ub}(T)$, or equivalently $acc\sigma_{ap}(T) \subseteq \sigma_{uw}(T)$, where $\sigma_{ub}(T)$ and $\sigma_{uw}(T)$ are the upper semi-Browder and upper semi-Weyl spectra of T.

In the sequel, we characterize the equality between the generalized Drazin-Riesz invertible(surjective, bounded below) spectrum and generalized Drazin-Riesz Weyl(upper-lower Weyl) spectrum by means of the Browder's theorem(a-Browder's theorems), which give new characterizations for Browder's and a-Browder's theorems.

Theorem 2.5. *Let $T \in \mathcal{B}(X)$, then*
1) a-Browder's theorem holds for T if and only if $\sigma_{gDR\mathcal{M}}(T) = \sigma_{gDR\mathcal{W}_+}(T)$.
2) a-Browder's theorem holds for T^* if and only if $\sigma_{gDR\mathcal{Q}}(T) = \sigma_{gDR\mathcal{W}_-}(T)$.
3) Browder's theorem holds for T if and only if $\sigma_{gDR}(T) = \sigma_{gDR\mathcal{W}}(T)$.

Proof. 1) Suppose that a-Browder's theorem holds for T implies $\sigma_{ub}(T) = \sigma_{uw}(T)$.
Using [26, Theorems 2.4 and 2.6], we conclude that

$$\begin{aligned} \lambda \notin \sigma_{gDR\mathcal{M}}(T) &\iff T-\lambda I \text{ is generalized Drazin Riesz bounded below} \\ &\iff T-\lambda I \text{ admits a GKRD and } \lambda \notin acc\sigma_{ub}(T) \\ &\iff T-\lambda I \text{ admits a GKRD and } \lambda \notin acc\sigma_{uw}(T) \\ &\iff T-\lambda I \text{ is generalized Drazin Riesz upper semi-Weyl} \\ &\iff \lambda \notin \sigma_{gDR\mathcal{W}_+}(T). \end{aligned}$$

Hence $\sigma_{gDR\mathcal{M}}(T) = \sigma_{gDR\mathcal{W}_+}(T)$. Conversely, if $\sigma_{gDR\mathcal{M}}(T) = \sigma_{gDR\mathcal{W}_+}(T)$, from Proposition 2.1, T has SVEP at every $\lambda \notin \sigma_{gDR\mathcal{W}_+}(T)$. Since $\sigma_{gDR\mathcal{W}_+}(T) \subseteq \sigma_{uw}(T)$, T has SVEP at every $\lambda \notin \sigma_{uw}(T)$, so a-Browder's theorem holds for T, see [2, Theorem 4.34].

2) Suppose that a-Browder's theorem holds for T^* then $\sigma_{lb}(T) = \sigma_{lw}(T)$.
Using [26, Theorems 2.5 and 2.6] we have

$$\begin{aligned} \lambda \notin \sigma_{gDR\mathcal{Q}}(T) &\iff T-\lambda I \text{ is generalized Drazin Riesz surjective} \\ &\iff T-\lambda I \text{ admits a GKRD and } \lambda \notin acc\sigma_{lb}(T) \\ &\iff T-\lambda I \text{ admits a GKRD and } \lambda \notin acc\sigma_{lw}(T) \\ &\iff T-\lambda I \text{ is generalized Drazin Riesz lower semi-Weyl} \\ &\iff \lambda \notin \sigma_{gDR\mathcal{W}_-}(T). \end{aligned}$$

Hence $\sigma_{gDR\mathcal{Q}}(T) = \sigma_{gDR\mathcal{W}_-}(T)$. Conversely, if $\sigma_{gDR\mathcal{Q}}(T) = \sigma_{gDR\mathcal{W}_-}(T)$, from Proposition 2.2, T^* has SVEP at every $\lambda \notin \sigma_{gDR\mathcal{W}_-}(T)$. Since $\sigma_{gDR\mathcal{W}_-}(T) \subseteq \sigma_{lw}(T)$, T^* has SVEP at every $\lambda \notin \sigma_{lw}(T)$, so a-Browder's theorem holds for T^*, see [2, Theorem 4.34].

3) Suppose that Browder's theorem holds for T then $\sigma_b(T) = \sigma_w(T)$.
Using [26, Theorems 2.6 and 2.3] we have

$$\begin{aligned} \lambda \notin \sigma_{gDR}(T) &\iff T-\lambda I \text{ is generalized Drazin Riesz invertible} \\ &\iff T-\lambda I \text{ admits a GKRD and } \lambda \notin acc\sigma_{b}(T) \\ &\iff T-\lambda I \text{ admits a GKRD and } \lambda \notin acc\sigma_{w}(T) \\ &\iff T-\lambda I \text{ is generalized Drazin Riesz Weyl} \\ &\iff \lambda \notin \sigma_{gDR\mathcal{W}}(T). \end{aligned}$$

Hence $\sigma_{gDR}(T) = \sigma_{gDR\mathcal{W}}(T)$. Conversely, if $\sigma_{gDR}(T) = \sigma_{gDR\mathcal{W}}(T)$, from Corollary 2.3, T and T^* has SVEP at every $\lambda \notin \sigma_{gDR\mathcal{W}}(T)$. Since $\sigma_{gDR\mathcal{W}}(T) \subseteq \sigma_w(T)$, T has SVEP at every $\lambda \notin \sigma_w(T)$, so Browder's theorem holds for T, see [2, Theorem 4.23].
□

It will be said that generalized Browder's theorem holds for $T \in \mathcal{B}(X)$ if $\sigma_{BW}(T) = \sigma(T)\backslash\Pi(T)$, equivalently, $\sigma_{BW}(T) = \sigma_D(T)$, where $\Pi(T)$ is the set of all poles of the resolvent of T ([4]). A classical result of M. Amouch and H. Zguitti [9, Theorem 2.1] shows that Browder's theorem and generalized Browder's theorem are equivalent. According to the previous results, [6, Theorem 2.2], [3, Theorem 2.3] an the equivalent between Browder's theorem and generalized Browder's theorem [9, Theorem 2.1] [10][Proposition 2.2] we have the following theorem.

Theorem 2.6. *Let $T \in \mathcal{B}(X)$. The statements are equivalent:*
1) Browder's theorem holds for T;
2) Browder's theorem holds for T^*;
3) T has SVEP at every $\lambda \notin \sigma_w(T)$;
4) T^* has SVEP at every $\lambda \notin \sigma_w(T)$;
5) T has SVEP at every $\lambda \notin \sigma_{BW}(T)$;
6) generalized Browder's theorem holds for T;
7) T or T^* has SVEP at every $\lambda \notin \sigma_{gDR\mathcal{W}}(T)$;
8) $\sigma_{gDR}(T) = \sigma_{gDR\mathcal{W}}(T)$;
9) T or T^* has SVEP at every $\lambda \notin \sigma_{gD\mathcal{W}}(T)$;
10) $\sigma_{gD}(T) = \sigma_{pBW}(T)$.

In the same way we have the following result.

Theorem 2.7. *Let $T \in \mathcal{B}(X)$. The statements are equivalent:*
1) a-Browder's theorem holds for T;
2) generalized a-Browder's theorem holds for T;
3) T has SVEP at every $\lambda \notin \sigma_{gDR\mathcal{W}_+}(T)$;
4) $\sigma_{gDR\mathcal{M}}(T) = \sigma_{gDR\mathcal{W}_+}(T)$.
5) T has SVEP at every $\lambda \notin \sigma_{gD\mathcal{W}_+}(T)$;
6) $\sigma_{gD\mathcal{M}}(T) = \sigma_{gD\mathcal{W}_+}(T)$.

We denote by $\sigma_{lf}(T)$ and $\sigma_{uf}(T)$, $T \in \mathcal{B}(X)$, respectively the lower and upper semi-Fredholm spectra. Note that $\sigma_{gDR\Phi_+}(T) \subset \sigma_{gDR\mathcal{M}}(T)$, $\sigma_{gDR\Phi_-}(T) \subset \sigma_{gDR\mathcal{Q}}(T)$ and $\sigma_{gDR\Phi}(T) \subset \sigma_{gDR}(T)$ are strict [26]. In this case we have the following theorems:

Theorem 2.8. *Let $T \in \mathcal{B}(X)$. The statements are equivalent:*
1) $\sigma_{uf}(T) = \sigma_{ub}(T)$;
2) T has SVEP at every $\lambda \notin \sigma_{uf}(T)$;
3) T has SVEP at every $\lambda \notin \sigma_{gDR\Phi_+}(T)$;
4) $\sigma_{gDR\mathcal{M}}(T) = \sigma_{gDR\Phi_+}(T)$.

Proof. 1) $\Longleftrightarrow$ 2): Suppose that T has SVEP at every $\lambda \notin \sigma_{uf}(T)$. $\lambda \notin \sigma_{uf}(T)$, $T - \lambda I$ is upper semi-Fredholm. T has SVEP at λ, then $a(T - \lambda I) < \infty$, see [1, Theorem 3.16]. So $\lambda \notin \sigma_{ub}(T)$. Now, Suppose that $\sigma_{uf}(T) = \sigma_{ub}(T)$. Let $\lambda \notin \sigma_{uf}(T)$, $\lambda \notin \sigma_{ub}(T)$ then $a(T - \lambda I) < \infty$, hence T has SVEP at λ by [1].

3) $\Longleftrightarrow$ 4): Suppose that T has SVEP at every $\lambda \notin \sigma_{gDR\Phi_+}(T)$. If $\lambda \notin \sigma_{gDR\Phi_+}(T)$, $T - \lambda I$ is generalized Drazin Riesz upper Fredholm, then there exists $(M, N) \in Red(T)$ such that $(T - \lambda I)_{|M}$ is semi-regular and $(T - \lambda I)_{|N}$ is Riesz. T has SVEP at every $\lambda \notin \sigma_{gDR\Phi_+}(T)$ implies $(T - \lambda I)_{|M}$ has the SVEP at 0, it follows that $(T - \lambda I)_{|M}$ is bounded below, see [18, Corollary 3.1.7]. Hence $T - \lambda I$ is generalized Drazin Riesz bounded below, $\lambda \notin \sigma_{gDR\mathcal{M}}(T)$, and since the reverse implication holds for every operator we conclude that $\sigma_{gDR\mathcal{M}}(T) = \sigma_{gDR\Phi_+}(T)$. Conversely, assume that $\sigma_{gDR\mathcal{M}}(T) = \sigma_{gDR\Phi_+}(T)$. If $\lambda \notin \sigma_{gDR\Phi_+}(T)$ then $T - \lambda I$ is generalized Drazin Riesz bounded below so T has the SVEP at λ, by [26, Theorem 2.4].

1) $\Longleftrightarrow$ 4): Suppose that $\sigma_{uf}(T) = \sigma_{ub}(T)$.
According to [26, Theorems 2.4 and 2.6] we have

$$\begin{aligned}
\lambda \notin \sigma_{gD\mathcal{M}}(T) &\Longleftrightarrow T - \lambda I \text{ is generalized Drazin Riesz bounded below} \\
&\Longleftrightarrow T - \lambda I \text{ admits a GKRD and } \lambda \notin acc\sigma_{ub}(T) \\
&\Longleftrightarrow T - \lambda I \text{ admits a GKRD and } \lambda \notin acc\sigma_{uf}(T) \\
&\Longleftrightarrow T - \lambda I \text{ is generalized Drazin Riesz Fredholm} \\
&\Longleftrightarrow \lambda \notin \sigma_{gDR\Phi_+}(T).
\end{aligned}$$

Hence $\sigma_{gDR\mathcal{M}}(T) = \sigma_{gDR\Phi_+}(T)$. Conversely, if $\sigma_{gDR\mathcal{M}}(T) = \sigma_{gDR\Phi_+}(T)$, then by 3) $\Longleftrightarrow$ 4), T has SVEP at every $\lambda \notin \sigma_{gDR\Phi_+}(T)$. Since $\sigma_{gDR\Phi_+}(T) \subseteq \sigma_{uf}(T)$, T has SVEP at every $\lambda \notin \sigma_{uf}(T)$, 1) $\Longleftrightarrow$ 2) gives the result.
□

Theorem 2.9. *Let $T \in \mathcal{B}(X)$. The statements are equivalent:*
1) $\sigma_{lf}(T) = \sigma_{lb}(T)$;
2) T^* has SVEP at every $\lambda \notin \sigma_{lf}(T)$;
3) T^* has SVEP at every $\lambda \notin \sigma_{gDR\Phi-}(T)$;
4) $\sigma_{gDRQ}(T) = \sigma_{gDR\Phi-}(T)$.

Proof. 1) $\Longleftrightarrow$ 2): Suppose that T^* has SVEP at every $\lambda \notin \sigma_{lf}(T)$. $\lambda \notin \sigma_{lf}(T)$ implies that $T - \lambda I$ is lower semi-Fredholm. T^* has SVEP at λ, then $d(T - \lambda I) < \infty$, see [1, Theorem 3.17]. So $\lambda \notin \sigma_{lb}(T)$. Now, Suppose that $\sigma_{lf}(T) = \sigma_{lb}(T)$. Let $\lambda \notin \sigma_{lf}(T)$, $\lambda \notin \sigma_{lb}(T)$ then $d(T - \lambda I) < \infty$, hence T^* has SVEP at λ by [1].

3) $\Longleftrightarrow$ 4): Suppose that T^* has SVEP at every $\lambda \notin \sigma_{gDR\Phi-}(T)$. If $\lambda \notin \sigma_{gDR\Phi-}(T)$, $T - \lambda I$ admits GKRD and $\lambda \notin acc\sigma_{lf}(T)$ by [26, Theorem 2.6]. T^* has SVEP at every $\lambda \notin \sigma_{gDR\Phi-}(T)$, it follows that T^* has SVEP at every $\lambda \notin \sigma_{lf}(T)$, then $\sigma_{lb}(T) = \sigma_{lf}(T)$ so $\lambda \notin acc\sigma_{lb}(T)$. Therefore, $T - \lambda I$ is generalized Drazin Riesz surjective [26, Theorem 2.5], $\lambda \notin \sigma_{gDRQ}(T)$ and since the reverse implication holds for every operator we conclude that $\sigma_{gDRQ}(T) = \sigma_{gDR\Phi-}(T)$. Conversely, suppose that $\sigma_{gDRQ}(T) = \sigma_{gDR\Phi-}(T)$, if $\lambda \notin \sigma_{gDR\Phi-}(T)$ then $T - \lambda I$ is generalized Drazin surjective, so T^* has SVEP at λ, by [26, Theorem 2.5].

1) $\Longleftrightarrow$ 4): Suppose that $\sigma_{lf}(T) = \sigma_{lb}(T)$.
According to [14, Theorems 2.5 and 2.6] we have

$$\begin{aligned} \lambda \notin \sigma_{gDQ}(T) &\Longleftrightarrow T - \lambda I \text{ is generalized Drazin Riesz surjective} \\ &\Longleftrightarrow T - \lambda I \text{ admits a GKRD and } \lambda \notin acc\sigma_{lb}(T) \\ &\Longleftrightarrow T - \lambda I \text{ admits a GKRD and } \lambda \notin acc\sigma_{lf}(T) \\ &\Longleftrightarrow T - \lambda I \text{ is generalized Drazin Riesz lower Fredholm} \\ &\Longleftrightarrow \lambda \notin \sigma_{gDR\Phi-}(T). \end{aligned}$$

Hence $\sigma_{gDRQ}(T) = \sigma_{gDR\Phi-}(T)$. Conversely, if $\sigma_{gDRQ}(T) = \sigma_{gDR\Phi-}(T)$, by 3) $\Longleftrightarrow$ 4), T^* has SVEP at every $\lambda \notin \sigma_{gDR\Phi-}(T)$. Since $\sigma_{gDR\Phi-}(T) \subseteq \sigma_{lf}(T)$, T has SVEP at every $\lambda \notin \sigma_{lf}(T)$, according to 1) $\Longleftrightarrow$ 2) we obtain the result. □

As a direct consequence of the Theorems 2.8, 2.9 and [6, Corollary 2.1] we have the following corollary.

Corollary 2.10. *Let $T \in \mathcal{B}(X)$. The statements are equivalent:*
1) $\sigma_e(T) = \sigma_b(T)$;
2) T and T^* have SVEP at every $\lambda \notin \sigma_e(T)$;
3) $\sigma_{BF}(T) = \sigma_D(T)$;
4) T and T^* have SVEP at every $\lambda \notin \sigma_{BF}(T)$;
5) $\sigma_{gD}(T) = \sigma_{pBF}(T)$.
6) T and T^* have SVEP at every $\lambda \notin \sigma_{pBF}(T)$;
7) $\sigma_{gDR}(T) = \sigma_{gDR\Phi}(T)$.
8) T and T^* have SVEP at every $\lambda \notin \sigma_{gDR\Phi}(T)$;

Acknowledgement: The authors thank the referee for his suggestions, remarks and comments thorough reading of the manuscript.

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