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Reverse Order Law of Drazin Inverse for Bounded Linear Operators*

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Abstract. In this paper, the reverse order law of Drazin inverse is investigated under some conditions in a Banach space. Moreover, the Drazin invertibility of sum for two bounded linear operators are also obtained.

1. Introduction

It is well known that, for the ordinary inverse, the equality $(PQ)^{-1} = Q^{-1}P^{-1}$ always holds if P and Q are invertible, which is called the reverse order law. However, it is not necessarily true for most generalized inverses such as K-inverse (K$\in \{\{1\},\{1,2\},\{1,3\},\{1,2,3\},\{1,3,4\}\}$), Moore-Penrose inverse and Drazin inverse. For example, let $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}$, then $P^D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $Q^D = \begin{pmatrix} \frac{1}{2} & \frac{3}{4} \\ 0 & 0 \end{pmatrix}$ and $(PQ)^D = \begin{pmatrix} \frac{1}{2} & \frac{3}{4} \\ 0 & 0 \end{pmatrix}$. Obviously, $(PQ)^D \neq Q^D P^D$. Thus, many authors were attracted to investigate the condition under which reverse order laws hold for various generalized inverses and some interesting results were obtained [1–15].

For bounded linear operators, Deng [4, 14] presented necessary and sufficient conditions for the reverse order law of group inverse and Drazin inverse, respectively. Mosić [8, 10] investigated the reverse order law for the generalized Drazin inverse in Banach algebras.

In this paper, we consider the reverse order law of the Drazin inverse for bounded linear operators. Precisely, we give some reverse order laws and commutation relations for Drazin invertible bounded linear operators P and Q under the condition (1)$P^2Q = QP$, (2)$PQ^2 = QP$ and (3)$P^2Q = QP = PQ^2$, respectively. In particular, we obtain that $(PQ)^D = Q^D P^D$ and $(P+Q)^D = P^D + Q^D - \frac{3}{2}P^D PQ^D$ when $P^2Q = QP = PQ^2$.

Throughout this paper, $\mathcal{X}$ and $\mathcal{Y}$ denote Banach spaces, and the set $\mathcal{B}(\mathcal{X})$ consists of all bounded linear operators on $\mathcal{X}$.

Recall that an operator $T \in \mathcal{B}(\mathcal{X})$ is said to be Drazin invertible, if there exists an operator $T^D \in \mathcal{B}(\mathcal{X})$ satisfying

$$TT^D = T^D T, \;\; T^D = T(T^D)^2, \;\text{ and }\; T^{k+1}T^D = T^k \text{ for some integer } k \geq 0.$$

Here T^D is called the Drazin inverse of T and the smallest integer k, denoted by ind(T), is called the index of T.

In the following, we list some basic facts about the Drazin inverse, which will be used in later proofs.

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Lemma 1.1. *Let* $T, S \in \mathcal{B}(\mathcal{X})$.

(1) *If* T *is Drazin invertible, then* T *has the operator matrix form*

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & N_1 \end{pmatrix}$$

with respect to the space decomposition $\mathcal{H} = \mathcal{N}(T^\pi) \oplus \mathcal{R}(T^\pi)$, *where* T_1 *is invertible,* N_1 *is nilpotent and* $T^\pi = I - TT^D$.

(2) *If* S *is invertible, then* T *is Drazin invertible if and only if* $S^{-1}TS$ *is Drazin invertible, and* $(S^{-1}TS)^D = S^{-1}T^DS$.

2. Main results

In this section, we investigate various types of reverse order laws and commutation relations. The following lemma is necessary.

Lemma 2.1. *Let* $N, Q \in \mathcal{B}(\mathcal{X})$. *If* $N^2Q = QN$ *and* N *is nilpotent, then* $N^kQ, QN^k, NQ^k (k = 1, 2, \cdots)$ *and* NQN *are all nilpotent, and we further have*

(1) *if* Q *is Drazin invertible, then* $Q^DN = 0$.

(2) *if* Q *is invertible, then* $N = 0$.

Proof. We only prove the case $k = 1$, and the other cases are similar. By $N^2Q = QN$, we have

$$QN^i = QNN^{i-1} = N^2QN^{i-1} = N^2QNN^{i-2} = N^4QN^{i-2} = \cdots = N^{2i}Q, i = 1, 2, \cdots .$$

Then, for $n = 2, 3, \cdots$,

$$\begin{aligned}(NQ)^n &= (NQ)^{n-2}NQNQ \\ &= (NQ)^{n-2}N^3Q^2 \\ &= (NQ)^{n-3}NQN^3Q^2 \\ &= (NQ)^{n-3}N^7Q^3 \\ &= \cdots \\ &= N^{2^n-1}Q^n,\end{aligned}$$

which implies NQ is nilpotent, since N is nilpotent. In view of $(QN)^n = Q(NQ)^{n-1}N$, QN is also nilpotent.

On the other hand, if Q is Drazin invertible, then, by $Q^D = (Q^D)^2Q$, it follows that

$$Q^DN = (Q^D)^2QN = (Q^D)^2N^2Q = Q^D(Q^DN)NQ,$$

and we further have

$$Q^DN = (Q^D)^n(Q^DN)(NQ)^n,\ n = 1, 2, \cdots .$$

Since NQ is nilpotent, $Q^DN = 0$.

If Q is invertible, then $N = Q^{-1}N^2Q$. Thus, $N = (Q^{-1})^kN^{2^k}Q^k,\ k = 1, 2, \cdots$, which demonstrates $N = 0$ since N is nilpotent. □

Theorem 2.2. *Let* $P, Q \in \mathcal{B}(\mathcal{X})$, P *be Drazin invertible with* $\mathrm{ind}(P) = n$, *and* $P^2Q = QP$. *Denote by*

$$\mathcal{M} = \{P^kQ, QP^k, PQP, P^DQ, QP^D, PP^DQ, QPP^D, PQP^D, P^DQP, PP^DPQ\}, k = 1, 2, 3, \cdots .$$

(1) *If one of the elements in the set* $\mathcal{M}$ *is Drazin invertible, then all the elements are Drazin invertible.*

(2) *If* PQ *is Drazin invertible, then the following statements are true.*

(i) $(PQ)^D = PP^D(PQ)^D = (PQ)^DPP^D = (P^DQP)^D = P(QP)^DP^D$
$= (P^D)^2(P^DQ)^DP^2 = (P^D)^3(QP^D)^DP^3 = P^D(PP^DQ)^DP;$

(ii) $(P^kQ)^D = P^D(P^{k-1}Q)^DP = (P^D)^{k-1}(PQ)^DP^{k-1};$

(iii) $(P^kQ)^D(P^D)^kQP^k$ *is idempotent;*
(iv) $(PP^DPQ)^D = (PQ)^DPP^D = PP^D(PQ)^D$;
(v) $(PQ)^DP^DQP = P^DQP(PQ)^D$;
(v) $(PQ)^DPP^\pi = 0$.

Proof. Since P is Drazin invertible with $\operatorname{ind}(P) = n$, by Lemma 1.1 (i), P has the following operator matrix form

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & N_1 \end{pmatrix} \tag{1}$$

with respect to the space decomposition $\mathcal{H} = \mathcal{N}(P^\pi) \oplus \mathcal{R}(P^\pi)$, where P_1 is invertible, $N_1^n = 0$, and

$$P^D = \begin{pmatrix} P_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Let the operator matrix form of Q with respect to the above space decomposition be given by

$$Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}.$$

From $P^2Q = QP$, it follows that $P^{2n}Q = QP^n$, then

$$\begin{pmatrix} P_1^{2n}Q_1 & P_1^{2n}Q_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} Q_1P_1^n & 0 \\ Q_3P_1^n & 0 \end{pmatrix}.$$

Note that, since P_1 is invertible, we have $Q_2 = 0$ and $Q_3 = 0$ from $P_1^{2n}Q_2 = 0$ and $Q_3P_1^n = 0$. Thus,

$$Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_4 \end{pmatrix}. \tag{2}$$

Moreover, by $P^2Q = QP$, we also have that

$$P_1^2Q_1 = Q_1P_1, \quad N_1^2Q_4 = Q_4N_1, \tag{3}$$

which shows that

$$P_1Q_1 = P_1^{-1}Q_1P_1 \tag{4}$$

and N_1Q_4 is nilpotent by Lemma 2.1.

(1) Through computations, we get that

$$P^kQ = \begin{pmatrix} P_1^kQ_1 & 0 \\ 0 & N_1^kQ_4 \end{pmatrix}, \ QP^k = \begin{pmatrix} Q_1P_1^k & 0 \\ 0 & Q_4N_1^k \end{pmatrix}, \ k = 1, 2, \cdots.$$

According to Lemma 2.1, $N_1^kQ_4$ and $Q_4N_1^k$ are all nilpotent by (3), and hence $(N_1^kQ_4)^D = 0$ and $(Q_4N_1^k)^D = 0$. Thus, P^kQ and QP^k are Drazin invertible if and only if $P_1^kQ_1$ and $Q_1P_1^k$ are Drazin invertible, respectively. On the other hand, we obtain, from (4), that

$$P_1^kQ_1 = (P_1^{-1})^kQ_1P_1^k, \quad Q_1P_1^k = (P_1^{-1})^k(P_1^kQ_1)P_1^k = (P_1^{-1})^{2k}Q_1P_1^{2k}, \tag{5}$$

which implies that $P_1^kQ_1$ and $Q_1P_1^k$ are Drazin invertible if and only if Q_1 is Drazin invertible by Lemma 1.1 (2). Therefore, P^kQ and QP^k are Drazin invertible if and only if Q_1 is Drazin invertible. Also, $PQP = P^3Q$ by the assumption $P^2Q = QP$, and so, PQP is Drazin invertible if and only if Q_1 is Drazin invertible.

For P^DQ and QP^D, we observe that

$$P^DQ = \begin{pmatrix} P_1^{-1}Q_1 & 0 \\ 0 & 0 \end{pmatrix}, \ QP^D = \begin{pmatrix} Q_1P_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Applying (4), it follows that

$$P_1^{-1}Q_1 = P_1Q_1P_1^{-1}, \quad Q_1P_1^{-1} = P_1^2Q_1(P_1^{-1})^2. \tag{6}$$

Hence, P^DQ and QP^D are Drazin invertible if and only if Q_1 is Drazin invertible.

Obviously, $PP^DQ, QPP^D, PQP^D, P^DQP, PP^DPQ$ are all Drazin invertible if and only if Q_1 is Drazin invertible since $PP^DQ = QPP^D = \begin{pmatrix} Q_1 & 0 \\ 0 & 0 \end{pmatrix}$, $PQP^D = \begin{pmatrix} P_1Q_1P_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}$, $P^DQP = \begin{pmatrix} P_1^{-1}Q_1P_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $PP^DPQ = \begin{pmatrix} P_1Q_1 & 0 \\ 0 & 0 \end{pmatrix}$.

Therefore, the conclusion (1) is proven.

(2) If PQ is Drazin invertible, then, using (4), (5) and (6), we obtain that

$$\begin{aligned}
(PQ)^D &= \begin{pmatrix} (P_1Q_1)^D & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} P_1^{-1}Q_1^DP_1 & 0 \\ 0 & 0 \end{pmatrix}, \\
(QP)^D &= \begin{pmatrix} (Q_1P_1)^D & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (P_1^{-1})^2Q_1^DP_1^2 & 0 \\ 0 & 0 \end{pmatrix}, \\
(QP^D)^D &= \begin{pmatrix} (Q_1P_1^{-1})^D & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} P_1^2Q_1^D(P_1^{-1})^2 & 0 \\ 0 & 0 \end{pmatrix}, \\
(P^DQ)^D &= \begin{pmatrix} (P_1^{-1}Q_1)^D & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} P_1Q_1^DP_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \\
(P^kQ)^D &= \begin{pmatrix} (P_1^kQ_1)^D & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (P_1^{-1})^kQ_1^DP_1^k & 0 \\ 0 & 0 \end{pmatrix}, \\
(QP^k)^D &= \begin{pmatrix} (Q_1P_1^k)^D & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (P_1^{-1})^{2k}Q_1^DP_1^{2k} & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Thus, we can easily verify items (i)–(v). □

Similarly, we state the symmetrical formulation of Theorem 2.2.

Theorem 2.3. *Let $P, Q \in \mathcal{B}(\mathcal{X})$, Q be Drazin invertible with* $\text{ind}(Q) = t$, *and $PQ^2 = QP$. Denote by*

$$\mathcal{N} = \{PQ^k, Q^kP, QPQ, PQ^D, Q^DP, QQ^DP, QPQ^D, PQQ^D, QQ^DPQ\},\ k = 1, 2, 3, \cdots .$$

(1) *If one of the elements in the set $\mathcal{N}$ is Drazin invertible, then all the elements are Drazin invertible.*
(2) *If PQ is Drazin invertible, then the following statements are true.*
(i) $(PQ)^D = QQ^D(PQ)^D = (PQ)^DQQ^D = (QPQ^D)^D = Q^D(QP)^DQ$
$= (Q^2(PQ^D)^D(Q^D)^2 = Q^3(QP^D)^D(Q^D)^3 = Q(QQ^DP)^DQ^D$;
(ii) $(PQ^k)^D = Q(PQ^{k-1})^DQ^D = Q^{k-1}(PQ)^D(Q^D)^{k-1}$;
(iii) $(PQ^k)^DQ^kP(Q^D)^k$ *is idempotent;*
(iv) $(QQ^DPQ)^D = (PQ)^DQQ^D = QQ^D(PQ)^D$;
(v) $(PQ)^DQPQ^D = QPQ^D(PQ)^D$;
(v) $(PQ)^DQQ^\pi = 0$.

Next, we will give two sufficient conditions for the reverse order law.

Theorem 2.4. *Let $P, Q \in \mathcal{B}(\mathcal{X})$ be Drazin invertible, then*
(1) *if $P^2Q = QP$ and $PQ^D = Q^DP^2$, then $(PQ)^D = Q^DP^D$.*
(2) *if $PQ^2 = QP$ and $P^DQ = Q^2P^D$, then $(PQ)^D = Q^DP^D$.*

Proof. We only prove (1), and (2) is similar. If $P^2Q = QP$, then, from the proof of Theorem 2.2, the expressions (1) and (2) are valid, and $Q^D = \begin{pmatrix} Q_1^D & 0 \\ 0 & Q_4^D \end{pmatrix}$. Together with $PQ^D = Q^DP^2$, we have $P_1Q_1^D = Q_1^DP_1^2$, which concludes $P_1^{-1}Q_1^DP_1 = Q_1^DP_1^{-1}$. Also,

$$(PQ)^D = \begin{pmatrix} P_1^{-1}Q_1^DP_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q^DP^D = \begin{pmatrix} Q_1^DP_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Evidently, $(PQ)^D = Q^DP^D$. □

Theorem 2.5. *Let $P, Q \in \mathcal{B}(\mathcal{X})$ be Drazin invertible. If $P^2Q = QP = PQ^2$, then the following various types of reverse order laws and commutation relations hold.*

(1) $(PQQ^D)^D = Q(PQ)^D = (PQ)^DQ = (PP^DQ)^D = (Q^DP)^D$.
(2) $(PQQ)^D = Q^D(PQ)^D = (PQ)^DQ^D = (QPQ)^D$.
(3) $(QQP)^D = (QP)^DQ^D = Q^D(QP)^D$.
(4) PQ^D *is idempotent, and* $(PQ^D)^D = (PQ(Q^D)^2)^D = Q^2(PQ)^D = (PQ)^DQ^2$.
(5) $(Q^D)^2P$ *is idempotent, and* $(Q^DQ^DP)^D = (Q^DP)^DQ = Q(Q^DP)^D$.

Proof. If $P^2Q = QP = PQ^2$, then, from (1) and (2), we have $P_1^2Q_1 = Q_1P_1 = P_1Q_1^2$, which gives $P_1Q_1 = Q_1^2$, and so

$$(PQ)^D = \begin{pmatrix} (Q_1^D)^2 & 0 \\ 0 & 0 \end{pmatrix}. \tag{7}$$

Also, $N_1^2Q_4 = Q_4N_1 = N_1Q_4^2$ follows. Then,

$$N_1Q_4^D = N_1Q_4^2(Q_4^D)^3 = N_1^2Q_4(Q_4^D)^3 = N_1^2(Q_4^D)^2.$$

Further, we have $N_1Q_4^D = N_1^n(Q_4^D)^n, n = 1, 2, \cdots$, and so $N_1Q_4^D = 0$. Again, by $P_1Q_1 = Q_1^2$, $Q_1P_1 = P_1Q_1^2 = Q_1^3$ and $Q_1^D = Q_1(Q_1^D)^2$, we get that

$$Q_1^DP_1 = (Q_1^D)^2Q_1P_1 = (Q_1^D)^2Q_1^3 = Q_1^DQ_1^2, \tag{8}$$

$$P_1Q_1^D = P_1Q_1(Q_1^D)^2 = Q_1^2(Q_1^D)^2 = Q_1Q_1^D.$$

Moreover, $Q_4^DN_1 = 0$ according to (3) and Lemma 2.1. Hence,

$$\begin{aligned} PQQ^D &= \begin{pmatrix} P_1Q_1Q_1^D & 0 \\ 0 & N_1Q_4Q_4^D \end{pmatrix} = \begin{pmatrix} Q_1^2Q_1^D & 0 \\ 0 & 0 \end{pmatrix}, \\ Q^DP &= \begin{pmatrix} Q_1^DP_1 & 0 \\ 0 & Q_4^DN_1 \end{pmatrix} = \begin{pmatrix} Q_1^2Q_1^D & 0 \\ 0 & 0 \end{pmatrix}, \\ QP &= \begin{pmatrix} Q_1P_1 & 0 \\ 0 & Q_4N_1 \end{pmatrix} = \begin{pmatrix} Q_1^3 & 0 \\ 0 & Q_4N_1 \end{pmatrix}, \end{aligned} \tag{9}$$

$$\begin{aligned} QPQ &= \begin{pmatrix} Q_1P_1Q_1 & 0 \\ 0 & Q_4N_1Q_4 \end{pmatrix} = \begin{pmatrix} Q_1^3 & 0 \\ 0 & N_1Q_4^3 \end{pmatrix}, \\ Q^2P &= \begin{pmatrix} Q_1^2P_1 & 0 \\ 0 & Q_4^2N_1 \end{pmatrix} = \begin{pmatrix} Q_1^4 & 0 \\ 0 & Q_4^2N_1 \end{pmatrix}, \\ PQ^D &= \begin{pmatrix} P_1Q_1^D & 0 \\ 0 & N_1Q_4^D \end{pmatrix} = \begin{pmatrix} Q_1Q_1^D & 0 \\ 0 & 0 \end{pmatrix}, \\ (Q^D)^2P &= \begin{pmatrix} (Q_1^D)^2P_1 & 0 \\ 0 & (Q_4^D)^2N_1 \end{pmatrix} = \begin{pmatrix} Q_1Q_1^D & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{10}$$

Note that $(Q_1^2 Q_1^D)^D = Q_1^D$, and $Q_4 N_1, N_1 Q_4^3, Q_4^2 N_1$ are all nilpotent. Thus, items (1)–(5) can be proven. □

Finally, we show that the reverse order law $(PQ)^D = Q^D P^D$ holds under the condition $P^2Q = QP = Q^2P$.

Theorem 2.6. *Let $P, Q \in \mathcal{B}(\mathcal{X})$ be Drazin invertible with* $\mathrm{ind}(P) = n$ *and* $\mathrm{ind}(Q) = t$. *If $P^2Q = QP = PQ^2$, then the following reverse order law and commutation relations hold.*

(1) $(PQ)^D = Q^D P^D = P^D Q^D$.
(2) $PQ^D = Q^D P$.
(3) $P^D Q = Q P^D$.

Proof. By Lemma 1.1, Q_1, Q_4 in (2) have the following operator matrix forms

$$Q_1 = \begin{pmatrix} Q_{11} & 0 \\ 0 & Q_{14} \end{pmatrix}, \quad Q_4 = \begin{pmatrix} Q_{41} & 0 \\ 0 & Q_{44} \end{pmatrix} \tag{11}$$

with respect to the space decompositions $\mathcal{N}(P^\pi) = \mathcal{N}(Q_1^\pi) \oplus \mathcal{R}(Q_1^\pi)$ and $\mathcal{R}(P^\pi) = \mathcal{N}(Q_4^\pi) \oplus \mathcal{R}(Q_4^\pi)$, where Q_{11}, Q_{41} are invertible, and Q_{14}, Q_{44} are nilpotent. Let

$$P_1 = \begin{pmatrix} P_{11} & P_{12} \\ P_{13} & P_{14} \end{pmatrix}, \quad N_1 = \begin{pmatrix} N_{11} & N_{12} \\ N_{13} & N_{14} \end{pmatrix}$$

in (1) be operator matrix forms of P_1, N_1 with respect to the above two space decompositions.

From (3), it follows that $P_1^{2^t} Q_1^t = Q_1^t P_1$ and $N_1^{2^t} Q_4^t = Q_4^t N_1$. Let $P_1^{2^t} = \begin{pmatrix} \widetilde{P}_{11} & \widetilde{P}_{12} \\ \widetilde{P}_{13} & \widetilde{P}_{14} \end{pmatrix}$, then $\begin{pmatrix} \widetilde{P}_{11} Q_{11}^t & 0 \\ \widetilde{P}_{13} Q_{11}^t & 0 \end{pmatrix} = \begin{pmatrix} Q_{11}^t P_{11} & Q_{11}^t P_{12} \\ 0 & 0 \end{pmatrix}$. This deduces that $P_{12} = 0$ and $\widetilde{P}_{13} = 0$, i.e,

$$P_1 = \begin{pmatrix} P_{11} & 0 \\ P_{13} & P_{14} \end{pmatrix} \text{ and } P_1^{2^t} = \begin{pmatrix} \widetilde{P}_{11} & \widetilde{P}_{12} \\ 0 & \widetilde{P}_{14} \end{pmatrix},$$

and it can be seen that $\widetilde{P}_{12} = 0$. Thus, $P_1^{2^t} = \begin{pmatrix} P_{11}^{2^t} & \\ & P_{14}^{2^t} \end{pmatrix}$, which indicates that P_{11}, P_{14} are invertible since P_1 is invertible. Similarly, from $N_1^{2^t} Q_4^t = Q_4^t N_1$, we can conclude that $N_{12} = 0$, and $N_1^{2^t} = \begin{pmatrix} N_{11}^{2^t} & \\ & N_{14}^{2^t} \end{pmatrix}$. So N_{11}, N_{14} are nilpotent since N_1 is nilpotent. Also, by $N_1^2 Q_4 = Q_4 N_1$, we get $N_{11}^2 Q_{41} = Q_{41} N_{11}$. Then $N_{11} = 0$ follows from Lemma 2.1. Thus,

$$N_1 = \begin{pmatrix} 0 & 0 \\ N_{13} & N_{14} \end{pmatrix}.$$

Through computations, we get that

$$P^2Q = \begin{pmatrix} P_{11}^2 Q_{11} & 0 & 0 & 0 \\ P_{13}P_{11}Q_{11} + P_{14}P_{13}Q_{11} & P_{14}^2 Q_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & N_{14}N_{13}Q_{41} & N_{14}^2 Q_{44} \end{pmatrix},$$

$$QP = \begin{pmatrix} Q_{11}P_{11} & 0 & 0 & 0 \\ Q_{14}P_{13} & Q_{14}P_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & Q_{44}N_{13} & Q_{44}N_{14} \end{pmatrix},$$

$$PQ^2 = \begin{pmatrix} P_{11}Q_{11}^2 & 0 & 0 & 0 \\ P_{13}Q_{11}^2 & P_{14}Q_{14}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & N_{13}Q_{41}^2 & N_{14}Q_{44}^2 \end{pmatrix}.$$

Hence, according to $P^2Q = QP = PQ^2$, the following equalities are obtained,

$$P_{11}^2 Q_{11} = Q_{11}P_{11} = P_{11}Q_{11}^2, \tag{12}$$
$$P_{14}^2 Q_{14} = Q_{14}P_{14} = P_{14}Q_{14}^2, \tag{13}$$
$$N_{14}^2 Q_{44} = Q_{44}N_{14} = N_{14}Q_{44}^2, \tag{14}$$
$$N_{14}N_{13}Q_{41} = Q_{44}N_{13} = N_{13}Q_{41}^2, \tag{15}$$
$$Q_{14}P_{13} = P_{13}Q_{11}^2. \tag{16}$$

From (12), we obtain $P_{11} = Q_{11}$. Notice that $N_{13} = N_{13}Q_{41}^2(Q_{41}^2)^{-1} = Q_{44}N_{13}(Q_{41}^2)^{-1}$ by (15). Then $N_{13} = Q_{44}^t N_{13}(Q_{41}^{2t})^{-1} = 0$ since Q_{44} is nilpotent. Also, $Q_{14} = P_{14}Q_{14}^2 P_{14}^{-1}$ from (13), then we get

$$Q_{14} = P_{14}(P_{14}Q_{14}^2 P_{14}^{-1})^2 P_{14}^{-1} = P_{14}^2 Q_{14}^4 (P_{14}^2)^{-1},$$

and we further have $Q_{14} = P_{14}^t Q_{14}^{2^t}(P_{14}^t)^{-1}$, which concludes $Q_{14} = 0$. Hence, $P_{13} = 0$ by (16). Thus, we have that

$$P = \begin{pmatrix} P_{11} & 0 & 0 & 0 \\ 0 & P_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_{14} \end{pmatrix}, \quad Q = \begin{pmatrix} Q_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & Q_{41} & 0 \\ 0 & 0 & 0 & Q_{44} \end{pmatrix}. \tag{17}$$

Moreover, (14) shows that $N_{14}Q_{44}$ is nilpotent. Therefore,

$$(PQ)^D = \begin{pmatrix} Q_{11}^{-1}P_{11}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad P^D = \begin{pmatrix} P_{11}^{-1} & 0 & 0 & 0 \\ 0 & P_{14}^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q^D = \begin{pmatrix} Q_{11}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & Q_{41}^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Obviously, items (1)–(3) hold. □

From the proof of Theorem 2.6, we can obtain the Drazin invertibility of $P + Q$.

Theorem 2.7. *Let $P, Q \in \mathcal{B}(\mathcal{X})$ be Drazin invertible. If $P^2Q = QP = PQ^2$, then $P + Q$ is Drazin invertible, and $(P + Q)^D = P^D + Q^D - \frac{3}{2}P^D PQ^D$.*

Proof. Since N_{14} and Q_{44} are nilpotent, it is easy to check that $N_{14} + Q_{44}$ is also nilpotent by (14). Then, the conclusion can be obtained according to the proof of Theorem 2.6. □

The following results are obtained immediately from Theorem 2.7.

Corollary 2.8. *Let $P, Q \in \mathcal{B}(\mathcal{X})$ be Drazin invertible. If $PQ = QP = 0$, then $P + Q$ is Drazin invertible, and $(P + Q)^D = P^D + Q^D$.*

Corollary 2.9. *Let $P, Q \in \mathcal{B}(\mathcal{X})$ be idempotent. If $PQ = QP$, then $P + Q$ is Drazin invertible, and $(P + Q)^D = P + Q - \frac{3}{2}PQ$.*

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