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Study of a Second-Order Nonlinear Elliptic Problem Generated by a Divergence Type Operator on a Compact Riemannian Manifold

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Abstract. In this paper, we will study a second-order nonlinear elliptic problem generated by an operator of divergence type (or type leray-Lion) :

$$(P1)\begin{cases} A(u) = f & in \quad M \\ u = 0 & on \quad \Gamma \end{cases} \tag{1}$$

on (M, g) a compact Riemannian manifold et Γ its border.

1. Introduction

Let (M, g) be a compact Riemannian manifold of dimension n ($n \geq 2$) and Γ its border, we consider the nonlinear elliptic problem of the following Dirichlet type (1)
with

$$A(u) = -div_g(a(x, u, \nabla u)) \qquad = -\sum_{i,j=1}^{N} \frac{\partial a_i}{\partial x_i} + a_j \Gamma^i_{ij}$$

where Γ^i_{ij} represents the symbol of Christoffel and

$$a = \sum_{i=1}^{N} a_i (\frac{\partial}{\partial x_i})$$

is a family of functions defined on $M \times \mathbb{R} \times \mathbb{R}^N$ has value in $\mathbb{R}$.

The goal of this work is to show that the Dirichlet type problem (P1) admits at least one solution in the variational case (ie, when the second member f is in a dual space).

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The main interest of Sobolev space theory lies in the existence of continuous embeddings of Sobolev, Poincare inequalities, and in the existence of the embedding We add to this list the existence of the regularity theorems very important and particularly important in the study of the equations to the partial linear and nonlinear partial defects, edge problems, calculation of the variations. The importance of these spaces is broader, including questions of differential geometry, analytic topology, complex analysis and probability theory. The Sobolev space theory classics on the Euclidean space has become integrated into other geometric frameworks.

The first understanding of the Sobolev spaces on the Riemannian manifolds is due to Thierry Aubin in 1976[1],[4][5][6]. He used his results in connection with the non-linear EDPs on the manifolds. Sobolev spaces on compact manifolds have been used for a long time (Ebin works). They do not essentially differ from the Sobolev spaces on a ball of $\mathbb{R}^n$. The case of the complete non-compact Riemannian manifolds is more delicate. Equations to partial derivatives allow to approach from a mathematical point of view the phenomena observed, for example in the fields of physics and chemistry. The situations of time depend more particularly on evolution equations taking into account the possible interactions between objects and events.

In Euclidean domains, the theory of Lebesgue–Sobolev spaces has applications in non-linear elastic mechanics [2], electrorheological uids[3] . The study of equations with elliptic partial equations is one of the research topics of great importance in the analysis of variability and development of these models. last years in many works [8] [9][10][11][12][13]. The resolution of elliptic partial differential equations and the problems related to conformal geometry led to the development of tools non-linear analysis, such as the "variational method" for solving the problem of Yamabe, the problem of scalar curvature.

The work presented in this paper concerns an equation with partial derivatives of the elliptic type involving the divergence operator $A(u) = -div(a(x, u, \nabla u))$, where $a = (a_i)_{1\leq i\leq N}$ is a family of functions defined on $M \times \mathbb{R} \times \mathbb{R}^N$ has value in $\mathbb{R}$. is a verifying field of hypotheses of the Leray-Lions type.

2. Fundamental theorems of existence

Theorem 2.1. *See,[14][15] Let X be a Banach Reflexive space and let $A : X \to X'$ an operator having the following properties:*

(P1) A is bounded hemicontinuous .

(P2) A is monotone .

(P3) A is coercive ,e.i, $\frac{<A(v),v>}{\|v\|} \to \infty$ *if* $\|v\| \to \infty$

Then A is surjective of $X \to X'$, i.e, for everything $f \in X'$, it exists $u \in X$ such as:

$$(Pv)\left\{ \; A(u)=f \right.$$

Theorem 2.2. *let X be a reflexive Banach space and let $A : X \to X'$ an operator having the following properties :*

(P1) A is pseudo-monotone .

(P2) A is coercive ,e.i, $\frac{<A(v),v>}{\|v\|} \to \infty$ *if* $\|v\| \to \infty$.

Then A is surjective of $X \to X'$,i.e , for everything $f \in X'$, it exists $u \in X$ such as:

$$(Pv)\left\{ \; A(u)=f \right.$$

3. Preliminaries

This section devoted to the presentation of sobolev spaces on the Riemannian manifolds.

3.1. First definitions

Let (M,g) a Riemannian manifold, for an integer k and $u \in C^{\infty}(M)$, $\nabla^k u$ represents the $k-th$ of the covariant derivative of u (with the Convention $\nabla^0 u = u$) . and the norm of $k-th$ of covariant derivative on a local map is given by the following formula :

$$|\nabla^k u| = g^{i_1 j_1} ....g^{i_k j_k}(\nabla^k u)_{i_1....i_k}(\nabla^k u)_{j_1....j_k}$$

where the Einstein summons is adopted.

3.2. Space of lebesgue and sobolev on the Riemennian manifolds

See[18]

Let be $p \geq 1$ a real, and k a positive integer.

$$L^p(M) = \{u : M \to \mathbb{R} \quad measurable \ / \int_M |u|^p d\sigma_g < \infty\}$$

$C^p_k(M)$ functions space $u \in C^{\infty}$ such as $|\nabla^j u| \in L^p(M)$ for $j = 0, ..., k$

$$C^p_k(M) = \{u \in C^{\infty} \ /\forall j = 0, ..., k \quad \int_M |\nabla^j u|^p d\sigma_g < \infty\}$$

Definition 3.1. *The Sobolev space $W^{k,p}(M)$is the complete space $C^p_k(M)$ for the norm*

$$||u||_{W^{k,p}(M)} = \sum_{j=0}^{k} ||\nabla^j u||_{L^p(M)}$$

$$||u||_{W^{1,p}(M)} = ||\nabla u||_p + ||u||_p$$

Definition 3.2. *We must recall the notion of the geodesic distance for every curve :*

$$\Upsilon : \ [a, b\ ] \to M$$

We define the length of Υ by :

$$l(\Upsilon) = \int_a^b \sqrt{g(\Upsilon(t))(\frac{d\Upsilon}{dt}, \frac{d\Upsilon}{dt})}dt$$

Remark 3.3. *For $x, y \in M$ defining a distance d_g by :*

$$d_g(x, y) = inf\{l(\Upsilon) : \Upsilon : [0, 1] \to M \quad . \quad \Upsilon(0) = x \quad , \quad \Upsilon(1) = y\}$$

By the theorem of Hopf-Rinow, we obtain that if M a Riemannian manifold then compact for all x, y in M can be joined by a courbe minimisant Υ i.e $l(\Upsilon) = d_g(x, y)$

Proposition 3.4. *If $p = 2$, space $W^{k,2}(M)$ is a Hilbert space for the following scalar product*

$$(u, v)_{H^k} = \sum_{j=0}^{k} \left(\nabla^j u, \nabla^j v\right)_{L^2} .$$

Proposition 3.5. *If $p > 1$ then $W^{k,p}(M)$ is reflexive.*

Proposition 3.6. *Any reflex normalized space is a Banach space. Then if $p > 1$ then $W^{k,p}(M)$ is Banach.*

Definition 3.7. *The Sobolev space $W^{k,p}_0(M)$ is the closure of $\mathcal{D}(M)$ in $W^{k,p}(M)$.*

Theorem 3.8. *If (M, g) is complete, then for all $p \geq 1$ $W^{1,p}_0(M) = W^{1,p}(M)$.*

3.3. Embeddings of Sobolev:

See[18].

Lemma 3.9. *Let (M,g) a complete Riemannian manifold of dimension n. Suppose that inclusion $W^{1,1}(M) \subset L^{\frac{n}{(n-1)}}(M)$ is valid. Then, for a whole real number $1 \leq q < p$ and an integer $0 \leq m < k$ which verify $\frac{1}{p} = \frac{1}{q} - \frac{(k-m)}{n}$, $W^{k,q}(M) \subset W^{m,p}(M)$.*

Proof. It is shown that if $W^{1,1}(M) \subset L^{n/(n-1)}(M)$ then, for all $1 \leq q < n$ and $1/p = 1/q - 1/n$, $W^{1,q}(M) \subset L^p(M)$.
Let $A \in \mathbb{R}$ such as $\forall \; u \in W^{1,1}(M)$ we have,

$$\left(\int_M |u|^{n/(n-1)} d\sigma_g\right)^{(n-1)/n} \leq A \int_M (|\nabla u| + |u|)\, d\sigma_g.)$$

let $\phi = |u|^{p(n-1)/n}$. We apply the Holder inequality, we have

$$\begin{aligned}
\left(\int_M |u|^p d\sigma_g\right)^{(n-1)/n} &= \left(\int_M |\phi|^{n/(n-1)} d\sigma_g\right)^{(n-1)/n} \\
&\leq A \int_M \Big(|\nabla\phi| + |\phi|\Big)\, d\sigma_g \\
&= \frac{Ap(n-1)}{n} \int_M |u|^{p'} |\nabla u| d\sigma_g + A \int_M |u|^{p(n-1)} d\sigma_g \\
&\leq \frac{Ap(n-1)}{n} \left(\int_M |u|^{p'q'} d\sigma_g\right)^{1/q'} \left(\int_M |\nabla u|^q d\sigma_g\right)^{1/q} + A \left(\int_M |u|^{p'q'} dx\right)^{1/q'} \\
&\quad \left(\int_M |u|^q d\sigma_g\right)^{1/q}.
\end{aligned}$$

Where $1/q + 1/q' = 1$ and $p' = p(n-1)/n - 1$. But $p'q' = p$ since $1/p = 1/q - 1/n$. Therefore, for all $u \in \mathcal{D}(M)$,

$$\left(\int_M |u|^p d\sigma_g\right)^{1/p} \leq \frac{Ap(n-1)}{n} \left\{ \left(\int_M |\nabla u|^q d\sigma_g\right)^{1/q} + \left(\int_M |u|^q d\sigma_g\right)^{1/q} \right\}$$

According to Theorem 3.8, which ends the □

Remark 3.10. *Note that the proof of the Lemma 3.9 shows that if $A \in \mathbb{R}$ is such that $\forall \; u \in W^{1,1}(M)$,*

$$\left(\int_M |u|^{n/(n-1)} d\sigma_g\right)^{(n-1)/n} \leq A \int_M (|\nabla u| + |u|)\, d\sigma_g)$$

So, for all $1 \leq q < n$ and all $u \in W^{1,q}(M)$,

$$\left(\int_M |u|^p d\sigma_g\right)^{1/p} \leq \frac{Ap(n-1)}{n} \left\{ \left(\int_M |\nabla u|^q d\sigma_g\right)^{1/q} + \left(\int_M |u|^q d\sigma_g\right)^{1/q} \right\}$$

Where $1/p = 1/q - 1/n$.

Theorem 3.11. *Let (M,g) a compact Riemannian manifold of dimension n. For a real number $1 \leq q < p$ and an integer $0 \leq m < k$ which verify $\frac{1}{p} = \frac{1}{q} - \frac{(k-m)}{n}$, $W^{k,q}(M) \subset W^{m,p}(M)$.*

Theorem 3.12. ***(Rellich-Kondrakov's Theorem):*** *Let (M, g) a compact Riemannian manifold of n dimension , $j \geq 0$ and $m \geq 1$ two integers, $q \geq 1$ and p two real numbers that verify $1 \leq p < nq/(n - mq)$, the inclusion $W^{j+m,q}(M) \subset W^{j,p}(M)$ is compact*

Corollary 3.13. *Let (M, g) a compact Riemannian manifold of n dimension. For everything $1 \leq q < n$ and $p \geq 1$ such as $\frac{1}{p} > \frac{1}{q} - \frac{1}{n}$, the inclusion $W^{1,q}(M) \subset L^p(M)$ is compact.*

Lemma 3.14. ***(Inequality of Poincare):*** *Let D a regular domain is bounded in a Riemannian manifold M and $1 \leq p < \infty$. Then there is a constant A such as:*

$$\left(\int_D |u - u_D|^p d\sigma_g\right)^{\frac{1}{p}} \leq A \left(\int_D |\nabla u|^p d\sigma_g\right)^{\frac{1}{p}},$$

for everything $u \in W^{1,p}_{loc}(M)$, where $u_D = \frac{1}{vol(D)} \int_D u d\sigma_g$ is the mean value of u on D

Proof. Let us first suppose that $p > 1$. In order to show the lemma, it suffices to show that

$$\inf_{u \in \mathcal{H}} \int_D |\nabla u|^p \, d\sigma_g > 0$$

Where

$$\mathcal{H} = \left\{ u \in W^{1,p}_{loc}(M) \text{ tel que } \int_D |u|^p \, d\sigma_g = 1 \text{ et } \int_d u d\sigma_g = 0 \right\}$$

Let $(u_k) \in \mathcal{H}$ such as

$$\lim_{k\to\infty} \int_D |\nabla u_k|^p d\sigma_g = \inf_{u\in\mathcal{H}} \int_D |\nabla u|^p d\sigma_g$$

Combining the fact that $W^{1,p}_{loc}(M)$ is reflexive for $p > 1$ and the Theorem of Rellich-Kondrachov, there exists a sub-sequence (u_k) of (u_k) which converges weakly in $W^{1,p}_{loc}(M)$ and strongly in $L^p(M) \cap L^1(M)$.

Note v its limit, the strong convergence in $L^p(M) \cap L^1(M)$ implies that $v \in \mathcal{H}$, and weak convergence implies that

$$\int_D |\nabla v|^p d\sigma_g \leq \lim_{k\to\infty} \int_D |\nabla u_k|^p d\sigma_g$$

Therefore ,

$$\inf_{u\in\mathcal{H}} \int_D |\nabla u|^p d\sigma_g > 0$$

This shows the inequality of Poincare for $p > 1$. When $p = 1$ (see Lemma 3.8) □

By combining this lemma with the Holder inequality, we obtain:

Corollary 3.15. *There exists a constant $c = c_D$ such that*

$$\int_D |u - u_D| d\sigma_g \leq c_D \left(\int_M |\nabla u|^p d\sigma_g\right)^{\frac{1}{p}} \qquad \forall\ u \in W^{1,p}_{loc}(M)$$

Proof. We apply the inequality of Hölder, we will have

$$\begin{aligned}\int_D 1.|u - u_D| d\sigma_g &\leq \left(\int_D 1^{p'} d\sigma_g\right)^{\frac{1}{p'}} \left(\int_D |u - u_D|^p d\sigma_g\right)^{\frac{1}{p}} \\ &\leq A vol(D)^{\frac{1}{p'}} \left(\int_M |\nabla u|^p d\sigma_g\right)^{\frac{1}{p}}\end{aligned}$$

□

4. Formulation of the problem

Let (M, g) be a compact Riemannian manifold of dimension n $(n \geq 2)$ et Γ its border . We consider the nonlinear elliptic problem of the following Dirichlet type :

$$(P1)\left\{ \; A(u) = f \quad on \quad M u = 0 \quad on \quad \Gamma \right.$$

with

$$A(u) = -div_g(a(x, u, \nabla u)) = -\sum_{i,j=1}^{N} \frac{\partial a_i}{\partial x_i} + a_j \Gamma^i_{ij}$$

where

$$a = \sum_{i=1}^{N} a_i(\frac{\partial}{\partial x_i})$$

is a family of functions defined on $M \times \mathbb{R} \times \mathbb{R}^N$ to value in $\mathbb{R}$.

The goal of this work is to show that the Dirichlet type problem (P1) allows at least one solution in the variational case (i.e, when the second member f is in a dual space).

4.1. Hypotheses

H_1 : Growth Condition:

Each $a_i(x, \eta, \xi)$ a caratheodory function, i.e x for everything (η, ξ)fixed in $\mathbb{R} \times \mathbb{R}^N$ and continues in $(\eta, \xi)$for everything x fixed in M.

For some $p > 1$ there is a constant $C_1 > 0$ and a function $K \in L^{p'}(M)$ such as :

$$|a_i(x, \eta, \xi| \leq C_1(K(x) + |\eta|^{p-1} + |\xi|^{\frac{q}{p'}})$$

For almost everywhere $x \in M$, all $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$, all $i = 1, 2, ..., N et$ Where

$$\begin{cases} 1 \leq q < \frac{NP}{N-p} & if \quad p < N \\ 1 \leq q < \infty & if \quad p = N \end{cases}$$

Where, respectively,

$$|a_i(x, \eta, \xi| \leq h(|\eta|)(K(x) + |\xi|^{p^{p-1}})$$

Where $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function , if $p > N$.

H_2 : Large monotony condition :

For almost everywhere.$x \in M$ and all $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$ we suppose that

$$\sum_{i=1}^{N} (a_i(x, u, \xi) - a_i(x, u, \xi^*))(\xi_i - \xi^*_i) \geq 0$$

H_3 : Elipticity condition :

It exists a constant $C_0 > 0$ and a function $K_0 \in L^1(M)$ such as

$$\sum_{i=1}^{N} (a_i(x, u, \xi^*)\xi^*_i) \geq C_0|\xi|^p - K_0(x)$$

For almost everywhere $x \in M$ and all $(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^N$.

Remark 4.1. *We can associate to the operator A the semi-linear form of Dirichlet defined by :*

$$(4.1) \quad b(u,v) = \sum_{i=1}^{N} \int_M (a_i(x,u,\nabla u))\omega_i v(x) d\sigma_g \quad \forall u, v \in W_0^{1,p}(M)$$

Where

$$\omega_i v = \sum_{i=1}^{N} g^{ij} \frac{\partial v}{\partial x_i}$$

With g^{ij} represents the inverse matrix of g_{ij}, and σ_g volume measurement on M induced by the metric g .

Proposition 4.2. *If the hypothesis $(H1)$ is satisfied then the semi-linear form $b(.,.)$ is bounded in $W_0^{1,p}(M)$.*

Proof. According to the inequality of Holder, we can write

$$|b(u,v)| \leq \sum_{i=1}^{N} \|(a_i(x,u,\nabla u))\|_{p'} \|\omega_i v(x)\|_p$$

with

$$\begin{aligned} |(a_i(x,u,\nabla u))|^{p'} &\leq C(|k(x)|^{p'} + |u(x)|^q + |\nabla u(x)|^p), if \quad p \leq N \\ &\leq Ch(|u(x)|)^{p'} (|k(x|^{p'} + |\nabla u(x)|^p) \quad if \quad p > N \end{aligned}$$

using the Sobolev embedding , we obtain :

$$\left(\int_M |u(x)|^q d\sigma_g \right)^{\frac{1}{q}} \leq C\|u\|_{1,p} \quad if \quad p \leq N$$

and

$$\sup_{x\in M} |u(x)| \leq C\|u\|_{1,p} \quad if \quad p \leq N$$

and

$$\sup_{x\in M} |u(x)| \leq C\|u\|_{1,p} \quad if \quad p > N$$

Therefore ; $|(a_i(x,u,\nabla u))|_{p'}$ is bounded if $\|u\|_{1,p}$ is bounded .

Hence, the operator A induces a bounded T function defined by $W_0^{1,p}(M)$ to its dual $W^{-1,p'}(M)$ by :

$$< T(u,v) >= b(u,v), \forall u, v \in W_0^{1,p}(M)$$

□

Proposition 4.3. *If the hypothesis $(H1)$ is satisfied, then the functional T is continuous .*

4.2. Result of existence

We show the problem of Dirichlet $(P1)$ admits at least one weak solution in the following sense :

Definition 4.4. *We say that $u \in W_0^{1,p}(M)$ is a weak solution of Dirichlet problem $(P1)$ if*

$$\sum_{i=1}^{N} \int_M (a_i(x,u,\nabla u)) D_i v(x) dx = \int_M f(x) v(x) dx \quad \forall u, v \in W_0^{1,p}(M).$$

Theorem 4.5. *Let (M,g) be a compact Riemannian manifold of n dimension. suppose that the hypotheses $(H1)$, $(H2)$, $(H3)$ are satisfied. Then for any $f \in W^{-1,p'}(M)$, the problem $(P1)$ allows at least a weak solution $u \in W_0^{1,p}(M)$.*

4.3. Proof

We show the result for $p \leq N$ and the same approach applies for $p > N$ using the corresponding Sobolev embedding.

Let T the functional associated with the operator A defined by the functional (4.2) .i.e ,

$$(4.3) < T(u), v >= \sum_{i=1}^{N} \int_{M} (a_i(x, u, \nabla u))\omega_i v(x) d\sigma_g \quad \forall u, v \in W_0^{1,p}(M)$$

Step 1:

We show that the functional T associated to A is peudo-monotone .

indeed , Let (U_n) a sequence of elements of the space $W_0^{1,p}(M)$ such that

$$\begin{cases} (4.4) \quad U_n \rightharpoonup u \quad in \quad W_0^{1,p}(M) \\ (4.5) \quad \lim sup < T(U_n), U_n - u > \leq 0 \end{cases}$$

Assertion 1:

It is asserted that $T(U_n) \rightharpoonup T(u)$ in $W^{-1,p'}(M)$. According to the (4.4) hypothesis and the compact embedding $W_0^{1,p}(M) \hookrightarrow\hookrightarrow L^q(M)$

We have :

$$\begin{cases} (4.6) \quad \omega_i(U_n) \rightharpoonup \omega_i(u) \quad in \quad L^q(M) \quad \forall i \in \{0; 1; ...., N\} \\ (4.7) \quad U_n \longrightarrow u \quad in \quad L^q(M) \\ (4.8) \quad U_n(x) \longrightarrow u(x) \quad almost \quad all \quad in \quad M \end{cases}$$

We note that the last two convergences are for a subsequence of (U_n) not again (U_n).

Since T is bounded, then one can write for a subsequence denoted by (U_n)

$$(4.9) \quad T(U_n) \rightharpoonup S \quad in \quad W^{-1,p'}(M)$$

$$(4.10) \quad a_i(., U_n, \nabla U_n) \rightharpoonup S_i \quad L^{p'}(M) \quad \forall i \in \{1; ...., N\}.$$

Where the action of S is given by :

$$< S, v >= \sum_{i=1}^{N} \int_{M} S_i(x) d_i v(x) d\sigma_g \quad \forall u, v \in W_0^{1,p}(M)$$

Moreover, by virtue (2.6) and (2.10) we obtain :

$$(4, 11) \quad \lim sup < T(U_n), U_n > \quad \leq \quad < S, u >$$

as the condition of large monotony $(H1)$ allows to write

$$\sum_{i=1}^{N} \int_{M} (a_i(x, U_n, \nabla v) - a_i(x, U_n, \nabla U_n))(\omega_i v - \omega_i U_n) d\sigma_g \geq 0 \quad Pour \quad tout \quad v \in W_0^{1,p}(M)$$

So ,

$$(4, 12) \quad \sum_{i=1}^{N} \int_{M} a_i(x, U_n, \nabla U_n)\omega_i U_n d\sigma_g$$

$$\geq \sum_{i=1}^{N} \int_{M} a_i(x, U_n, \nabla U_n)\omega_i v(x) d\sigma_g + \sum_{i=1}^{N} \int_{M} a_i(x, U_n, \nabla v)\omega_i U_n d\sigma_g$$

$$-\sum_{i=1}^{N}\int_M a_i(x,U_n,\nabla v)\omega_i v(x)d\sigma_g$$

Thus, by applying (4.7); (4.10) and (4.11) where will have

$$\sum_{i=1}^{N}\int_M S_i(x)\omega_i u(x)d\sigma_g \geq limsup\sum_{i=1}^{N}\int_M a_i(x,U_n,\nabla U_n)\omega_i U_n d\sigma_g$$

$$\geq \sum_{i=1}^{N}\int_M S_i(x)\omega_i v(x)d\sigma_g + \sum_{i=1}^{N}\int_M (a_i(x,u,\nabla v)\omega_i u(x)d\sigma_g - \sum_{i=1}^{N}\int_M (a_i(x,u,\nabla v)\omega_i v(x)d\sigma_g$$

Therefore,we will have:

$$\sum_{i=1}^{N}\int_M (a_i(x,u,\nabla v) - S_i)(\omega_i v(x) - \omega_i u(x)d\sigma_g \geq 0 \quad \forall v \in W_0^{1,p}(M)$$

take $v = u + tw$ with $t > 0$ and $w \in W_0^{1,p}(M)$

We will have :

$$\sum_{i=1}^{N}\int_M (a_i(x,u,\nabla u + t\nabla w) - S_i)\omega_i w(x)d\sigma_g \geq 0 \quad \forall w \in W_0^{1,p}(M)$$

make tender t to 0^+ , we conclude that for all $i = \{1,...,N\}$

We have $a_i(x,u(x),\nabla u(x) = S_i(x)$ almost everywhere in M .

Therefore $T(x) = S$ and then the assertion $T(U_n) \rightharpoonup T(u)$ is proved.

Assertion 2:

it is asserted that $< T(U_n), U_n > \longrightarrow < T(u), u >$since it has already been shown that

$$\lim sup \quad < T(U_n), U_n > \quad \leq \quad < S, u > \quad = \quad < T(u), u >$$

it is enough to show that

$$\lim inf \quad < T(U_n), U_n > \quad \geq \quad < T(u), u >$$

Indeed , by taking $v = u$ in the inequality (2.13) we obtain from the above,

$$\lim inf \quad < T(U_n), U_n > \quad = \lim inf \quad \sum_{i=1}^{N}\int_M a_i(x,U_n,\nabla U_n)d_i U_n d\sigma_g$$

$$\geq \sum_{i=1}^{N}\int_M S_i(x)\omega_i(u(x))d\sigma_g + \sum_{i=1}^{N}\int_M a_i(x,u,\nabla u)\omega_i(u(x))d\sigma_g - \sum_{i=1}^{N}\int_M a_i(x,u,\nabla v)\omega_i(v(x))d\sigma_g$$

$$=< T(u), u >$$

So assertion 2 is proved.

2nd step :

We show that the functional T is coercive,

Indeed, by virtue of the condition of ellipticity e $(H3)$ and the inequality of the poincare we can write:

$$< T(u), u > \quad = \sum_{i=1}^{N}\int_M a_i(x,u,\nabla u)\omega_i(u(x))d\sigma_g$$

$$\geq C_0 \int_M |\nabla u|^p d\sigma_g - \int_M k(x) d\sigma_g \geq C_1 \|u\|_{1,p} - C_2.$$

which gives directly the coercivity of the functional T .
And by applying the existence theorems we deduce that the problem $(P1)$ admits at least one solution $u \quad \in W_0^{1,p}(M)$.

Remark 4.6. *It should be noted that in the demonstration of the pseudo-monotone of* T*, we have used only the hypotheses* $(H1)$; $(H2)$ *and not* $(H3)$.

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