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Ricci Solitons on η-Einstein Contact Manifolds

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Abstract. The object of the present paper is to study Ricci solitons on η-Einstein contact manifolds. As a consequence of the main result we deduce some important corollaries.

1. Introduction

In 1982, R. S. Hamilton [25] introduced the notion of Ricci flow to find a canonical metric on a Riemannian manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}. \tag{1}$$

Ricci solitons are special solutions of the Ricci flow equation (1) of the form $g_{ij} = \sigma(t)\psi_t^* g_{ij}$ with the initial condition $g_{ij}(0) = g_{ij}$, where ψ_t are diffeomorphisms of M and $\sigma(t)$ is the scaling function.
A Ricci soliton is a generalization of an Einstein metric. We follow the notion of Ricci soliton according to [14]. On the manifold M, a Ricci soliton is a triple (g, V, λ) with g, a Riemannian metric, V a vector field, called potential vector field and λ a real scalar such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{2}$$

where $\pounds$ is the Lie derivative and S is the Ricci tensor. Metrics satisfying (2) are interesting and useful in physics and are often referred as quasi-Einstein metrics ([8],[9]).

Theoretical physicists have also been looking into the equation of Ricci soliton in relation with string theory. The initial contribution in this direction is due to Friedan [22] who discusses some aspects of it.
The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive. If the vector field V is the gradient of a potential function $-f$, then g is called a gradient Ricci soliton and equation (2) takes the form

$$\nabla\nabla f = S + \lambda g,$$

where ∇ denotes the Levi-Civita connection.

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A Ricci soliton on a compact manifold has constant curvature in dimension 2 (Hamilton [25]), and also in dimension 3 (Ivey [26]). For details we refer to Chow and Knoff [15]. We also recall the following significant result of Perelman [31]: A Ricci soliton on a compact manifold is a gradient Ricci soliton.
Since the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians. In particular, it has become more important after Grigory Perelman solved the Poincare conjecture. An Einstein manifold is a trivial example of a gradient Ricci soliton with constant potential function and therefore it is called a trivial Ricci soliton. There exist many non-trivial examples of Ricci solitons compact as well as non-compact ([15],[26], [27]).
There are two aspects of the study of Ricci solitons, one looking at the influence on the topology by the Ricci soliton structure of the Riemannian manifold ([19],[37]) and the other looking at its influence on its geometry ([20], [21]).
On the other hand, the roots of contact geometry lie in differential equations as in 1872 Sophus Lie introduced the notion of contact transformation as a geometric tool to study systems of differential equations. This subject has multiple connections with the other fields of pure mathematics , and substantial applications in applied areas such as mechanics, optics, phase space of dynamical system, thermodynamics and control theory.
In a recent paper Wang et al. [34] studied Ricci solitons on three dimensional η-Einstein almost Kenmotsu manifolds. Also Ghosh [24] studied η-Einstein Kenmotsu metric as a Ricci soliton. However Ricci soliton on η-Einstein contact metric manifold have not been studied. Also Ricci solitons and gradient Ricci solitons on some kinds of almost contact metric manifolds of dimension three were studied by many authors. For instances, De et al. [18] and Turan et al. [33] investigated Ricci solitons and gradient Ricci solitons on three-dimensional normal almost contact metric manifolds and three-dimensional trans- Sasakian manifolds respectively. Moreover, A. Ghosh [23] and J. T. Cho [10] classified Ricci solitons on three-dimensional Kenmotsu manifolds respectively. In addition, Ricci solitons on f-Kenmotsu manifolds and N(k)-quasi-Einstein manifolds were also studied by C. Calin and M. Crasmareanu [14] and M. Crasmareanu [13] respectively

In a recent paper J.T. Cho [12] studied Ricci solitons on almost contact geometry and proved that a three dimensional contact Ricci soliton (g, ξ) is Sasakian and of constant curvature +1. Ricci solitons have been studied by several authors such as ([6], [10], [11], [12], [17], [34], [35]) and many others. Motivated by these circumstances, in this paper we study Ricci solitons on η-Einstein contact manifolds.
In a contact metric manifold $M^{(2n+1)}(\phi, \xi, \eta, g)$, the $(1, 1)$ tensor field h is defined by $h = \frac{1}{2}\pounds_{\xi}\phi$. During the last five decades the notion of locally symmetric manifolds have been weakened by many authors in several ways to a different extent such as recurrent manifolds introduced by Walker [36].
In a contact metric manifold the $(1, 1)$ tensor field h is said to be recurrent if it satisfies the condition

$$(\nabla_X h)(Y) = \eta(X)hY,$$

where η is the 1-form of the contact metric manifold.
A contact manifold is said to be η-Einstein if the Ricci tensor S of type $(0, 2)$ satisfies the condition

$$S(X, Y) = \alpha g(X, Y) + \beta\eta(X)\eta(Y), \tag{3}$$

where α, β are non-constant smooth function. Such a structure in a Riemannian manifold is called quasi-Einstein. Mantica et al. ([29], [30]) have obtained the physical interpretation of quasi-Einstein manifold in perfect fluid space-time.
The paper is organized as follows: After introduction in section 2 we discuss some preliminaries of contact metric manifolds. Section 3 is devoted to study our main result. Our main Theorem can be presented as follows:

Main Theorem:

Theorem 1.1. *Let the metric g of an η-Einstein connected contact metric manifold be a Ricci soliton (g, V). If the tensor h is recurrent, then the manifold is an Einstein manifold.*

As a consequence of the main Theorem we obtain the following corollary:

Corollary 1.2. *Let the metric g of an η-Einstein connected contact metric manifold be a Ricci soliton (g, V). If the tensor h is recurrent, then V is a homothetic vector field.*

2. Contact metric manifolds

A $(2n+1)$-dimensional manifold M is said to admit an almost contact structure if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying ([1],[2])

$$(a)\ \ \phi^2 = -I + \eta \otimes \xi,\ \ (b)\ \ \eta(\xi) = 1,\ \ (c)\ \ \phi\xi = 0,\ \ (d)\ \ \eta \circ \phi = 0. \tag{4}$$

An almost contact structure is said to be normal if the almost complex structure J on the product manifold defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$$

is integrable, where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M \times \mathbb{R}$. Let g be a compatible Riemannian metric with almost contact metric structure (ϕ, η, ξ), that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

Then M admits an almost contact metric structure (ϕ, ξ, η, g). From (4) it can be easily seen that

$$(a)\quad g(X, \phi Y) = -g(\phi X, Y), \qquad (b)\quad g(X, \xi) = \eta(X). \tag{5}$$

for all vector fields X, Y. An almost contact metric structure becomes a contact metric structure if

$$g(X, \phi Y) = d\eta(X, Y),$$

for all vectors fields X, Y. The 1-form η is called a contact metric form and ξ is its characteristic vector field. We define a $(1,1)$ tensor field h by $h = \frac{1}{2}\pounds_\xi\phi$, where $\pounds$ denote the Lie derivative. Then h is symmetric and satisfies the conditions $h\phi = -\phi h$, $Tr.h = Tr.\phi h = 0$ and $h\xi = 0$. Also

$$\nabla_X\xi = -\phi X - \phi h X,$$

holds in a contact metric manifold. A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is a Sasakian manifold if and only if

$$(\nabla_X\phi)(Y) = g(X, Y)\xi - \eta(Y)X,$$

where $X, Y \in \chi(M)$ and ∇ is the Levi-Civita connection of the Riemannian metric g. A contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ for which ξ is a Killing vector field is said to be a K-contact metric manifold. A Sasakian manifold is K-contact but not conversely. However a 3-dimensional K-contact metric manifold is Sasakian [28].
Given the contact metric manifold (M, η, ξ, ϕ, g), we have the following identities ([1], [2]):

$$h\xi = 0, \qquad h\phi + \phi h = 0,$$

$$\nabla_X\xi = -\phi X - \phi h X,$$

$$\nabla_\xi\phi = 0,$$

$$R(\xi, X)\xi - \phi R(\xi, \phi X)\xi = 2(h^2 + \phi^2)X,$$

$$(\nabla_\xi h)X = \phi X - h^2\phi X + \phi R(\xi, X)\xi,$$

$$S(\xi,\xi) = 2n - \operatorname{Tr} h^2,$$

$$R(X,Y)\xi = -(\nabla_X\phi)Y + (\nabla_Y\phi)X - (\nabla_X\phi h)Y + (\nabla_Y\phi h)X. \tag{6}$$

Here, ∇ is the Levi-Civita connection and R the Riemannian curvature tensor of (M,g) with the sign convention

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

for vector fields X, Y, Z on M. The tensor $l = R(.,\xi)\xi$ is the Jacobi operator with respect to the characteristic field ξ. Contact metric manifolds have been studied by several authors such as ([3], [4], [5], [7], [32]) and many others.

Let (M^n, g); $(n = dimM)$ be a Riemannian manifold, i.e., a manifold M with the Riemannian metric g and let ∇ be the Levi-Civita connection of (M^n, g). A Riemannian manifold is called locally symmetric [16] if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M^n, g). The class of Riemannian symmetric manifolds is very natural generalization of the class of manifolds of constant curvature.

3. Proof of the Main Theorem

In view of equation (3) the Ricci tensor is given by

$$S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y), \tag{7}$$

where α, β are non-constant smooth functions. Again from (2) we have

$$\begin{aligned}(\pounds_V g)(Y,Z) &= -2S(Y,Z) - 2\lambda g(Y,Z)\\ &= -2\{\alpha g(Y,Z) + \beta\eta(Y)\eta(Z)\} - 2\lambda g(Y,Z)\\ &= -2(\alpha+\lambda)g(Y,Z) - 2\beta\eta(Y)\eta(Z).\end{aligned}$$

Taking Covariant differentiation with respect to X, we get

$$\begin{aligned}(\nabla_X\pounds_V g)(Y,Z) = &- 2(X\alpha)g(Y,Z) - 2(X\beta)\eta(Y)\eta(Z)\\ &- 2\beta(\nabla_X\eta)(Y)\eta(Z) - 2\beta\eta(Y)(\nabla_X\eta)(Z).\end{aligned} \tag{8}$$

Using $(\nabla_X\eta)(Y) = g(X + hX, \phi Y)$ in (8), we obtain

$$\begin{aligned}(\nabla_X\pounds_V g)(Y,Z) = &- 2(X\alpha)g(Y,Z) - 2(X\beta)\eta(Y)\eta(Z)\\ &- 2\beta g(X+hX,\phi Y)\eta(Z) - 2\beta\eta(Y)g(X+hX,\phi Z),\end{aligned} \tag{9}$$

for any vector field X, Y, Z. According to Yano ([38], pp-23), the following formula

$$(\pounds_V\nabla_X g - \pounds_X\nabla_V g - \nabla_{[V,X]}g)(Y,Z) = -g((\pounds_V\nabla)(X,Y),Z) - g((\pounds_V\nabla)(X,Z),Y)$$

is well known for any vector fields X, Y, Z on M. As g is parallel with respect to the Levi-Civita connection ∇, then the above relation becomes

$$(\nabla_X\pounds_V g)(Y,Z) = g((\pounds_V\nabla)(X,Y),Z) + g((\pounds_V\nabla)(X,Z),Y) \tag{10}$$

for any vector fields X, Y, Z. Since $\pounds_V\nabla$ is symmetric tensor of type $(1,2)$, i.e., $(\pounds_V\nabla)(X,Y) = (\pounds_V\nabla)(Y,X)$, then it follows from (10) that

$$g((\pounds_V\nabla)(X,Y),Z) = \frac{1}{2}(\nabla_X\pounds_V g)(Y,Z) + \frac{1}{2}(\nabla_Y\pounds_V g)(X,Z) - \frac{1}{2}(\nabla_Z\pounds_V g)(X,Y). \tag{11}$$

Using (9) in (11) we obtain

$$\begin{aligned} g((\pounds_V\nabla)(X,Y),Z) = &- (X\alpha)g(Y,Z) - (X\beta)\eta(Y)\eta(Z) \\ &- \beta g(X+hX,\phi Y)\eta(Z) - \beta\eta(Y)g(X+hX,\phi Z) \\ &- (Y\alpha)g(X,Z) - (Y\beta)\eta(X)\eta(Z) \\ &- \beta g(Y+hY,\phi X)\eta(Z) - \beta\eta(X)g(Y+hY,\phi Z) \\ &+ (Z\alpha)g(X,Y) + (Z\beta)\eta(X)\eta(Y) \\ &+ \beta g(Z+hZ,\phi Y)\eta(X) - \beta\eta(Y)g(Z+hZ,\phi X). \end{aligned} \tag{12}$$

From (5) and (12) we get

$$\begin{aligned} g((\pounds_V\nabla)(X,Y),Z) = &- (X\alpha)g(Y,Z) - (X\beta)\eta(Y)\eta(Z) - (Y\alpha)g(X,Z) \\ &- (Y\beta)\eta(X)\eta(Z) + (Z\alpha)g(X,Y) + (Z\beta)\eta(X)\eta(Y) \\ &- 2\beta g(hX,\phi Y)\eta(Z) - 2\beta g(X,\phi Z)\eta(Y) \\ &- 2\beta g(Y,\phi Z)\eta(X). \end{aligned}$$

The above equation gives

$$\begin{aligned} (\pounds_V\nabla)(X,Y) = &- (X\alpha)Y - (X\beta)\eta(Y)\xi - (Y\alpha)X - (Y\beta)\eta(X)\xi \\ &+ (D\alpha)g(X,Y) + (D\beta)\eta(X)\eta(Y) - 2\beta g(hX,\phi Y)\xi \\ &+ 2\beta\eta(Y)\phi X + 2\eta(X)\phi Y, \end{aligned} \tag{13}$$

where $X\alpha = g(D\alpha, X)$, D denotes the gradient operator with respect to g. Thus replacing X by Y and Y by Z in (13) we have

$$\begin{aligned} (\pounds_V\nabla)(Y,Z) = &- (Y\alpha)Z - (Y\beta)\eta(Z)\xi - (Z\alpha)Y \\ &- (Z\beta)\eta(Y)\xi + (D\alpha)g(Y,Z) + (D\beta)\eta(Y)\eta(Z) \\ &- 2\beta g(hY,\phi Z)\xi + 2\beta\eta(Z)\phi Y + 2\eta(Y)\phi Z. \end{aligned} \tag{14}$$

Taking covariant derivative of (14) with respect to X, we get

$$\begin{aligned} (\nabla_X\pounds_V\nabla)(Y,Z) = &- g(\nabla_X D\alpha, Y)Z - g(\nabla_X D\beta, Y)\eta(Z)\xi \\ &- (Y\beta)(\nabla_X\eta)Z\xi - (Y\beta)\eta(Z)\nabla_X\xi \\ &- g(\nabla_X D\alpha, Z)Y - g(\nabla_X D\beta, Z)\eta(Y)\xi \\ &- (Z\beta)(\nabla_X\eta)Y\xi - (Z\beta)\eta(Y)\nabla_X\xi \\ &+ \nabla_X D\alpha g(Y,Z) + \nabla_X D\beta\eta(Y)\eta(Z) \\ &+ D\beta(\nabla_X\eta)Y\eta(Z) + D\beta(\nabla_X\eta)(Z)\eta(Y) \\ &- 2(X\beta)g(hY,\phi Z)\xi + 2\beta g((\nabla_X h)Y,\phi Z)\xi \\ &- 2\beta g(hY,(\nabla_X\phi)Z,\xi) - 2\beta g(hY,\phi Z)\nabla_X\xi \\ &+ 2(X\beta)\eta(Z)\phi Y + 2\beta(\nabla_X\eta)Z\phi Y \\ &+ 2\beta\eta(Z)(\nabla_X\phi)Y + 2(X\beta)\eta(Y)\phi Z \\ &+ 2\beta(\nabla_X\eta)Y\phi Z + 2\beta\eta(Y)(\nabla_X\phi)Z. \end{aligned} \tag{15}$$

Again,

$$(\pounds_V R)(X,Y)Z = (\nabla_X\pounds_V\nabla)(Y,Z) - (\nabla_Y\pounds_V\nabla)(X,Z). \tag{16}$$

Now we suppose that h is recurrent, that is,

$$(\nabla_X h)(Y) = \eta(X)hY. \tag{17}$$

Using (15), (17) in (16) yields

$$
\begin{aligned}
(\pounds_V R)(X,Y)Z &= -(Y\beta)g(X+hX,\phi Z)\xi-(Y\beta)\eta(Z)(-\phi X-\phi hX)\\
&-g(\nabla_X D\alpha,Z)Y-g(\nabla_X D\beta,Z)\eta(Y)\xi\\
&-(Z\beta)g(X+hX,\phi Y)\xi-(Z\beta)\eta(Y)(-\phi X-\phi hX)\\
&+\nabla_X D\alpha g(Y,Z)+\nabla_X D\beta\eta(Y)\eta(Z)\\
&+D\beta g(XhX,\phi Y)\eta(Z)+D\beta\eta(Y)g(X+hX,\phi Z)\\
&-2(X\beta)g(hY,\phi Z)\xi+2\beta\eta(Z)g(hY,X+hX)\xi\\
&+2\beta g(hY,\phi Z)(hX+\phi hX)+2(X\beta)\eta(Z)\phi Y\\
&+2\beta g(X+hX,\phi Z)\phi Y+2\beta\eta(Z)\{g(X+hX,Y)\xi\\
&-\eta(Y)(X+hX)\}+2(X\beta)\eta(Y)\phi Z\\
&+2\beta g(X+hX,\phi Y)\phi Z+2\beta\eta(Y)\{g(X+hX,Z)\xi\\
&-\eta(Z)(X+hX)\}+(X\beta)g(Y+hY,\phi Z)\xi\\
&+(X\beta)\eta(Z)(-\phi Y-\phi hY)+g(\nabla_Y D\alpha,Z)X\\
&+g(\nabla_Y D\beta,Z)\eta(X)\xi+(Z\beta)g(Y+hY,\phi X)\xi\\
&+2\beta\eta(X)(-\phi Y-\phi hY)-\nabla_Y D\alpha g(X,Z)\\
&-\nabla_Y D\beta\eta(X)\eta(Z)-D\beta g(Y+hY,\phi X)\eta(Z)\\
&-D\beta g(Y+hY,\phi Z)\eta(X)+2(Y\beta)g(hX,\phi Z)\xi\\
&-2\beta\eta(Z)g(hX,Y+hY)\xi+2\beta g(hX,\phi Z)(-\phi Y-\phi hY)\\
&-2(Y\beta)\eta(Z)\phi X-2\beta g(y+hY,\phi Z)\phi X\\
&-2\beta\eta(Z)\{g(Y+hY,X)\xi-\eta(X)(Y+hY)\}\\
&-2(Y\beta)\eta(X)\phi Z-2\beta g(Y+hY,\phi X)\phi Z\\
&-2\beta\eta(X)\{g(Y+hY,Z)\xi-\eta(Z)(Y+hY)\}\\
&-2\beta\eta(X)g(hY,\phi Z)\xi+2\beta\eta(Y)g(hX,\phi Z)\xi.
\end{aligned}
\tag{18}
$$

Contracting X in (18), we have

$$
\begin{aligned}
(\pounds_V S)(Y,Z) &= -g(\nabla_Y D\alpha,Z)-g(\nabla_\xi D\beta,Z)\eta(Y)\\
&-\Delta\alpha g(Y,Z)-\Delta\beta\eta(Y)\eta(Z)\\
&+g(D\beta,\phi Y+\phi hY)\eta(Z)+g(D\beta,\phi Z+h\phi Z)\eta(Y)\\
&-2(\xi\beta)g(hY,\phi Z)+2g(D\beta,\phi Y)\eta(Z)\\
&+2\beta g(\phi Y+h\phi Y,\phi Z)-4n\beta\eta(Y)\eta(Z)\\
&+2g(D\beta,\phi Z)\eta(Y)+2\beta g(\phi Z+h\phi Z,\phi Y)\\
&-4n\beta\eta(Y)\eta(Z)+(\xi\beta)g(Y+hY,\phi Z)\\
&+\eta(Z)g(D\beta,-\phi Y-\phi hY)+(2n+1)g(\nabla_Y D\alpha,Z)\\
&+g(\nabla_Y D\beta,Z)-g(\nabla_Y D\alpha,Z)-g(\nabla_Y D\beta,\xi)\eta(Y)\\
&+g(D\beta,\phi Y+\phi hY)\eta(Z)-(\xi\beta)g(Y+hY,\phi Z)\\
&+2\beta g(h\phi Z,-\phi Y-\phi hY)+2\beta g(\phi Y+\phi hY,\phi Z)\\
&-2\beta\{g(Y+hY,Z)-\eta(Y)\eta(Z)\}-2\beta(hY,\phi Z).
\end{aligned}
\tag{19}
$$

Putting $Y = \phi Y$, $Z = \phi Z$ in (19) yields

$$\begin{aligned}(\pounds_V S)(\phi Y, \phi Z) \quad = \quad & -g(\nabla_{\phi Y} D\alpha, \phi Z) - \Delta\alpha g(\phi Y, \phi Z) \\ & -2(\xi\beta) g(h\phi Y, \phi^2 Z) + 2\beta g(\phi^2 Y + h\phi^2 Y, \phi^2 Z) \\ & +2\beta g(\phi^2 Z + h\phi^2 Z, \phi^2 Y) + (\xi\beta) g(\phi Y + h\phi Y, \phi^2 Z) \\ & +(2n+1) g(\nabla_{\phi Y} D\alpha, \phi Z) + g(\nabla_{\phi Y} D\beta, \phi Z) \\ & -g(\nabla_{\phi} Y D\alpha, \phi Z) - (\xi\beta) g(\phi Y + h\phi Y, \phi^2 Z) \\ & +2\beta g(h\phi^2 Z, -\phi^2 Y - \phi h\phi Y) + 2\beta g(\phi^2 Y + \phi h\phi Y, \phi^2 Z) \\ & -2\beta g(\phi Y + h\phi Y, \phi Z) + 2\beta g(hZ, \phi Y).\end{aligned} \tag{20}$$

On the other hand, from (7) we get

$$\begin{aligned}(\pounds_V S)(\phi Y, \phi Z) = & (V\alpha) g(\phi Y, \phi Z) \\ & + \alpha\{g(\nabla_{\phi Y} V, \phi Z) + g(\nabla_{\phi Z} V, \phi Y)\}\end{aligned} \tag{21}$$

Therefore from (20) and (21) we obtain

$$\begin{aligned} & (V\alpha) g(\phi Y, \phi Z) + \alpha\{g(\nabla_{\phi Y} V, \phi Z) + g(\nabla_{\phi Z} V, \phi Y)\} \\ = & - g(\nabla_{\phi Y} D\alpha, \phi Z) - \Delta\alpha g(\phi Y, \phi Z) - 2(\xi\beta) g(h\phi Y, \phi^2 Z) \\ & + 2\beta g(\phi^2 Y + h\phi^2 Y, \phi^2 Z) + 2\beta g(\phi^2 Z + h\phi^2 Z, \phi^2 Y) \\ & + (\xi\beta) g(\phi Y + h\phi Y, \phi^2 Z) + (2n+1) g(\nabla_{\phi Y} D\alpha, \phi Z) \\ & + g(\nabla_{\phi Y} D\beta, \phi Z) - g(\nabla_{\phi} Y D\alpha, \phi Z) - (\xi\beta) g(\phi Y + h\phi Y, \phi^2 Z) \\ & + 2\beta g(h\phi^2 Z, -\phi^2 Y - \phi h\phi Y) + 2\beta g(\phi^2 Y + \phi h\phi Y, \phi^2 Z) \\ & - 2\beta g(\phi Y + h\phi Y, \phi Z) + 2\beta g(hZ, \phi Y).\end{aligned} \tag{22}$$

Interchanging Y and Z in (22) and then subtracting from (22) [by using $g(\nabla_X D\alpha, Y) = g(\nabla_Y D\alpha, X)$] we have

$$\begin{aligned}0 = & - 2(\xi\beta)\{g(h\phi Y, \phi^2 Z) - g(h\phi Z, \phi^2 Y)\} \\ & + 2\beta\{g(h\phi^2 Y + \phi^2 Y, \phi^2 Z) - g(\phi^2 Z + h\phi^2 Z, \phi^2 Y\} \\ & + 2\beta\{g(h\phi^2 Z + \phi^2 Z, \phi^2 Y) - g(\phi^2 Y + h\phi^2 Y, \phi^2 Z\} \\ & + (\xi\beta)\{g(\phi Y + h\phi Y, \phi^2 Z - g(\phi Z + h\phi Z, \phi^2 Y))\} \\ & - (\xi\beta)\{g(\phi Y + h\phi Y, \phi^2 Z - g(\phi Z + h\phi Z, \phi^2 Y))\} \\ & + 2\beta\{g(h\phi^2 Z, -\phi^Y - \phi h\phi Y) - g(h\phi^2 Y, -\phi^2 Z - \phi h\phi Z)\} \\ & + 2\beta\{g(\phi^2 Y + \phi h\phi Y, \phi^2 Z) - g(\phi^2 Z + \phi h\phi Z, \phi^2 Y)\} \\ & - 2\beta\{g(\phi Y + h\phi Y, \phi Z - g(\phi Z + h\phi Z, \phi Y))\} \\ & + 2\beta\{g(hZ, \phi Z) - g(hY, \phi Z)\}.\end{aligned}$$

This implies that

$$2\beta\{g(\phi Y, \phi^2 Z) - g(\phi Z, \phi^2 Y)\} = 0,$$

and hence

$$4\beta g(\phi Y, \phi Z) = 0,$$

which implies

$$\beta = 0.$$

On the other hand, from (13) we obtain

$$0 = \sum_{i=1}^{2n+1} (\pounds_V \nabla)(e_i, e_i)$$
$$= -D\alpha - (\xi\beta)\xi - D\alpha - (\xi\beta)\xi + (2n+1)D\alpha + D\beta - 2\beta \sum_{i=1}^{2n+1} g(he_i, \phi e_i)\xi.$$

Since $\beta = 0$, from the above equation it follows that $(2n-1)D\alpha = 0$. Therefore we have $g(D\alpha, X) = 0$. This implies $(X\alpha) = 0$. Hence α = constant. Therefore our main Theorem 1.1 is proved.

Also from (2) we obtain

$$\pounds_V g + 2S + 2\lambda g = 0,$$

from which it follows that

$$\begin{aligned}(\pounds_V g)(X, Y) &= -2S(X, Y) - 2\lambda g(X, Y)\\ &= -2(\alpha + \lambda)g(X, Y)\\ &= -2\rho g(X, Y),\end{aligned}$$

where $\rho = -(\alpha + \lambda)$ = constant. Thus V is a homothetic vector field. Hence the corollary 1.2 is proved.

We suppose that contact metric manifold admits a Ricci soliton (g, ξ).
Then from (2) we get

$$\frac{1}{2}\{g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X)\} + S(X, Y) - \lambda g(X, Y) = 0.$$

This implies

$$\begin{aligned}&\{g(-\phi X - \phi h X, Y) + g(-\phi Y - h\phi Y, X)\}\\ &+ 2S(X, Y) - 2\lambda g(X, Y) = 0.\end{aligned} \tag{23}$$

Since $h\phi = -\phi h$, from the above equation (23) we have

$$g(X, h\phi Y) + S(X, Y) - \lambda g(X, Y) = 0.$$

It follows that

$$QY = \lambda Y - h\phi Y. \tag{24}$$

Substituting $Y = \xi$ in the above equation (24) yields

$$Q\xi = \lambda\xi.$$

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References

[1] Blair, D. E., *Contact manifolds in Riemannian geometry*, Lecture note in Math., 509, Springer-Verlag, Berlin-New York, 1976.
[2] Blair, D. E., *Riemannian geometry of contact and symplectic manifolds*, Progress in Math., 203, Birkhauser Boston, Inc., Boston, 2002.
[3] Baikoussis, C. and Koufogiorgos, T., *On a type of contact manifolds*, J. of Geometry, 46(1993), 1-9.
[4] Blair, D. E., *Two remarks on contact metric structures*, Tohoku Math. J., 29(1977), 319-324.
[5] Blair, D. E., Koufogiorgos, T. and Papantoniou, B. J., *Contact metric manifolds satisfying a nullity condition*, Israel J. Math., 91(1995), 189-214.

[6] Bajan, C. L. and Crasmareanu, M., *Second order parallel tensors and Ricci solitons in 3-dimensional normal paracontact geometry*, Ann. Glob. Anal. Geom, 46 (2014), 117-127.

[7] Boeckx, E. and Cho, J. T., *Locally symmetric contact metric manifolds*, Monatsh. Math. 148(2006), 269-381.

[8] Chave, T. and Valent, G., *Quasi-Einstein metrics and their renoirmalizability properties*, Helv. Phys. Acta. 69(1996) 344-347.

[9] Chave, T. and Valent, G., *On a class of compact and non-compact quasi-Einstein metrics and their renoirmalizability properties*, Nuclear Phys. B. 478(1996), 758-778.

[10] Cho, J. T., *Almost contact 3-manifolds and Ricci solitons*, Int. J. Geom. Methods Mod. Phys., 10(1)(2013), 1220022(7pages).

[11] Cho, J. T., *Notes on contact Ricci solitons*, Proc. Edinb. Math. Soc., 54(2011), 47-53.

[12] Cho, J. T., *Ricci solitons on almost contact geometry*, Proc. of 17th Int. workshop on Diff. Geom. and related fields, 17(2013), 85-95.

[13] Crasmareanu, M., *Second order Parallel tensors and Ricci solitons in N(k)-quasi Einstein manifolds*, Indian J. Pure Appl. Math., 43(4) (2012), 359-369.

[14] Calin, C. and Crasmareanu, M., *From the Eisenhart problem to Ricci solitons in f-Kenmotsu manifolds*, Bull. Malays. Math. Soc. no. 3, 33(2010), 361-368.

[15] Chow, B. and Knoff, D., *The Ricci flow: An introduction, Mathematical surveys and Monographs 110*, American Math. Soc. 2004.

[16] Cartan, E., *Sur une classes remarquable d'espaces de Riemannian*, Bull. Soc. Math. France, 54(1926), 214-264.

[17] De, U. C. and Matsuyama, Y., *Ricci solitons and gradient Ricci solitons in a Kenmotsu manifold*, Southeast Asian Bull. Math., 37 (2013), 691-697.

[18] De, U. C., Turan, M., Yildiz, A. and De, A., *Ricci solitons and gradient Ricci solitons on 3-dimensional normal almost contact metric manifolds*, Publ. Math. Debrecen, 80(2012), 127-142.

[19] Derdzinski, A., *A Myers-type theorem and compact Ricci solitons*, Proc. Amer. Math. Soc. 134, 12 (2006), 3645-3648.

[20] Deshmukh, S., *Jacobi-type vector fields on Ricci solitons*, Bull. Math. Soc. Sci. Math. Roumanie 55(103), 1 (2012), 41-50.

[21] Deshmukh, S., Alodan, H. and Al-Sodais, H., *A Note on Ricci Soliton*, Balkan J. Geom. Appl. 16, 1 (2011), 48-55.

[22] Friedan, D., *Non linera models in $2 + \epsilon$ dimensions*, Ann. Phys. 163(1985), 318-419.

[23] Ghosh, A., *Kenmotsu 3-metric as a Ricci soliton*, Chaos Solitons Fractals, 44 (2011), 647-650.

[24] Ghosh, A., *An η-Einstein Kenmotsu metric as a Ricci soliton*, Publ. Math. Debrecen, 82(3-4) (2013), 591-598.

[25] Hamilton, R.S., *The Ricci flow on surfaces, Mathematics and general relativity* (Santa Cruz, CA, 1986), 237-262.Contemp. Math.71, American Math. Soc., 1988.

[26] Ivey, T., *Ricci solitons on compact 3-manifolds*, Diff. Geom. Appl. 3(1993),301-307.

[27] Ivey, T., *New examples of complete Ricci solitons*, Proc. Amer. Math. Soc. 122, 1 (1994), 241-245.

[28] Jun, J. B. and Kim, U. K., *On 3-dimensional almost contact metric manifolds*, Kyungpook Math. J. 34, 2 (1994), 293-301.

[29] Mantica, C. A., Molinari, L. G. and De, U. C., *A condition for a perfect-fluid space-time to be a generalized Robertson-Walker space-time*, J. Math. Phys. 57, 4(2016) 022508.

[30] Mantica, C. A. and Suh, Y. J., *Pseuo Z-symmetric space-times*, J. Math. Phys. 55, 4(2014) 042502.

[31] Perelman, G., *The entropy formula for the Ricci flow and its geometric applications*, Preprint, http://arXiv.org/abs/Math.DG/0211159.

[32] Sharma, R. and Koufogiorgos, T., *Locally symmetric and Ricci symmetric contact metric manifolds*, Ann. Global Anal. Geom. 9(1991), 177-182.

[33] Turan, M., De, U. C. and Yildiz, A., *Ricci solitons and gradient Ricci solitons in three dimensional trans-Sasakian manifolds*, Filomat, 26(2) (2012), 363-370.

[34] Wang, Y. and Liu, X., *Ricci solitons on three dimensional η-Einstein almost Kenmotsu manifolds*, Taiwanese J. of Math., 19(2015), 91-100.

[35] Wang, Y., *Gradient Ricci almost solitons on two classes of almost Kenmotsu manifolds*, J. Korean. Math. Soc., 53(2016), 1101-1114.

[36] Walker, A. G., *On Ruse's space of recurrent curvature*, Proc. of London Math. Soc., 52(1950), 36-54.

[37] Wylie, W., *Complete shrinking Ricci solitons have finite fundamental group*, Proc. Amer. Math. Soc. 136, 5 (2008), 1803-1806.

[38] Yano, K., *Integral Formulas in Riemannian Geometry*, Marcel Dekker, New York, 1970.