Filomat 32:13 (2018), 4587–4590
https://doi.org/10.2298/FIL1813587T

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Fixed Point Method to Solve the Nonlinear Complementarity Problem for a Class of Monotone Operators

Dinu Teodorescu[a], Mohammad Saeed Khan[b]

[a]*Department of Mathematics, Valahia University of Targoviste, Boulevard Unirii 18, Targoviste, 130024, Romania*
[b]*Department of Mathematics & Statistics, Sultan Qaboos University, P. O. Box 36, Al-Khoud 123, Muscat, Sultanate of Oman*

Abstract. In this paper, using the classic Banach fixed point theorem, we study the nonlinear complementarity problem for a class of monotone operators in real Hilbert space.

1. Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm denoted by $\|\cdot\|$. Let $C \subset H$ be a closed convex cone with the vertex at 0 and let C^* be the dual cone of C defined by

$$C^* = \{y \in H \ / \ \langle x, y \rangle \geq 0 \text{ for all } x \in C\}.$$

Let $A : C \to H$ be a nonlinear operator. To solve the nonlinear complementarity problem for A is to find an element $u \in H$ so that

$$u \in C, \ Au \in C^* \text{ and } \langle u, Au \rangle = 0.$$

Solving the nonlinear complementarity problem for certain classes of nonlinear operators provides strong tools in the approach of some problems from optimization theory. Also, the nonlinear complementarity problem has applications in mechanics and economics(see [3]).

The complementarity problem for linear operators was proposed by R. W. Cottle and G. B. Dantzig in 1968(see [2]). More exactly Cottle and Dantzig studied the complementarity problem for $n \times n$ matrices, i.e. for linear operators from the euclidian space $\mathbf{R}^n$ into itself. An approach of the nonlinear complementarity problem for nonlinear mappings between finite dimensional spaces was proposed by S. Karamardian in [5]. In the paper [6], Sribatsa Nanda and Sudarsan Nanda established the existence and uniqueness of solution of the nonlinear complementarity problem for Lipschitz strongly monotone operators in Hilbert space.

In this paper we prove an existence and uniqueness result for the nonlinear complementarity problem for a strongly monotone operator satisfying simultaneous the inverse strong monotonicity condition. Our proof uses a similar technique as in [6], based on the Banach fixed point theorem, but in a more clear and elegant style.

2010 *Mathematics Subject Classification*. Primary 47H05; Secondary 47H09, 47H10
Keywords. nonlinear complementarity problem, strongly monotone operator, inverse strongly monotone operator
Received: 10 November 2017; Accepted: 15 February 2018
Communicated by Vladimir Rakočević
Email addresses: dteodorescu2003@yahoo.com (Dinu Teodorescu), mohammad@squ.edu.om (Mohammad Saeed Khan)

2. Preliminaries

In the following H denotes a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm denoted by $\|\cdot\|$.

Definition 2.1. *Let $A : D(A) \subset H \to H$ be an operator and $\alpha, \beta > 0$. A is said to be*

1) α-strongly monotone if $\langle Ax - Ay, x - y \rangle \geq \alpha \left\| x - y \right\|^2$ for all $x, y \in D(A)$;

2) β-inverse strongly monotone if $\langle Ax - Ay, x - y \rangle \geq \beta \left\| Ax - Ay \right\|^2$ for all $x, y \in D(A)$;

3) nonexpansive if $\left\| Ax - Ay \right\| \leq \left\| x - y \right\|$ for all $x, y \in D(A)$;

4) firmly nonexpansive if $\langle Ax - Ay, x - y \rangle \geq \left\| Ax - Ay \right\|^2$ for all $x, y \in D(A)$.

Clearly, due to the Schwartz inequality, every firmly nonexpansive operator is nonexpansive.

The definition of monotone operator was first given by Kachurovski[4] and iterative methods for strongly monotone operators in Hilbert space satisfying a Lipschitz condition were first given by Zarantonello [8] and Vainberg [7].(see Browder and Petryshyn[1]). The notion of inverse strongly monotone operator appears firstly in 1967(Browder and Petryshyn[1]).

To prove our main result we apply a technique used in the study of some variational inequalities and we need the notion of metric projection.

Let K be a closed convex nonempty subset of H and P_K be the metric projection onto K. The projection operator P_K is characterized by

$$P_K x \in K : \langle P_K x - x, y - P_K x \rangle \geq 0 \text{ for all } y \in K.$$

It follows from this characterization that for $x, y \in H$

$$\begin{aligned}
&\langle P_K x - P_K y, x - y \rangle \\
&= \langle P_K x - P_K y, x - P_K x \rangle + \left\| P_K x - P_K y \right\|^2 + \langle P_K x - P_K y, P_K y - y \rangle \\
&\geq \left\| P_K x - P_K y \right\|^2,
\end{aligned}$$

so P_K is a firmly nonexpansive operator, thus P_K is nonexpansive. Also $\|x - P_K x\| = d(x; K) = \inf\{\left\| x - y \right\| / y \in K\}$.

3. Main Result

In the paper [6] the authors prove the following result:

Theorem 3.1. *Let $C \subset H$ be a closed convex cone with the vertex at 0, $F : C \to H$ be a c-strongly monotone and k-lipschitzian operator($c, k > 0$). If $k^2 < 2c < k^2 + 1$, then the nonlinear complementarity problem for F has a unique solution.*

If F satisfies the conditions from the above theorem, using the Schwartz inequality we obtain

$$\begin{aligned}
&c \left\| x - y \right\|^2 \leq \langle Fx - Fy, x - y \rangle \leq \left\| Fx - Fy \right\| \cdot \left\| x - y \right\| \\
&\leq k \left\| x - y \right\|^2 \text{ for all } x, y \in C,
\end{aligned}$$

thus $c \leq k$. Then $c^2 < 2c$, so $c \in (0, 2)$. Now $k^2 < 2c < 4$ and we have necessarily $k \in (0, 2)$.

If for example $c = 1$, from the condition $k^2 < 2c < k^2 + 1$, we have necessarily $k \geq 1$ and $k^2 < 2$, so we obtain that $k \in [1, \sqrt{2})$.

Clearly the result obtained in [6] is not so general as it seems, because the authors ignore the necessary condition $c \leq k$.

Our result, presented below, relate to an operator satisfying simultaneous the α-strong monotonicity condition and the β-inverse strong monotonicity condition, so the constants α and β satisfy $\alpha \leq \frac{1}{\beta}$.

Now we are in position to give the main result of this paper:

Theorem 3.2. *Let $C \subset H$ be a closed convex cone with the vertex at 0, C^* be the dual cone of C and $\alpha \in (0, \frac{1}{2}]$, $\beta \in [1, 2]$. If $A : C \to H$ is α-strongly monotone and β-inverse strongly monotone, then there exists a unique $u \in C$ such that*

$$u \in C,\ Au \in C^* \text{ and } \langle u, Au \rangle = 0.$$

Proof. Let $T : C \to C$ be the operator defined by

$$Tx = P_C(x - Ax),$$

where P_C is the projection operator onto C. From the properties of the projection operator we obtain:

$$\left\|Tx - Ty\right\| = \left\|P_C(x - Ax) - P_C(y - Ay)\right\|$$
$$\leq \left\|(x - Ax) - (y - Ay)\right\|$$
$$= \left\|x - y - (Ax - Ay)\right\|.$$

It results that

$$\left\|Tx - Ty\right\|^2 \leq \left\|x - y - (Ax - Ay)\right\|^2$$
$$= \left\|x - y\right\|^2 + \left\|Ax - Ay\right\|^2 - 2\left\langle Ax - Ay, x - y\right\rangle.$$

Now it follows from the assumptions from the hypothesis that

$$\left\|Tx - Ty\right\|^2$$
$$\leq \left\|x - y\right\|^2 + \left\|Ax - Ay\right\|^2 - \langle Ax - Ay, x - y\rangle - \langle Ax - Ay, x - y\rangle$$
$$\leq \left\|x - y\right\|^2 + \left\|Ax - Ay\right\|^2 - \alpha\left\|x - y\right\|^2 - \beta\left\|Ax - Ay\right\|^2$$
$$= (1 - \alpha)\left\|x - y\right\|^2 + (1 - \beta)\left\|Ax - Ay\right\|^2.$$

Necessarily $\alpha \leq \frac{1}{\beta}$, because the operator A satisfies simultaneous the α-strong monotonicity condition and the β-inverse strong monotonicity condition.

If $\alpha \in (0, \frac{1}{2}]$, $\beta \in [1, 2]$, clearly we have $\alpha \leq \frac{1}{\beta}$, $1 - \alpha \in (0, 1)$ and $1 - \beta \leq 0$.

Consequently we obtain

$$\left\|Tx - Ty\right\|^2 \leq (1 - \alpha)\left\|x - y\right\|^2,$$

thus

$$\left\|Tx - Ty\right\| \leq \sqrt{1 - \alpha}\left\|x - y\right\|$$

for all $x, y \in C$.

So $T : C \to C$ is a contraction($\sqrt{1 - \alpha} \in (0, 1)$ for $\alpha \in (0, \frac{1}{2}]$) and, due to the Banach fixed point theorem, we conclude that T has a unique fixed point $u \in C$.

It follows from $Tu = u = P_C(u - Au)$ that

$$0 \leq \langle P_C(u - Au) - u + Au, y - P_C(u - Au)\rangle = \langle Au, y - u\rangle \text{ for all } y \in C.$$

Now we take $y = 0 \in C$ and $y = 2u \in C$ and we have

$$\langle Au, -u\rangle \geq 0 \text{ and } \langle Au, u\rangle \geq 0,$$

so $\langle Au, u\rangle = 0$.
Since $\langle Au, y-u\rangle \geq 0$ for all $y \in C$, it results that $\langle Au, y\rangle - \langle Au, u\rangle = \langle Au, y\rangle \geq 0$ for all $y \in C$, then $Au \in C^*$.
If $v \in C$ satisfies $\langle Av, v\rangle = 0$ and $Av \in C^*$, then we have

$$\langle v-(v-Av), y-v\rangle = \langle Av, y-v\rangle = \langle Av, y\rangle \geq 0 \text{ for all } y \in C,$$

and, using the properties of the projection operator, it results that $v = P_C(v-Av) = Tv$.
Consequently v is a fixed point of T, therefore $v = u$.
Thus the uniqueness of the point $u \in C$ satisfying the conditions from the conclusion of our theorem is proved and the proof of Theorem 3.2. is complete.□

The unique solution $u \in C$ of the nonlinear complementarity problem for the operator A, obtained by Theorem 3.2, can be approximated using the Picard iteration associated to the contraction T.

The values of α and β from the statement of Theorem 3.2 can be changed in many ways, the result still remaining true. Indeed, for example, the following result holds:

Theorem 3.3. *Let $C \subset H$ be a closed convex cone with the vertex at* 0*, C^* be the dual cone of C and $\alpha \in (0,1)$,* $\beta = 1$. If $A : C \to H$ *is α-strongly monotone and β-inverse strongly monotone(in other words the operator A is firmly nonexpansive, satisfying simultaneous a strong monotonicity condition), then there exists a unique $u \in C$ such that*

$$u \in C,\ Au \in C^* \text{ and } \langle u, Au\rangle = 0.$$

Proof. We construct the operator $T : C \to C$ as in the proof of Theorem 3.1 and we have

$$\begin{aligned}
&\left\|Tx-Ty\right\|^2\\
&\leq \left\|x-y\right\|^2 + \left\|Ax-Ay\right\|^2 - 2\langle Ax-Ay, x-y\rangle\\
&= \left\|x-y\right\|^2 + \left\|Ax-Ay\right\|^2 - \langle Ax-Ay, x-y\rangle - \langle Ax-Ay, x-y\rangle\\
&\leq \left\|x-y\right\|^2 + \left\|Ax-Ay\right\|^2 - \alpha\left\|x-y\right\|^2 - \left\|Ax-Ay\right\|^2\\
&= (1-\alpha)\left\|x-y\right\|^2.
\end{aligned}$$

Clearly T is a contraction and the proof continues similarly to the proof of Theorem 3.2.□

References

[1] F. E. Browder and W. V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, J. Math. Anal. Appl. 20(1967) 197-228
[2] R.W. Cottle and G.B. Dantzig, Complementary pivot theory of mathematical programming, Linear Algebra Appl. 1(1968) 103-125
[3] M.C. Ferris and J.S. Pang, Engineering and Economic Applications of Complementarity Problems, SIAM Rev. 39(1997) 669-713
[4] R. I. Kachurovski, On monotone operators and convex functionals, Uspekhi Mat. 15(1960) 213-215
[5] S. Karamardian, The nonlinear complementarity problem with applications I, II, J. Optimization Theory Appl. 4(1969) 87-98 167-181
[6] S. Nanda and S. Nanda, A nonlinear complementarity problem in mathematical programming in Hilbert space, Bull. Austral. Math. Soc. 20(1979) 233-236
[7] M.M. Vainberg, On the convergence of the method of steepest descent for nonlinear equations, Sibir. Mat. J. 2(1961) 201-220
[8] E.H. Zarantonello, The closure of the numerical range contains the spectrum, Bull. Amer. Math. Soc. 70(1964) 771-778