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Hyers-Ulam Stability of Hyperbolic Möbius Difference Equation

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Abstract. Hyers-Ulam stability of the difference equation with the initial point z_0 as follows

$$z_{i+1} = \frac{az_i + b}{cz_i + d}$$

is investigated for complex numbers a, b, c and d where $ad - bc = 1$, $c \neq 0$ and $a + d \in \mathbb{R} \setminus [-2, 2]$. The stability of the sequence $\{z_n\}_{n \in \mathbb{N}_0}$ holds if the initial point is in the exterior of a certain disk of which center is $-\frac{d}{c}$. Furthermore, the region for stability can be extended to the complement of some neighborhood of the line segment between $-\frac{d}{c}$ and the repelling fixed point of the map $z \mapsto \frac{az+b}{cz+d}$. This result is the generalization of Hyers-Ulam stability of Pielou logistic equation.

1. Introduction

Difference equation is the recurrence relation which defines the sequence and each of which terms determines the proceeding terms. For the introduction of difference equation, for example, see [4]. The first order difference equation is of the following form

$$z_{i+1} = g(i, z_i)$$

for all integer $i \geq 0$. In 1940, Ulam [13] suggested the problem concerning the stability of group homomorphisms: Given a metric group $(G, \cdot, d)$, a positive number ε, and a function $f : G \to G$ which satisfies the inequality $d\big(f(xy),\ f(x)f(y)\big) \leq \varepsilon$ for all $x, y \in G$, do there exist an homomorphism $a : G \to G$ and a constant δ depending only on G and ε such that $d\big(a(x), f(x)\big) \leq \delta$ for all $x \in G$? A first answer to this question was given by Hyers [5] in 1941 who proved that the Cauchy additive equation is stable in Banach spaces.

The difference equation has Hyers-Ulam stability if each terms of the sequence with the given relation has (small) error, this sequence is approximated by the sequence with same relation which has no error. Hyers-Ulam stability of difference equation is relatively recent topic. For example, see [8–11]. In particular, Pielou logistic difference equation has Hyers-Ulam stability only if the initial point of the sequence is contained in definite intervals in [10]. In the same paper, this result is extended to the following difference equation over $\mathbb{R}$

$$x_{i+1} = \frac{ax_i + b}{cx_i + d}$$

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where $ad - bc = 1$, $c \neq 0$ and $(a+d)^2 > 4$ for real numbers a, b, c and d. In this article, we generalize the result of difference equations on the complex plane $\mathbb{C}$ where $a+d$ is real and it satisfies that $(a+d)^2 > 4$ with complex numbers a, b, c and d.

Möbius map

Linear fractional map on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called *Möbius map* or *Möbius transformation*.

$$g(z) = \frac{az+b}{cz+d}$$

where $ad - bc \neq 0$ for $z \in \hat{\mathbb{C}}$.
The non-constant Möbius map $g(z) = \frac{az+b}{cz+d}$ has the following properties.

- Without loss of generality, we may assume that $ad - bc = 1$.
- $g(\infty)$ is defined as $\frac{a}{c}$ and $g\left(-\frac{d}{c}\right)$ is defined as ∞.
- The composition of two Möbius maps is also a Möbius map.
- The map g is the linear map if and only if ∞ is a fixed point of g.
- The image of circle or line under Möbius map is circle or line.

The matrix representation of Möbius map is useful to classify Möbius map qualitatively. In particular, the equation $\frac{az+b}{cz+d} = \frac{paz+pb}{pcz+pd}$ holds for all $p \neq 0$. We define the matrix representation of Möbius map $z \mapsto \frac{az+b}{cz+d}$ as follows $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $ad - bc = 1$. We denote the matrix representation of Möbius map g by also g unless it makes confusion. Denote the trace of the matrix representation of Möbius map g by $\mathrm{tr}(g)$.

Main content

In Section 3, Hyers-Ulam stability of the sequence defined by hyperbolic Möbius map on the exterior of the disk of which center is $g^{-1}(\infty)$ with a certain radius. This is the direct generalization of Hyers-Ulam stability of Pielou logistic equation in [10] on the complex plane. In Section 5, the *avoided region* at ∞ is defined as the complement of the closure of the neighborhood of the line segment between $g^{-1}(\infty)$ and the repelling fixed point of g. In Section 7, Hyers-Ulam stability of g is proved in the complement of an avoided region.

2. Hyperbolic Möbius map

The trace of matrix is invariant under conjugation. Thus qualitative classification of Möbius map depends on the trace of matrix representation.

Definition 2.1. *If the matrix representation of the non-constant Möbius map* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *has its trace* $a+d$, *say* $\mathrm{tr}(g)$, *is in the set* $\mathbb{R} \setminus [-2, 2]$, *then the map* g *is called the* hyperbolic *Möbius map.*

Denote the fixed points of g by α and β. If $|g'(\alpha)| < 1$, then α is called the *attracting* fixed point. If $|g'(\beta)| > 1$, then β is called the *repelling* fixed point.

Lemma 2.2. *Let* g *be the hyperbolic Möbius map such that* $g(z) = \frac{az+b}{cz+d}$ *where* $ad - bc = 1$ *and* $c \neq 0$. *Then* g *has two different fixed points, one of which is the attracting fixed point and the other is the repelling fixed point.*

Proof. The fixed points of g are the roots of the quadratic equation

$$cz^2 - (a-d)z - b = 0$$

Denote the fixed points of g as follows

$$\alpha = \frac{a-d+\sqrt{(a+d)^2-4}}{2c} \quad \text{and} \quad \beta = \frac{a-d-\sqrt{(a+d)^2-4}}{2c}. \tag{1}$$

Observe that $\alpha+\beta = \frac{a-d}{c}$ and $\alpha\beta = -\frac{b}{c}$. Thus we have the following equation

$$\begin{aligned}(c\alpha+d)(c\beta+d) &= c^2\alpha\beta + cd(\alpha+\beta) + d^2 \\ &= -bc + d(a-d) + d^2 \\ &= -bc + ad \\ &= 1 \end{aligned} \tag{2}$$

Since g is the hyperbolic Möbius map, that is, $a+d > 2$ or $a+d < -2$, without loss of generality we may assume that $a+d > 2$. Then we obtain the following inequality using the equation (1)

$$c\alpha + d = \frac{a+d+\sqrt{(a+d)^2-4}}{2} > \frac{a+d}{2} > 1. \tag{3}$$

Since $g'(z) = \frac{1}{(cz+d)^2}$ and by the equations (2) and (3), we obtain that $g'(\alpha) = \frac{1}{(c\alpha+d)^2} < 1$ and $g'(\beta) = \frac{1}{(c\beta+d)^2} > 1$. □

Lemma 2.3. *Let g and h are Möbius map as follows*

$$g(z) = \frac{az+b}{cz+d} \quad \text{and} \quad h(z) = \frac{z-\beta}{z-\alpha}$$

where α and β are the fixed points of g and $ad-bc=1$. If $\alpha \neq \beta$, then $h \circ g \circ h^{-1}(w) = kw$ where $k = \frac{1}{(c\beta+d)^2}$. In particular, if g is the hyperbolic Möbius map and β is the repelling fixed point, then $k > 1$.

Proof. The maps g and h are Möbius map. Thus so is $h \circ g \circ h^{-1}$. By the direct calculation, we obtain that $h^{-1}(w) = \frac{\alpha w - \beta}{w-1}$. Observe that $h^{-1}(0) = \beta$, $h^{-1}(\infty) = \alpha$ and $h^{-1}(1) = \infty$. Then we have

$$\begin{aligned} h \circ g \circ h^{-1}(0) &= h \circ g(\beta) = h(\beta) = 0 \\ h \circ g \circ h^{-1}(\infty) &= h \circ g(\alpha) = h(\alpha) = \infty \end{aligned}$$

The points 0 and ∞ are fixed points of $h \circ g \circ h^{-1}$. So $h \circ g \circ h^{-1}(w) = kw$ for some $k \in \mathbb{C}$. Since $k = h \circ g \circ h^{-1}(1)$, the following equation holds by (1) and (2)

$$\begin{aligned} k &= h \circ g \circ h^{-1}(1) = h \circ g(\infty) = h\left(\frac{a}{c}\right) \\ &= \frac{\frac{a}{c} - \beta}{\frac{a}{c} - \alpha} = \frac{a - c\beta}{a - c\alpha} \\ &= \frac{a+d+\sqrt{(a+d)^2-4}}{a+d-\sqrt{(a+d)^2-4}} \\ &= \frac{c\alpha+d}{c\beta+d} \\ &= \frac{1}{(c\beta+d)^2}. \end{aligned}$$

If g is the hyperbolic Möbius map, then $k = \frac{1}{(c\beta+d)^2} = g'(\beta) > 1$ by the proof of Lemma 2.2. □

Lemma 2.4. *Let g be the hyperbolic Möbius map on $\hat{\mathbb{C}}$. Let α and β be the attracting and the repelling fixed point respectively. Then*

$$\lim_{n\to\infty} g^n(z) \to \alpha \quad as \quad n \to +\infty$$

for all $z \in \hat{\mathbb{C}} \setminus \{\beta\}$.

Proof. By the classification of Möbius map, the hyperbolic Möbius map has both the attracting and the repelling fixed points. Let h be the linear fractional map as follows

$$h(z) = \frac{z-\beta}{z-\alpha}.$$

Then $f = h \circ g \circ h^{-1}$ is the dilation with the repelling fixed point at zero, that is, $f(w) = kw$ for $k > 1$. Thus 0 is the repelling fixed point of f. Since h is a bijection on $\hat{\mathbb{C}}$, the orbit, $\{g^n(z)\}_{n\in\mathbb{Z}}$ corresponds to the orbit, $\{f^n(h(z))\}_{n\in\mathbb{Z}}$ by conjugation h. Observe that

$$f^n(z) \to \infty \quad \text{as} \quad n \to +\infty$$

for all $z \in \hat{\mathbb{C}} \setminus \{0\}$. Hence,

$$g^n(z) \to \alpha \quad \text{as} \quad n \to +\infty$$

for all $z \in \hat{\mathbb{C}} \setminus \{\beta\}$. □

Corollary 2.5. *Let g be the map defined in Lemma 2.4. Then*

$$\lim_{n\to\infty} g^{-n}(z) \to \beta \quad as \quad n \to +\infty$$

for all $z \in \hat{\mathbb{C}} \setminus \{\alpha\}$.

Proof. Observe that g^{-1} is also hyperbolic Möbius transformation and β and α are the attracting and the repelling fixed point under g^{-1} respectively. Thus we apply the proof of Lemma 2.4 to the map g^{-1}. It completes the proof. □

We collect the notions throughout this paper as follows

- The Möbius map g is the hyperbolic Möbius map and $g(z) = \frac{az+b}{cz+d}$ where $ad - bc = 1$ and $c \neq 0$.
- The Möbius map h is defined as $h(z) = \frac{z-\beta}{z-\alpha}$ where α and β are the attracting and the repelling fixed points of g.
- Without loss of generality, we may assume that the hyperbolic Möbius map g has the matrix representation with $\mathrm{tr}(g) > 2$.
- Since the trace of matrix is invariant under conjugation, we obtain that $\mathrm{tr}(g) = \mathrm{tr}(h \circ g \circ h^{-1})$. By Lemma 2.3, if $\mathrm{tr}(g) > 2$, then

$$\mathrm{tr}(g) = \mathrm{tr}\begin{pmatrix} \sqrt{k} & 0 \\ 0 & \frac{1}{\sqrt{k}} \end{pmatrix} = \sqrt{k} + \frac{1}{\sqrt{k}} > 2.$$

3. Hyers-Ulam stability on the exterior of disk

Let F be the function from $\mathbb{N}_0 \times \mathbb{C}$ to $\mathbb{C}$. Suppose that for a given positive number ε, a complex valued sequence $\{a_n\}_{n\in\mathbb{N}_0}$ satisfies the inequality

$$|a_{i+1} - F(i, a_i)| \leq \varepsilon$$

for all $i \in \mathbb{N}_0$ where $|\cdot|$ is the absolute value of complex number. If there exists the sequence $\{b_n\}_{n\in\mathbb{N}_0}$ which satisfies that

$$b_{i+1} = F(i, b_i)$$

for each $i \in \mathbb{N}_0$, and $|a_i - b_i| \leq G(\varepsilon)$ for all $i \in \mathbb{N}_0$ where the positive number $G(\varepsilon)$ converges to zero as $\varepsilon \to 0$, then we say that the sequence $\{b_n\}_{n\in\mathbb{N}_0}$ has *Hyers-Ulam stability*. Denote $F(i, z)$ by $F_i(z)$ if necessary.

The set S is called an invariant set under F (or S is invariant under F) where for any $s \in S$ we obtain that $F(i, s) \in S$ for all $i \in \mathbb{N}_0$.

Lemma 3.1. *Let $F\colon \mathbb{N}_0 \times \mathbb{C} \to \mathbb{C}$ be a function satisfying the condition*

$$|F(i, u) - F(i, v)| \leq K|u - v| \tag{4}$$

for all $i \in \mathbb{N}_0$, $u, v \in \mathbb{C}$ and for $0 < K < 1$. For a given an $\varepsilon > 0$ suppose that the complex valued sequence $\{a_i\}_{i\in\mathbb{N}_0}$ satisfies the inequality

$$|a_{i+1} - F(i, a_i)| \leq \varepsilon \tag{5}$$

for all $i \in \mathbb{N}_0$. Then there exists a sequence $\{b_i\}_{i\in\mathbb{N}_0}$ satisfying

$$b_{i+1} = F(i, b_i) \tag{6}$$

and

$$|b_i - a_i| \leq K^i|b_0 - a_0| + \frac{1 - K^i}{1 - K}\,\varepsilon$$

for $i \in \mathbb{N}_0$. If the whole sequence $\{a_i\}_{i\in\mathbb{N}_0}$ is contained in the invariant set $S \subset \mathbb{C}$ under F , then $\{b_i\}_{i\in\mathbb{N}_0}$ is also in S under the condition, $a_0 = b_0$.

Proof. By induction suppose that

$$|b_{i-1} - a_{i-1}| \leq K^{i-1}|b_0 - a_0| + \frac{1 - K^{i-1}}{1 - K}\,\varepsilon.$$

If $i = 0$, then trivially $|b_0 - a_0| \leq \varepsilon$. Morover,

$$\begin{aligned}
|b_i - a_i| &\leq |b_i - F(i-1, a_{i-1})| + |a_i - F(i-1, a_{i-1})| \\
&\leq |F(i-1, b_{i-1}) - F(i-1, a_{i-1})| + |a_i - F(i-1, a_{i-1})| \\
&= K|b_{i-1} - a_{i-1}| + \varepsilon \\
&\leq K\left\{K^{i-1}|b_0 - a_0| + \frac{1 - K^{i-1}}{1 - K}\,\varepsilon\right\} + \varepsilon \\
&= K^i|b_0 - a_0| + \frac{1 - K^i}{1 - K}\,\varepsilon.
\end{aligned}$$

Moreover, if $a_0 = b_0$, then the sequence $\{b_i\}_{i\in\mathbb{N}_0}$ satisfies the inequality (5) without error under F. Hence, $\{b_i\}_{i\in\mathbb{N}_0}$ is contained in the invariant set S. $\square$

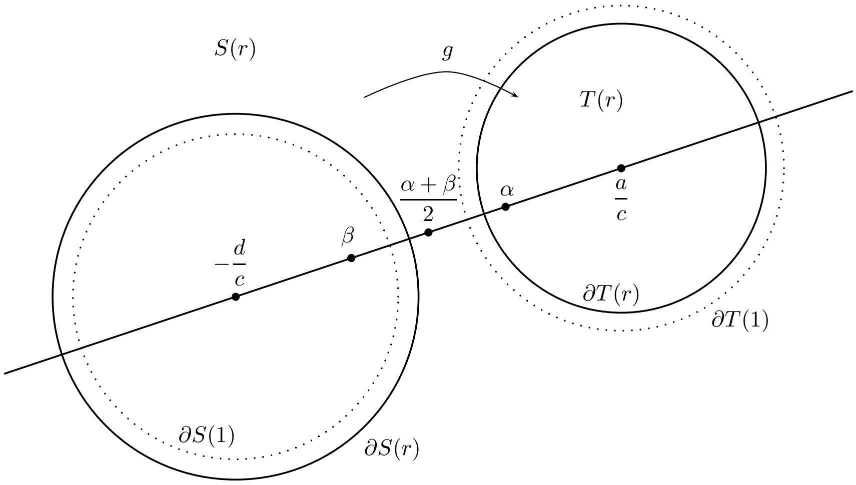


Figure 1: Image of the disk under hyperbolic Möbius map

The real version of the following lemma is proved in [10] as g is the map defined on the real line.

Lemma 3.2. *Let g be the Möbius map $g(z) = \frac{az+b}{cz+d}$ where a, b, c and d are complex numbers, $ad - bc = 1$ and $c \neq 0$. Let the region $S(r)$ and $T(r)$ be as follows*

$$S(r) = \left\{ z \in \mathbb{C} \colon \left| z + \frac{d}{c} \right| > \frac{r}{|c|} \right\}, \quad T(r) = \left\{ z \in \mathbb{C} \colon \left| z - \frac{a}{c} \right| < \frac{1}{r|c|} \right\}$$

for $r > 0$. Then $g(S(r)) = T(r) \setminus \left\{ \frac{a}{c} \right\}$ for any $r > 0$. Moreover, if g is hyperbolic Möbius map and $r + \frac{1}{r} < |\mathrm{tr}(g)|$, then the closure of $T(r)$ is contained in $S(r)$.

Proof. The set $S(r)$ is contained in $\mathbb{C}$ and $g(\infty) = \frac{a}{c}$. Then $S(r)$ does not have $\frac{a}{c}$. The equation $g(S(r)) = T(r) \setminus \left\{ \frac{a}{c} \right\}$ for any $r > 0$ is shown by the following equivalent conditions

$$\begin{aligned} g(z) \in T(r) \setminus \left\{ \frac{a}{c} \right\} &\iff 0 < \left| \frac{az+b}{cz+d} - \frac{a}{c} \right| < \frac{1}{r|c|} \\ &\iff 0 < \left| \frac{ad-bc}{cz^2+cd} \right| < \frac{1}{r|c|} \\ &\iff 0 < \frac{1}{\left| z + \frac{d}{c} \right|} < \frac{|c|}{r} \\ &\iff \left| z + \frac{d}{c} \right| > \frac{r}{|c|} \\ &\iff z \in S(r). \end{aligned}$$

Additionally, suppose that g is the hyperbolic map and $r + \frac{1}{r} < |\mathrm{tr}(g)|$. The closure of $T(r)$ is the set of points satisfying that $\left|z - \frac{a}{c}\right| \leq \frac{1}{r|c|}$. Then for any z in the closure of $T(r)$, we obtain that

$$\begin{aligned}|cz + d| &= |cz - a + a + d| \\ &\geq -|cz - a| + |a + d| \\ &= -|c| \cdot \left|z - \frac{a}{c}\right| + |\mathrm{tr}(g)| \\ &> -|c| \cdot \frac{r}{|c|} + r + \frac{1}{r} \\ &= \frac{1}{r}\end{aligned}$$

Then we have $\left|z + \frac{d}{c}\right| > \frac{1}{r|c|}$, that is, $z \in S(r)$. Hence, the closure of $T(r)$ is contained in $S(r)$ if $r + \frac{1}{r} < |\mathrm{tr}(g)|$. □

Proposition 3.3. *Let g be the hyperbolic Möbius map with* $\mathrm{tr}(g) = 2 + \tau$ *for $\tau > 0$. Let $S(r)$ be the region defined in Lemma 3.2. For a given $\varepsilon > 0$, let a complex valued sequence $\{a_i\}_{i\in\mathbb{N}_0}$ satisfies the inequality*

$$|a_{i+1} - g(a_i)| \leq \varepsilon$$

for all $i \in \mathbb{N}_0$. Suppose that $\varepsilon < \frac{t}{|c|(1+t)}$ and a_0 is in $S(1 + t)$ for $0 < t \leq \tau$. Then the sequence $\{a_n\}_{n\in\mathbb{N}_0}$ is contained in $S(1 + t)$. Moreover, there exists the sequence $\{b_i\}_{i\in\mathbb{N}_0}$ satisfying

$$b_{i+1} = g(b_i)$$

for each $i \in \mathbb{N}$ has Hyers-Ulam stability where $b_0 = a_0$.

Proof. For the map

$$g(z) = \frac{az + b}{cz + d}$$

we may assume that $ad - bc = 1$. Recall that $g'(z) = \frac{1}{(cz+d)^2}$. Thus $|g'|$ has a uniform upper bound in $S(1 + t)$ as follows

$$\begin{aligned}z \in S(1 + t) &\Longleftrightarrow \left|z + \frac{d}{c}\right| > \frac{1 + t}{|c|} \\ &\Longleftrightarrow |cz + d| > 1 + t \\ &\Longleftrightarrow |g'(z)| = \frac{1}{|cz + d|^2} < \frac{1}{(1 + \tau)^2} < 1. \end{aligned} \tag{7}$$

claim : If $a_0 \in S(1 + \tau)$ and $\varepsilon < \frac{\tau}{|c|(1+\tau)}$, then the whole sequence $\{a_i\}_{i\in\mathbb{N}_0}$ is also contained in $S(1 + \tau)$. By induction assume that $a_{i-1} \in S(1 + \tau)$. Then

$$\begin{aligned}\left|a_i - \left(-\frac{d}{c}\right)\right| &= \left|a_i - \frac{a}{c} + \frac{a}{c} - \frac{d}{c}\right| \\ &\geq -\left|a_i - \frac{a}{c}\right| + \left|\frac{a + d}{c}\right| \\ &\geq -\left|g(a_{i-1}) - \frac{a}{c}\right| - \varepsilon + \frac{2 + t}{|c|} \\ &> -\frac{1}{|c|(1 + \tau)} - \varepsilon + \frac{2 + t}{|c|} \qquad \text{by Lemma 3.2} \\ &= \frac{1}{|c|}\left(2 + t - \frac{1}{1 + t} - \varepsilon\right) \\ &> \frac{1}{|c|}\left(2 + t - \frac{1}{1 + t} - \frac{t}{1 + t}\right) \\ &= \frac{1}{|c|}(1 + t).\end{aligned}$$

Thus $\left|a_i + \frac{d}{c}\right| > \frac{1+t}{|c|}$, that is, $a_i \in S(1+t)$. Then the whole sequence $\{a_i\}_{i\in\mathbb{N}_0}$ is contained in $S(1+\tau)$.
The inequality (7) implies that g is the Lipschitz map with Lipschitz constant $\frac{1}{(1+t)^2}$ in $S(1+t)$. Then Lemma 3.1 implies that

$$|b_i - a_i| \leq K^i|b_0 - a_0| + \frac{1+K^i}{1-K}\,\varepsilon$$

where $K = \dfrac{1}{(1+t)^2} < 1$. Hence, the sequence $\{b_i\}_{i\in\mathbb{N}_0}$ has Hyers-Ulam stability where $b_0 = a_0$. □

4. Image of concentric circles under the conjugation h

In this section, we show that the image of concentric circles of which center is $-\frac{d}{c}$ under the map h defined as $h(z) = \frac{z-\beta}{z-\alpha}$. Denote a circle in the complex plane by C. Recall that the image of line or circle under Möbius map is line or circle. Moreover, since Möbius map is conformal, the end points of the diameter of C is mapped by h to the end points of the diameter of $h(C)$. However, the image of the center of C is not in general the center of $h(C)$. Recall the map $f = h \circ g \circ h^{-1}$ is the dilation defined as $f(w) = kw$ where

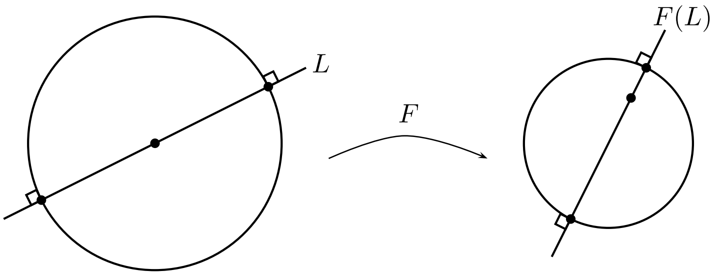


Figure 2: Circle and line under Möbius map

$k = \frac{1}{(c\beta+d)^2} > 1$ by Lemma 2.3. Let L be the straight line in $\mathbb{C}$. Define the *extended (straight) line* as $L \cup \{\infty\}$ and denote it by L_∞.

Lemma 4.1. *Let h be the Möbius map defined as $h(z) = \frac{z-\beta}{z-\alpha}$. Then the image of $-\frac{d}{c}$ under h as follows*

$$h\left(-\frac{d}{c}\right) = \frac{1}{k} \ \ \textit{and} \ \ -\frac{d}{c} = \frac{k\beta-\alpha}{k-1}.$$

Proof. The map h is the conjugation from g to f and $h(\infty) = 1$. The fact that $f \circ h = h \circ g$ implies that

$$f \circ h\left(-\frac{d}{c}\right) = h \circ g\left(-\frac{d}{c}\right) = h(\infty) = 1 = f\left(\frac{1}{k}\right).$$

Since f is a bijection on $\mathbb{C}$, $h(-\frac{d}{c}) = \frac{1}{k}$. Observe that the map $h^{-1}(w) = \frac{\alpha w-\beta}{w-1}$. Hence, we have

$$-\frac{d}{c} = h^{-1}\left(\frac{1}{k}\right) = \frac{k\beta-\alpha}{k-1}. \tag{8}$$

□

Lemma 4.2. *Let g be the hyperbolic Möbius map $g(z) = \frac{az+b}{cz+d}$ where $ad-bc = 1$ and $c \neq 0$ with the attracting and repelling fixed points α and β respectively. Denote the extended (straight) line which contains α and β by $L_\infty = \{\, t\alpha + (1-t)\beta \,:\, t \in \mathbb{R}\,\} \cup \{\infty\}$. Then L_∞ is invariant under g. In particular, $g(L_\infty) = L_\infty$.*

Proof. Lemma 4.1 implies the following equation

$$-\frac{d}{c} = \frac{k}{k-1}\beta - \frac{1}{k-1}\alpha$$

Thus $-\frac{d}{c}$ is in the line segment which contains both α and β. Then $-\frac{d}{c} \in L_\infty$. The fact that $g\left(-\frac{d}{c}\right) = \infty$ implies that ∞ is also contained in $g(L_\infty)$. Recall that any non constant Möbius map is bijective on $\hat{\mathbb{C}}$. Then $g(L_\infty)$ is the extended line which contains α and β. Hence, the extended line L_∞ is invariant under g and $g(L_\infty) = L_\infty$. □

Corollary 4.3. *Let L_∞ be the extended line defined in Lemma 4.2. Let h be the map defined as $\frac{z-\beta}{z-\alpha}$ where α and β are the attracting and the repelling fixed points of g respectively. Then $h(L_\infty)$ is the extended real line, $\mathbb{R} \cup \{\infty\}$.*

Proof. By the definition of h, $h(\alpha) = \infty$, $h(\beta) = 0$ and $h(\infty) = 1$. Hence, $h(L_\infty)$ is the extended line which contains $0, 1$ and ∞. Hence, $h(L_\infty)$ is the extended real line, $\mathbb{R} \cup \{\infty\}$. □

Recall the set $\partial S(r)$ is the circle of which center is $-\frac{d}{c}$ with radius $\frac{r}{|c|}$ for $c \neq 0$. The extended line L_∞ contains $-\frac{d}{c}$. Thus for any two points p and q in L_∞, if the midpoint of the line segment connecting p and q is $-\frac{d}{c}$, then p and q are the end points of the diameter of $\partial S(r)$ for some $r > 0$. In the following lemma the image of the endpoints of the diameter of $\partial S(r)$ for arbitrary radius.

Lemma 4.4. *Let h be the map $h(z) = \frac{z-\beta}{z-\alpha}$. Let $t\alpha + (1-t)\beta$ be the point in L_∞ and denote it by p_t for $t \in \mathbb{R}$. Then*

$$h(p_t) = \frac{t}{t-1} \quad \text{and} \quad h\left(-p_t - \frac{2d}{c}\right) = \frac{tk-t+2}{tk-t+k+1}.$$

Proof. The straightforward calculation shows that $h(p_t) = \frac{t}{t-1}$. So we omit the detail of the calculation. By the equation (8) in Lemma 4.1, we have $-\frac{d}{c} = \frac{k\beta-\alpha}{k-1}$. Then

$$\begin{aligned}
h\left(-p_t - \frac{2d}{c}\right) &= \frac{\left(-p_t - \frac{2d}{c}\right) - \beta}{\left(-p_t - \frac{2d}{c}\right) - \alpha} \\
&= \frac{-t\alpha - (1-t)\beta - \beta - \frac{2d}{c}}{-t\alpha - (1-t)\beta - \alpha - \frac{2d}{c}} \\
&= \frac{t\alpha + (2-t)\beta - \frac{2k\beta-2\alpha}{k-1}}{(1+t)\alpha + (1-t)\beta - \frac{2k\beta-2\alpha}{k-1}} && \text{by (8) in Lemma 4.1} \\
&= \frac{(tk-t+2)(\alpha-\beta)}{(tk-t+k+1)(\alpha-\beta)} \\
&= \frac{tk-t+2}{tk-t+k+1}.
\end{aligned}$$

It completes the proof. □

Corollary 4.5. *Let h be the Möbius map $h(z) = \frac{z-\beta}{z-\alpha}$. Then we obtain that*

$$h\left(-\beta - \frac{2d}{c}\right) = \frac{2}{k+1}, \quad \text{and} \quad h\left(-\frac{\alpha+\beta}{2} - \frac{2d}{c}\right) = \frac{k+3}{3k+1}.$$

Proof. Observe that $p_0 = \beta$ and $p_{\frac{1}{2}} = \frac{\alpha+\beta}{2}$ in Lemma 4.4. Put $t = 0$ for $h\left(-\beta - \frac{2d}{c}\right)$ and put $t = \frac{1}{2}$ for $h\left(-\frac{\alpha+\beta}{2} - \frac{2d}{c}\right)$ in Lemma 4.4. □

Lemma 4.6. *Let g be the hyperbolic Möbius map $g(z) = \frac{az+b}{cz+d}$ where $ad - bc = 1$ and $c \neq 0$. Let h be another Möbius map as follows*

$$h(z) = \frac{z-\beta}{z-\alpha}.$$

where α and β be the attracting and the repelling fixed point of g respectively. Then

$$\partial S\left(\frac{1}{\sqrt{k}}\right) = \left\{z\colon\ \left|z + \frac{d}{c}\right| = \left|\frac{d}{c} + \beta\right|\right\}$$

and

$$h\left(\partial S\left(\frac{1}{\sqrt{k}}\right)\right) = \left\{w\colon\ \left|w - \frac{1}{k+1}\right| = \frac{1}{k+1}\right\}$$

where $k = \frac{1}{(c\beta+d)^2} > 1$.

Proof. The fact that $k = \frac{1}{(c\beta+d)^2}$ implies $\frac{|c\beta+d|}{|c|} = \frac{1}{\sqrt{k}|c|}$. Thus by Lemma 3.2, we have

$$\partial S\left(\frac{1}{\sqrt{k}}\right) = \left\{z\colon\ \left|z + \frac{d}{c}\right| = \left|\frac{d}{c} + \beta\right|\right\}.$$

The extended straight line L_∞ contains β and $-\beta - \frac{2d}{c}$ by Lemma 4.1. The midpoint of the line segment between these two points is $-\frac{d}{c}$, which is the center of $\partial S\left(\frac{1}{\sqrt{k}}\right)$. The half of the distance between β and $-\beta - \frac{2d}{c}$ is the radius. Moreover, $\partial S\left(\frac{1}{\sqrt{k}}\right)$ meets L_∞ at two points β and $-\beta - \frac{2d}{c}$ at right angle because L_∞ goes through the center of $\partial S\left(\frac{1}{\sqrt{k}}\right)$. Since h is conformal, $F(L_\infty)$ also meets $F\left(\partial S\left(\frac{1}{\sqrt{k}}\right)\right)$ at $F(\beta)$ and $F\left(-\beta - \frac{2d}{c}\right)$ at right angle. $h(\beta) = 0$ by the definition of h. Corollary 4.5 implies that

$$h\left(-\beta - \frac{2d}{c}\right) = \frac{2}{k+1}. \tag{9}$$

Then center of the circle $h\left(\partial S\left(\frac{1}{\sqrt{k}}\right)\right)$ is the midpoint of the line segment between 0 and $\frac{2}{k+1}$. Hence, $h\left(\partial S\left(\frac{1}{\sqrt{k}}\right)\right)$ is the following circle

$$\left\{w :\ \left|w - \frac{1}{k+1}\right| = \frac{1}{k+1}\right\}.$$

□

Corollary 4.7. *The following equations hold*

$$\left|\frac{c\beta+d}{c}\right| = \frac{1}{k-1}\,|\alpha-\beta| \quad \text{and} \quad \frac{1}{|c|} = \frac{\sqrt{k}}{k-1}\,|\alpha-\beta|$$

where $k = \frac{1}{(c\beta+d)^2}$.

Proof. Observe that $h^{-1}(w) = \frac{\alpha w - \beta}{w-1}$. Thus we have that

$$h^{-1}\left(\frac{2}{k+1}\right) = -\frac{k+1}{k-1}\,(\alpha-\beta) + \alpha$$

and $h^{-1}(0) = \beta$. The distance between the above two points is the diameter of $\partial S\left(\frac{1}{\sqrt{k}}\right)$. Then the half of this distance is the radius of $\partial S\left(\frac{1}{\sqrt{k}}\right)$ as follows

$$\frac{1}{2}\left|-\frac{k+1}{k-1}(\alpha-\beta)+\alpha-\beta\right| = \frac{1}{k-1}|\alpha-\beta|.$$

However, $\left|\frac{c\beta+d}{c}\right|$ is also the radius of $\partial S\left(\frac{1}{\sqrt{k}}\right)$ by Lemma 4.6. Hence, we have $\left|\frac{c\beta+d}{c}\right| = \frac{1}{k-1}|\alpha-\beta|$ and moreover, since $\frac{1}{\sqrt{k}} = |c\beta+d|$, the equation $\frac{1}{|c|} = \frac{\sqrt{k}}{k-1}|\alpha-\beta|$ holds. $\square$

Remark 4.8. *The upper bound of $\frac{1}{|c|}$ for every $k>1$ is the distance between $-\frac{d}{c}$ and $\frac{\alpha+\beta}{2}$ because*

$$\left|\frac{\alpha+\beta}{2}-\left(-\frac{d}{c}\right)\right| = \left|\frac{a-d}{2c}+\frac{d}{c}\right| = \frac{a+d}{2|c|} > \frac{2}{2|c|} = \frac{1}{|c|}.$$

Recall that $\mathrm{tr}(g) = a+d = 2+\tau$. *Thus the equation, $\frac{a+d}{2|c|} = \frac{1+\frac{\tau}{2}}{|c|}$ holds. Then the disk $\hat{\mathbb{C}}\setminus S\left(1+\frac{\tau}{2}\right)$ as follows*

$$\hat{\mathbb{C}}\setminus S\left(1+\frac{\tau}{2}\right) = \left\{z\colon \left|z+\frac{d}{c}\right| \le \left|\frac{\alpha+\beta}{2}+\frac{d}{c}\right|\right\}. \tag{10}$$

Moreover, this disk contains compactly the disk $\hat{\mathbb{C}}\setminus S(1)$.

Proposition 4.9. *The circle $h\left(\partial S\left(1+\frac{\tau}{2}\right)\right)$ is as follows*

$$\left\{w\colon \left|w-\frac{1}{2}\left(\frac{k+3}{3k+1}-1\right)\right| = \frac{1}{2}\left(\frac{k+3}{3k+1}+1\right)\right\}. \tag{11}$$

Proof. By the definition of $\partial S(r)$ for $r>0$, the line connecting the fixed points α and β, say L, goes through the center of the circle $\partial S(r)$. In other words, the circle and line meet at two points at right angle. Since h is conformal, $h(\partial S(r))$ meets also the real line at two points at right angle.
Observe that two points in the set $\partial S\left(1+\frac{\tau}{2}\right)\cap L$ are $-\frac{\alpha+\beta}{2}-\frac{2d}{c}$ and $\frac{\alpha+\beta}{2}$. Thus it suffice to show that the image of two points in $\partial S\left(1+\frac{\tau}{2}\right)\cap L$ under h is the points $\frac{k+3}{3k+1}$ and -1. In Lemma 4.4, put $t=\frac{1}{2}$. Hence, we have the following equations

$$h\left(\frac{\alpha+\beta}{2}-\frac{2d}{c}\right) = \frac{k+3}{3k+1} \quad \text{and} \quad h\left(\frac{\alpha+\beta}{2}\right) = -1$$

for each $k>1$. Then the midpoint of the two points $\frac{k+3}{3k+1}$ and -1 is the center of the circle $h\left(\partial S\left(1+\frac{\tau}{2}\right)\right)$. Moreover, the half of the distance between these two points is the radius of $h\left(\partial S\left(1+\frac{\tau}{2}\right)\right)$. $\square$

Remark 4.10. *Observe that the map $y(k) = \frac{k+3}{3k+1}$ is a decreasing function for $k>0$ and $\frac{1}{3} < \frac{k+3}{3k+1} < 1$ for $k>1$. $y(1) = 1$ and $\lim_{k\to\infty}\frac{k+3}{3k+1} = \frac{1}{3}$. Then the circle in Corollary 4.9, $h\left(\partial S\left(1+\frac{\tau}{2}\right)\right)$ is contained in the unit disk $\{w\colon |w|\le 1\}$ for all $1<k<\infty$.*

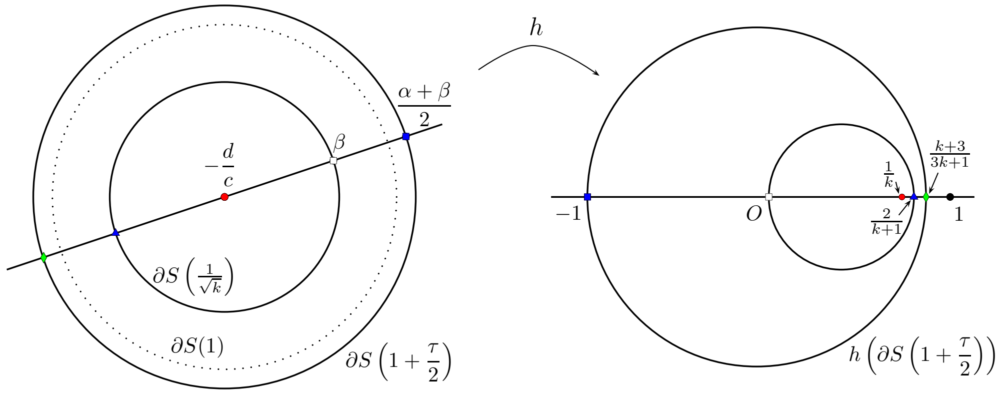


Figure 3: Concentric circles and its images under h

5. Avoided region

The map g is the hyperbolic Möbius map as follows

$$g(z) = \frac{az+b}{cz+d}$$

for $ad - bc = 1$ and $c \neq 0$. Since the point ∞ is not a fixed point of g, the preimage of ∞ under g, namely, $g^{-1}(\infty)$ is in the complex plane. For a given $\varepsilon > 0$, the sequence $\{a_i\}_{i\in\mathbb{N}_0}$ satisfying that

$$|a_{i+1} - g(a_i)| \leq \varepsilon$$

contains $g^{-1}(\infty)$, say a_k, then $|a_{k+1} - \infty|$ is not bounded where $|\cdot|$ is the absolute value of the complex number. In order to exclude $g^{-1}(\infty)$ in the whole sequence $\{a_i\}_{i\in\mathbb{N}_0}$, the region $\mathcal{R}_g$ is defined such that if the initial point of the sequence, a_0 is not in $\mathcal{R}_g$, then the whole sequence $\{a_i\}_{i\in\mathbb{N}_0}$ cannot be in the same region $\mathcal{R}_g$. Let the forward orbit of p under F be the set $\{F(p), F^2(p), \ldots, F^n(p), \ldots\}$ and denote it by $\mathrm{Orb}_{\mathbb{N}}(p, F)$.

Definition 5.1. *Let F be the map on $\hat{\mathbb{C}}$ which does not fix ∞. Avoided region for the sequence $\{a_i\}_{i\in\mathbb{N}_0}$ satisfying $|a_{i+1} - F(a_i)| \leq \varepsilon$ for a given $\varepsilon > 0$ which is denoted by $\mathcal{R}_F \subset \mathbb{C}$ is defined as follows*

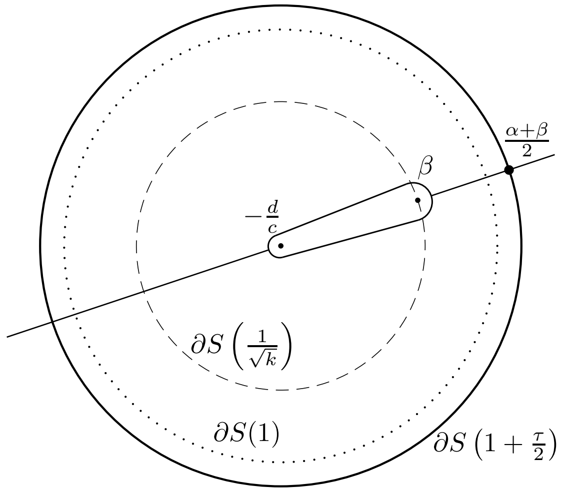
1. *$\hat{\mathbb{C}} \setminus \mathcal{R}_F$ is (forward) invariant under F, that is, $F(\hat{\mathbb{C}} \setminus \mathcal{R}_F) \subset \hat{\mathbb{C}} \setminus \mathcal{R}_F$.*
2. *For any given initial point a_0 in $\mathbb{C} \setminus \mathcal{R}_F$, all points in the sequence $\{a_i\}_{i\in\mathbb{N}_0}$ satisfying $|a_{i+1} - F(a_i)| \leq \varepsilon$ are in $\mathbb{C} \setminus \mathcal{R}_F$.*

If $\mathcal{R}_F$ contains $\mathrm{Orb}_{\mathbb{N}}(p, F^{-1})$ where $p \in \hat{\mathbb{C}}$, then it is called the avoided region at p and is denoted by $\mathcal{R}_F(p)$.

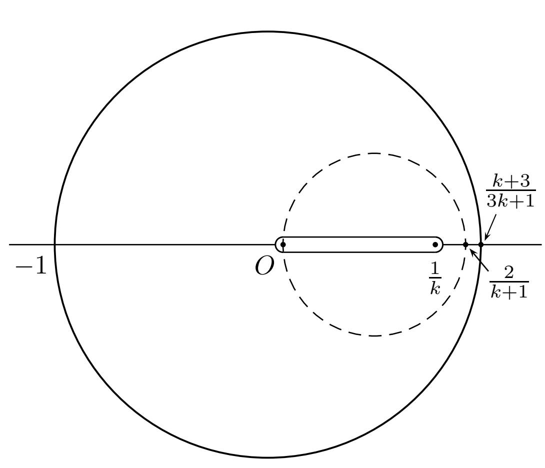
In the above definition, the avoided region does not have to be connected.

Remark 5.2. *The set $\hat{\mathbb{C}} \setminus S(1+t)$ in Proposition 3.3 is an avoided region at ∞. However, avoided region $\hat{\mathbb{C}} \setminus S\left(1+\frac{\tau}{2}\right)$ can be extended to some neighborhood of $\mathrm{Orb}_{\mathbb{N}}(\infty, g^{-1})$, which is denoted to be $\mathcal{R}_g(\infty)$ in the following proposition.*

Recall that for a complex number z, $\mathrm{Re}\, z$ is the real part of z and $\mathrm{Im}\, z$ is the imaginary part of z. The complex conjugate of z is denoted by $\bar{z}$.



(a) Avoided region $\mathcal{R}_g(\infty)$



(b) Avoided region $\mathcal{R}_f(1)$

Figure 4: Avoided regions for hyerbolic Möbius maps

Proposition 5.3. *Define the following regions*

$$\mathcal{R}_1^t = \left\{w\colon |w| \le \frac{t\delta}{k-1}\right\} \cap \{w\colon \operatorname{Re} w \le 0\}$$

$$\mathcal{R}_2^t = \left\{w\colon -\frac{t\delta}{k-1} \le \operatorname{Im} w \le \frac{t\delta}{k-1}\right\} \cap \left\{w\colon 0 \le \operatorname{Re} w \le \frac{1}{k}\right\}$$

$$\mathcal{R}_3^t = \left\{w\colon \left|w - \frac{1}{k}\right| \le \frac{t\delta}{k-1}\right\} \cap \left\{w\colon \operatorname{Re} w \ge \frac{1}{k}\right\}$$

for $t \ge 1$ and denote the union $\mathcal{R}_1^t \cup \mathcal{R}_2^t \cup \mathcal{R}_3^t$ by $\mathcal{R}^t$. Let the map f be the dilation, $f(w) = kw$ for the given number $k > 1$. Let $\{c_i\}_{i\in\mathbb{N}_0}$ be the sequence for a given $\delta > 0$ satisfying that

$$|c_{i+1} - f(c_i)| < \delta \tag{12}$$

for all $i \in \mathbb{N}_0$. If $\delta < \left(1 - \frac{1}{k}\right)^2$, then the set $\mathcal{R}^t$ is an avoided region for $\{c_i\}_{i\in\mathbb{N}_0}$, namely, $\mathcal{R}_f$. Furthermore, $\mathcal{R}_f$ contains the the orbit, $\operatorname{Orb}_{\mathbb{N}}(1, f^{-1})$*. Hence, the avoided region can be chosen as $\mathcal{R}_f(1)$.*

The proof of Proposition 5.3 requires the combined result of lemmas as follows.

Lemma 5.4. *Let f be the map $f(w) = kw$ for $k > 1$. The sequence $\{c_i\}_{i\in\mathbb{N}_0}$ is defined in* (12) *for $\delta > 0$. Consider the following regions*

$$\mathcal{D}_1^t = \left\{w\colon |w| \le \frac{t\delta}{k-1}\right\}, \quad \mathcal{D}_2^t = \left\{w\colon -\frac{t\delta}{k-1} \le \operatorname{Im} w \le \frac{t\delta}{k-1}\right\}$$

for $\delta > 0$ and $t \ge 1$. If $c_0 \in \hat{\mathbb{C}} \setminus \mathcal{D}_j^t$ for $j = 1, 2$, then the whole sequence $\{c_i\}_{i\in\mathbb{N}_0}$ is contained in the same set $\hat{\mathbb{C}} \setminus \mathcal{D}_j^t$ respectively.

Proof. Suppose firstly that $|c_0| > \frac{t\delta}{k-1}$. Thus $|f(c_0)| > \frac{kt\delta}{k-1}$. The inequality $|c_{i+1} - f(c_i)| < \delta$ implies that

$$\delta > |c_1 - f(c_0)| \ge \big|\,|c_1| - |f(c_0)|\,\big| \tag{13}$$

$$|c_1| > |f(c_0)| - \delta > \frac{kt\delta}{k-1} - t\delta = \frac{t\delta}{k-1} \tag{14}$$

for $t \geq 1$. Then $c_1 \in \hat{\mathbb{C}} \setminus \mathcal{D}_1^t$. By induction the whole sequence is also contained in $\hat{\mathbb{C}} \setminus \mathcal{D}_1^t$.
Similarly, suppose that $c_0 \in \mathcal{D}_2^t$. Since $|\mathrm{Im}(c_1 - f(c_0))| \leq |c_1 - f(c_0)| < \delta$, the assumption $c_0 \in \mathcal{D}_2^t$ implies the following inequality.

$$|\mathrm{Im}\, c_1| > |\mathrm{Im}\, f(c_0)| - \delta = k\,|\mathrm{Im}\, c_0| - \delta > \frac{kt\delta}{k-1} - t\delta = \frac{t\delta}{k-1}$$

for $t \geq 1$. Then $c_1 \in \mathcal{D}_2^t$. Hence, by induction the whole sequence is also contained in $\hat{\mathbb{C}} \setminus \mathcal{D}_2^t$. $\square$

Observe that $\mathcal{D}_1^t \subset \mathcal{D}_2^t$ for every $t \geq 1$ in Lemma 5.4. The region $\mathcal{R}_1^t$ is the half disk of $\mathcal{D}_1^t$ in the left half complex plane. Thus if a point c_j in $\hat{\mathbb{C}} \setminus \mathcal{R}_1^t$ and c_{j+1} is also contained in the set $\{w \colon \mathrm{Re}\, w \leq 0\}$ for some $j \in \mathbb{N}_0$, then c_{j+1} is contained in $\left(\hat{\mathbb{C}} \setminus \mathcal{R}_1^t\right) \cap \{w \colon \mathrm{Re}\, w \leq 0\}$. However, c_{j+1} may be in the right half complex plane. Thus in order to construct the avoided region in Proposition 5.3, another lemma is required as follows.

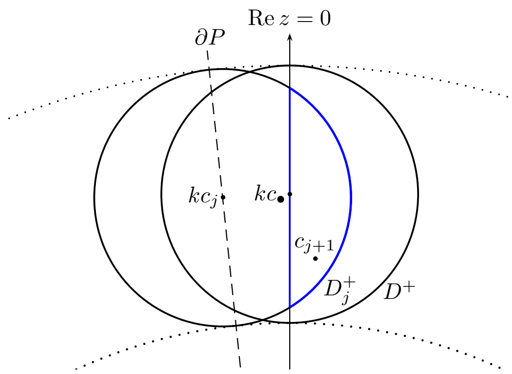


Figure 5

Lemma 5.5. *Let f be the map $f(w) = kw$ for $k > 1$. The sequence $\{c_i\}_{i\in\mathbb{N}_0}$ is defined in* (12) *for $\delta > 0$. Suppose that c_j for some $j \in \mathbb{N}_0$ is contained in the set $\left(\hat{\mathbb{C}} \setminus \mathcal{D}_1^t\right) \cap \{w \colon \mathrm{Re}\, w \leq 0\}$ but $\mathrm{Re}\, c_{j+1} \geq 0$. Then c_{j+1} satisfies that $|\mathrm{Im}\, c_{j+1}| > \frac{t\delta}{k-1}$, that is, $c_{j+1} \in \hat{\mathbb{C}} \setminus \mathcal{D}_2^t$ for $t \geq 1$.*

Proof. By the assumption we have

$$c_j \in \left\{ w \colon |w| > \frac{t\delta}{k-1} \right\} \cap \{w \colon \mathrm{Re}\, w \leq 0\}$$

for some given $j \in \mathbb{N}_0$. Let $c_\bullet$ be the purely imaginary number satisfying $|c_j| = |c_\bullet|$ and $\mathrm{Im}\, c_j$ and $\mathrm{Im}\, c_\bullet$ has the same sign. Without loss of generality, we may assume that $\mathrm{Im}\, c_j > 0$ and $\mathrm{Im}\, c_\bullet > 0$. The proof of other case is similar. Define the sets as follows

$$D^+ = \{w \colon |w - kc_\bullet| < \delta\} \cap \{w \colon \mathrm{Re}\, w \geq 0\}$$
$$D_j^+ = \{w \colon |w - kc_j| < \delta\} \cap \{w \colon \mathrm{Re}\, w \geq 0\}.$$

Consider the region

$$P = \{z \colon |z - kc_\bullet| < |z - kc_j|\}$$

We show that $D_j^+ \subseteq D^+$ where $\mathrm{Re}\, c_j \leq 0$ using the following claim.

claim: For any $z \in \left(\hat{\mathbb{C}} \setminus P\right) \cap \{z\colon \operatorname{Im} z \geq 0\}$, the real part of z is negative, namely, $\operatorname{Re} z \leq 0$. The number z satisfies the equivalent inequalities

$$\begin{aligned}
& |z - kc_\bullet| \geq |z - kc_j| \\
\iff\ & |z - kc_\bullet|^2 \geq |z - kc_j|^2 \\
\iff & - kc_\bullet \bar{z} - k\bar{c_\bullet} z \geq -kc_j \bar{z} - k\bar{c_j} z \\
\iff & - \operatorname{Re}(\bar{c_\bullet} z) \geq -\operatorname{Re}(\bar{c_j} z) \\
\iff\ & \operatorname{Re}((\overline{c_j - c_\bullet}) z) \geq 0
\end{aligned}$$

Recall that $\operatorname{Re} c_j \leq 0$, $\operatorname{Re} c_\bullet = 0$ and $\operatorname{Im} c_\bullet \geq \operatorname{Im} c_j > 0$ because $|c_j| = |c_\bullet|$. Thus $\overline{c_j - c_\bullet} = -a + bi$ for some $a,\ b > 0$. Denote $z = x + yi$ for $y > 0$. Then $\operatorname{Re}((\overline{c_j - c_\bullet})z) = \operatorname{Re}\big((-a + bi)(x + yi)\big) = -ax - by \geq 0$. Then $x \leq 0$, that is, $\operatorname{Re} z \leq 0$ is the necessary condition for the inequality. The proof of the claim is complete.
If $\operatorname{Re} w \geq 0$ and $\operatorname{Im} w \geq 0$, then $w \in P$. Thus

$$|w - kc_\bullet| < |w - kc_j| < \delta.$$

Then $D_j^+ \subseteq D^+$. For every $w \in D^+$, the following inequality holds

$$\begin{aligned}
& |\operatorname{Im} w - \operatorname{Im} kc_\bullet| \leq |w - kc_\bullet| < \delta \\
\implies & |\operatorname{Im} w| > k\,|\operatorname{Im} c_\bullet| - \delta \\
& = k|c_\bullet| - \delta = k|c_j| - \delta \\
& > \frac{tk\delta}{k-1} - t\delta \\
& = \frac{t\delta}{k-1}
\end{aligned}$$

Then D^+ is disjoint from $\mathcal{D}_2^t$. The fact that $c_{j+1} \in D_j^+$ implies that $c_{j+1} \notin \mathcal{D}_2^t$ for $t \geq 1$. Hence, $c_{j+1} \in \hat{\mathbb{C}} \setminus \mathcal{D}_2^t$ for $t \geq 1$. $\square$

Lemma 5.6. *Let f be the map $f(w) = kw$ for $k > 1$. The sequence $\{c_i\}_{i \in \mathbb{N}_0}$ is defined in (12) for $\delta > 0$. Suppose that $c_j \in \left(\hat{\mathbb{C}} \setminus \mathcal{R}_3^t\right) \cap \left\{w\colon \operatorname{Re} w \geq \frac{1}{k}\right\}$, that is,*

$$c_j \in \left\{w\colon \left|w - \frac{1}{k}\right| > \frac{t\delta}{k-1}\right\} \cap \left\{w\colon \operatorname{Re} w \geq \frac{1}{k}\right\} \tag{15}$$

for $t \geq 1$ and for some $j \in \mathbb{N}_0$. If $\delta < \left(\frac{k-1}{k}\right)^2$, then c_{j+1} is contained in $\hat{\mathbb{C}} \setminus \mathcal{R}_3^t \cap \left\{w\colon \operatorname{Re} w \geq \frac{1}{k}\right\}$ for $t \geq 1$.

Proof. Observe that $f(\hat{\mathbb{C}} \setminus \mathcal{R}_3^t)$ is the half disk. Since $|c_{j+1} - f(c_j)| < \delta$, we have that $|c_{j+1}| > |f(c_j)| - \delta \geq 1 - \delta$. Moreover, $|c_j| < \frac{1}{k} + \frac{t\delta}{k-1}$. Thus it suffice to show that $1 - \delta > \frac{1}{k} + \frac{t\delta}{k-1}$. Then

$$\begin{aligned}
1 - \delta > \frac{1}{k} + \frac{t\delta}{k-1} \iff 1 - \frac{1}{k} > \frac{t\delta}{k-1} + \delta &= \left(\frac{t}{k-1} + 1\right)\delta \\
&\geq \left(\frac{k+t-1}{k-1}\right)\delta.
\end{aligned}$$

Hence, $\delta < \frac{(k-1)^2}{k(k+t-1)} \leq \left(\frac{k-1}{k}\right)^2$ for all $t \geq 1$. $\square$

Proof. [proof of Proposition 5.3] The definition of $\mathcal{R}^t$ and $\mathcal{D}_2^t$ implies that $\mathcal{R}^t \subset \mathcal{D}_2^t$. If $c_j \in \mathcal{D}_2^t$, then $c_{j+1} \in \mathcal{D}_2^t$ by Lemma 5.4. Thus we may assume that $\operatorname{Re} c_j \leq 0$ or $\operatorname{Re} c_j \geq \frac{1}{k}$. If $c_j \in \hat{\mathbb{C}} \setminus R_1^t \cap \{w\colon \operatorname{Re} \leq 0\}$, then c_{j+1} is contained in the same set by Lemma 5.5. Similarly, if $c_j \in \hat{\mathbb{C}} \setminus R_3^t \cap \{w\colon \operatorname{Re} \leq 0\}$, then c_{j+1} is contained in the same set by Lemma 5.6. Hence, $\mathcal{R}^t = \mathcal{R}_1^t \cup \mathcal{R}_2^t \cup \mathcal{R}_3^t$ can be an avoided region $\mathcal{R}_f$ for $t \geq 1$. Moreover, $\operatorname{Orb}_{\mathbb{N}}(1, f^{-1})$ is contained in the line segment connecting 0 and $\frac{1}{k}$. □

We define the avoided region $\mathcal{R}_f(1)$ as $\mathcal{R}_1^k \cup \mathcal{R}_2^1 \cup \mathcal{R}_3^1$ and $\mathcal{R}_g(\infty)$ as $h^{-1}(\mathcal{R}_f(1))$.

6. Escaping time from the region

Let the sequence $\{c_i\}_{i\in\mathbb{N}_0}$ satisfies the following

$$|c_{i+1} - f(c_i)| \leq \delta \tag{16}$$

for all $i \in \mathbb{N}_0$. For the given region R, assume that $c_0 \in R$. If the distance between c_n and the closure of R is positive for all $n \geq N$ for some $N \in \mathbb{N}$, then N is called *escaping time* of the sequence $\{c_i\}_{i\in\mathbb{N}_0}$ from R under f. If the escaping time N is independent of the initial point c_0 in R, then the number N is called *uniformly* escaping time. Denote the ball of which center is the origin with radius $r > 0$ by $B(0, r)$.

Lemma 6.1. *Let $\{c_i\}_{i\in\mathbb{N}_0}$ be the sequence defined in the equation* (16) *where $f(w) = kw$ for $k > 1$. Suppose that $c_0 \in E_f$ where E_f is defined as the region $h\Big(\hat{\mathbb{C}} \setminus S\left(1 + \frac{\tau}{2}\right)\Big) \setminus B\left(0, \frac{k\delta}{k-1}\right)$. Then the (uniformly) escaping time N from the region E_f under f satisfies the following inequality*

$$N > \log\left(\frac{1}{\delta}\left(\frac{k+3}{3k+1} + 1\right) + 1\right) \Big/ \log k$$

for small enough $\delta > 0$.

Proof. By triangular inequality, we have

$$\begin{aligned}
|f^n(c_0) - c_0| &\leq |f^n(c_0) - f^{n-1}(c_1)| + |f^{n-1}(c_1) - f^{n-2}(c_2)| + \\
&\qquad \cdots + |f^2(c_{n-2}) - f(c_{n-1})| + |f(c_{n-1}) - c_n| + |c_n - c_0| \\
&= \sum_{j=1}^{n} k^{n-j} |f(c_{j-1}) - c_j| + |c_n - c_0| \\
&\leq \frac{k^n - 1}{k-1}\,\delta + |c_n - c_0|.
\end{aligned}$$

Thus we have

$$\begin{aligned}
|c_n - c_0| &\geq |f^n(c_0) - c_0| - \frac{k^n - 1}{k-1}\,\delta \\
&= (k^n - 1)|c_0| - \frac{k^n - 1}{k-1}\,\delta \\
&= (k^n - 1)\left(|c_0| - \frac{\delta}{k-1}\right) \\
&\geq (k^n - 1)\left(\frac{k\delta}{k-1} - \frac{\delta}{k-1}\right) \\
&= (k^n - 1)\,\delta
\end{aligned}$$

Hence, the escaping time N satisfies if the inequality $(k^N - 1)\,\delta > \frac{k+3}{3k+1} + 1$ holds, then $|c_n - c_0|$ is greater than the diameter of $h\Big(S\left(1 + \frac{\tau}{2}\right)\Big)$ for all $n \geq N$ by (11) in Proposition 4.9. Hence, N is the uniformly escaping time where $N > \frac{\log\left(\frac{1}{\delta}\left(\frac{k+3}{3k+1}+1\right)+1\right)}{\log k}$. □

Remark 6.2. *The inequality* $\frac{1}{3} < \frac{k+3}{3k+1} < 1$ *for* $k > 1$ *implies that a upper bound of the uniformly escaping time is* $N_0 > \log\left(\frac{2}{\delta} + 1\right) \Big/ \log k$.

The sequence $\{a_i\}_{i\in\mathbb{N}_0}$ is defined as the set each of which element $a_i = h(c_i)$ for every $i \in \mathbb{N}_0$ where $\{c_i\}_{i\in\mathbb{N}_0}$ is defined in (16). Recall that f is the map $h \circ g \circ h^{-1}$. Denote the radius of the ball $h^{-1}\big(B(c_j, \delta)\big)$ by ε_j for $j \in \mathbb{N}_0$. Then the sequence $\{a_i\}_{i\in\mathbb{N}_0}$ as follows

$$|a_{i+1} - g(a_i)| \leq \varepsilon_i. \tag{17}$$

corresponds $\{c_i\}_{i\in\mathbb{N}_0}$ by the conjugation h. Then the escaping time of $\{a_i\}_{i\in\mathbb{N}_0}$ from $h^{-1}(E_f)$ under g is the same as that of $\{c_i\}_{i\in\mathbb{N}_0}$ from E_f under f in Lemma 6.1. Furthermore, since h is uniformly continuous on the closure of $\hat{\mathbb{C}} \setminus S\left(1 + \frac{\tau}{2}\right)$ under Euclidean metric, there exists $\varepsilon > 0$ such that $h\big(B(a_j, \varepsilon)\big) \subset B(c_j, \delta)$ for $j = 1, 2, \dots N_1$ for all $c_j \in E_f$. Thus we obtain the following Proposition.

Proposition 6.3. *Let* $\{c_i\}_{i\in\mathbb{N}_0}$ *be the sequence satisfying*

$$|c_{i+1} - f(c_i)| \leq \delta$$

where $f(w) = kw$ *for* $k > 1$ *on* E_f *defined in Lemma 6.1. Let* N *be the (uniformly) escaping time from* E_f *Let* $\{a_i\}_{i\in\mathbb{N}_0}$ *be the sequence satisfying* $a_i = h(c_i)$ *for every* $i \in \mathbb{N}_0$*. Then there exists* $\varepsilon > 0$ *such that if* $\{a_i\}_{i\in\mathbb{N}_0}$ *satisfies that*

$$|a_{i+1} - g(a_i)| \leq \varepsilon \tag{18}$$

on $h^{-1}(E_f)$ *for* $i = 1, 2, \dots, N-1$*, then the escaping time from* $h^{-1}(E_f)$ *under* g *is also* N*.*

Remark 6.4. *The definition of* E_f *implies that the set* $h^{-1}(E_f)$ *is the set,* $\left(\hat{\mathbb{C}} \setminus S\left(1 + \frac{\tau}{2}\right)\right) \setminus h^{-1}\left(B\left(0, \frac{k\delta}{k-1}\right)\right)$.

7. Hyers-Ulam stability on the complement of the avoided region

Hyers-Ulam stability of hyperbolic Möbius map requires two different regions, one of which is $S(1 + \frac{\tau}{2})$ where $\mathrm{tr}(g) = 2 + \tau$ for $\tau > 0$. The other region is the $\left(\hat{\mathbb{C}} \setminus S\left(1 + \frac{\tau}{2}\right)\right) \setminus \mathcal{R}_g(\infty)$ where $\mathcal{R}_g(\infty) = h^{-1}\left(\mathcal{R}_f(1)\right)$ is the avoided region defined in Section 5. We show that Hyers-Ulam stability on the region $\left(\hat{\mathbb{C}} \setminus S\left(1 + \frac{\tau}{2}\right)\right) \setminus \mathcal{R}_g(\infty)$ for finite time bounded by the uniformly escaping time. Then this stability and Proposition 3.3 implies Hyers-Ulam stability of g on the set $\hat{\mathbb{C}} \setminus \mathcal{R}_g(\infty)$.

Lemma 7.1. *The avoided region* $h^{-1}\left(\mathcal{R}_f(1)\right)$ *contains the disk*

$$D_{\tilde{\varepsilon}} = \left\{ z \colon \left| z + \frac{d}{c} \right| < \tilde{\varepsilon} \right\}$$

where $\tilde{\varepsilon} = \frac{k^2\delta}{(k-1)^3}\,|\alpha - \beta|$.

Proof. The avoided region $\mathcal{R}_f(1)$ contains the disk

$$B\left(\frac{1}{k}, \frac{\delta}{k-1}\right) = \left\{ w \colon \left| w - \frac{1}{k} \right| \leq \frac{\delta}{k-1} \right\}.$$

Since $h^{-1}\left(\frac{1}{k}\right) = -\frac{d}{c}$, the avoided region $h^{-1}\left(\mathcal{R}_f(1)\right)$ contains a small disk of which center is $-\frac{d}{c}$. Thus the number $\tilde{\varepsilon}$ is either the radius of the circle, $h^{-1}\left(B\left(\frac{1}{k}, \frac{\delta}{k-1}\right)\right)$ or the distance between $-\frac{d}{c}$ and $h^{-1}\left(\frac{1}{k} + \frac{\delta}{k-1}\right)$. Denote the point $\frac{1}{k} + \frac{\delta}{k-1}$ by $\frac{t_0}{k}$, that is, choose $t_0 = 1 + \frac{k\delta}{k-1}$. Thus we have

$$h^{-1}\left(\frac{t_0}{k}\right) = \frac{k\beta - t_0\alpha}{k - t_0} = \left(1 + \frac{t_0}{k - t_0}\right)\beta - \frac{t_0}{k - t_0}\,\alpha. \tag{19}$$

By Lemma 4.1, we have

$$-\frac{d}{c} = \left(1 + \frac{1}{k-1}\right)\beta - \frac{1}{k-1}\,\alpha.$$

Then the distance between $-\frac{d}{c}$ and $h^{-1}\left(\frac{t_0}{k}\right)$ is as follows

$$\begin{aligned}
&\left|\left(1 + \frac{t_0}{k - t_0}\right)\beta - \frac{t_0}{k - t_0}\,\alpha - \left(1 + \frac{1}{k-1}\right)\beta - \frac{1}{k-1}\,\alpha\right| \\
=&\left|\frac{1}{k-1} - \frac{t_0}{k - t_0}\right| |\alpha - \beta| \\
=&\left|\frac{1}{k-1} - \frac{1 + \frac{k\delta}{k-1}}{k - 1 - \frac{k\delta}{k-1}}\right| |\alpha - \beta| \\
=&\left|\frac{1}{k-1} - \frac{k - 1 + k\delta}{(k-1)^2 - k\delta}\right| |\alpha - \beta| \\
=&\left|\frac{-k^2\delta}{(k-1)\{(k-1)^2 - k\delta\}}\right| |\alpha - \beta| \\
=&\frac{k^2\delta}{(k-1)\{(k-1)^2 - k\delta\}} |\alpha - \beta|
\end{aligned} \tag{20}$$

Another candidate for $\tilde{\varepsilon}$ is the half of diameter of $h^{-1}(\mathcal{R}_2)$. Thus take two points, $\frac{t_0}{k}$ and $\frac{t_1}{k}$ in $\mathcal{R}_2$ where $\frac{t_1}{k} = \frac{1}{k} - \frac{\delta}{k-1}$, that is, $t_1 = 1 - \frac{k\delta}{k-1}$. Since $\frac{t_0}{k} - \frac{t_1}{k}$ is the diameter of the circle $\mathcal{R}_2$, the half of the distance between $h^{-1}\left(\frac{t_0}{k}\right)$ and $h^{-1}\left(\frac{t_1}{k}\right)$ is the radius of $h^{-1}(\mathcal{R}_2)$. Then the calculation in (19) implies that

$$\begin{aligned}
&\frac{1}{2}\left|h^{-1}\left(\frac{t_0}{k}\right) - h^{-1}\left(\frac{t_1}{k}\right)\right| \\
=&\frac{1}{2}\left|\left(1 + \frac{t_0}{k - t_0}\right)\beta - \frac{t_0}{k - t_0}\,\alpha - \left(1 + \frac{t_1}{k - t_1}\right)\beta - \frac{t_1}{k - t_1}\,\alpha\right| \\
=&\frac{1}{2}\left|\frac{t_0}{k - t_0} - \frac{t_1}{k - t_1}\right| |\alpha - \beta| \\
=&\frac{1}{2}\left|\frac{k - 1 + k\delta}{(k-1)^2 - k\delta} - \frac{k - 1 - k\delta}{(k-1)^2 + k\delta}\right| |\alpha - \beta| \\
=&\frac{1}{2}\left|\frac{2k^2(k-1)\delta}{\{(k-1)^2 - k\delta\}\{(k-1)^2 + k\delta\}}\right| |\alpha - \beta| \\
=&\frac{k-1}{(k-1)^2 + k\delta} \cdot \frac{k^2\delta}{(k-1)^2 - k\delta}\, |\alpha - \beta|
\end{aligned} \tag{21}$$

The number $\tilde{\varepsilon}$ have to be smaller than the both number of equation (20) and (21). Hence, $\tilde{\varepsilon}$ can be chosen as follows

$$\begin{aligned}\tilde{\varepsilon} &= \frac{k^2\delta}{(k-1)^3}\,|\alpha-\beta| \\ &< \frac{k-1}{(k-1)^2+k\delta}\cdot\frac{k^2\delta}{(k-1)^2-k\delta}\,|\alpha-\beta| < \frac{k^2\delta}{(k-1)\{(k-1)^2-k\delta\}}|\alpha-\beta|.\end{aligned}$$

□

Corollary 7.2. *For every $z \in \mathbb{C}\setminus\mathcal{R}_g(\infty)$, the following inequality holds*

$$|g'(z)| \le \frac{(k-1)^4}{k^3\delta^2}.$$

Proof. It suffice to show the upper bound of $|g'|$ on the region $\mathbb{C}\setminus h^{-1}(\mathcal{R}_2)$ because $\mathcal{R}_g(\infty)$ contains $h^{-1}(\mathcal{R}_2)$ and moreover, contains D_ε in Lemma 7.1. Recall that

$$g'(z) = \frac{1}{(cz+d)^2}$$

Thus in the region $\mathbb{C}\setminus D_\varepsilon$, the inequality $|cz+d| \ge |c|\varepsilon$ holds. Then the upper bound of $|g'|$ is as follows

$$|g'(z)| = \frac{1}{|cz+d|^2} \le \frac{1}{|c|^2\varepsilon^2} = \frac{(k-1)^6}{|c|^2k^4\delta^2|\alpha-\beta|^2}$$

by Lemma 7.1. Moreover, Corollary 4.7 implies the equation

$$\frac{1}{|c|} = \frac{\sqrt{k}}{k-1}|\alpha-\beta|.$$

Hence,

$$\frac{(k-1)^6}{|c|^2k^4\delta^2|\alpha-\beta|^2} = \frac{(k-1)^6}{k^4\delta^2}\cdot\frac{k}{(k-1)^2} = \frac{(k-1)^4}{k^3\delta^2}.$$

□

The following is the mean value inequality for holomorphic function.

Lemma 7.3. *Let g be the holomorphic function on the convex open set B in $\mathbb{C}$. Suppose that $\sup_{z\in B}|g'| < \infty$. Then for any two different points u and v in B, we have*

$$\left|\frac{g(u)-g(v)}{u-v}\right| \le 2\sup_{z\in B}|g'|.$$

Proof. The complex mean value theorem implies that

$$\mathrm{Re}(g'(p)) = \mathrm{Re}\left(\frac{g(u)-g(v)}{u-v}\right) \quad\text{and}\quad \mathrm{Im}(g'(q)) = \mathrm{Im}\left(\frac{g(u)-g(v)}{u-v}\right)$$

where p and q are in the line segment between u and v. Hence, the inequality

$$|\mathrm{Re}(g'(p)) + i\,\mathrm{Im}(g'(q))| \le |\mathrm{Re}(g'(p))| + |\mathrm{Im}(g'(q))| \le 2\sup_{z\in B}|g'|$$

completes the proof. □

Proposition 7.4. *Let g be the hyperbolic Möbius map. For a given $\varepsilon > 0$, let a complex valued sequence $\{a_i\}_{i\in\mathbb{N}_0}$ satisfies the inequality*

$$|a_{i+1} - g(a_i)| \le \varepsilon$$

for all $i \in \mathbb{N}_0$. Suppose that $a_0 \in \left(\hat{\mathbb{C}} \setminus S\left(1 + \frac{\tau}{2}\right)\right) \setminus \mathcal{R}_g(\infty)$ where $\mathcal{R}_g(\infty)$ is the avoided region defined in Section 5. For a given small enough number $\varepsilon > 0$, there exists the sequence $\{b_i\}_{i\in\mathbb{N}_0}$ defined as

$$b_{i+1} = g(b_i)$$

for each $i = 0, 1, 2, \ldots, N$ which satisfies that

$$|a_N - b_N| \le \frac{M^N - 1}{M - 1}\,\varepsilon$$

where N is the uniformly escaping time from the region $\hat{\mathbb{C}} \setminus S\left(1 + \frac{\tau}{2}\right)$ and $M = \frac{2(k-1)^4}{k^3\delta^2}$.

Proof. $\frac{M}{2}$ is an upper bound of $|g'|$ in $\mathbb{C} \setminus \mathcal{R}_g(\infty)$ by Corollary 7.2. The triangular inequality and Lemma 7.3 implies that

$$\begin{aligned}|a_N - b_N| &\le |a_N - g(a_{N-1})| + |g(a_{N-1}) - g(b_{N-1})| + |g(b_{N-1}) - b_N| \\ &\le \varepsilon + M\,|a_{N-1} - b_{N-1}|\end{aligned}$$

where $M \ge \sup_{z\in\mathbb{C}\setminus\mathcal{R}_g(\infty)} 2|g'|$. Observe that if $\delta > 0$ is sufficiently small, then $M > 1$ in the region $\hat{\mathbb{C}} \setminus S\left(1 + \frac{\tau}{2}\right)$. Thus we have

$$|a_N - b_N| + \frac{\varepsilon}{M-1} \le M\left(|a_{N-1} - b_{N-1}| + \frac{\varepsilon}{M-1}\right).$$

Then $|a_N - b_N|$ is bounded above by the geometric sequence with rate M.

$$\begin{aligned}|a_N - b_N| &\le M^N\left(|a_0 - b_0| + \frac{\varepsilon}{M-1}\right) - \frac{\varepsilon}{M-1} \\ &= M^N\,|a_0 - b_0| + \frac{M^N - 1}{M-1}\,\varepsilon\end{aligned}$$

Hence, if we choose $b_0 = a_0$, then

$$|a_N - b_N| \le \frac{M^N - 1}{M - 1}\,\varepsilon.$$

□

We show the Hyers-Ulam stability of hyperbolic Möbius map outside the avoided region.

Theorem 7.5. *Let g be a hyperbolic Möbius map. For a given $\varepsilon > 0$, let a complex valued sequence $\{a_n\}_{n\in\mathbb{N}_0}$ satisfies the inequality*

$$|a_{i+1} - g(a_i)| \le \varepsilon$$

for all $i \in \mathbb{N}_0$. Suppose that a given point $a_0 \in \mathbb{C} \setminus \mathcal{R}_g(\infty)$ where $\mathcal{R}_g(\infty)$ is the avoided region defined in Section 5. For a small enough number $\varepsilon > 0$, there exists the sequence $\{b_i\}_{i\in\mathbb{N}_0}$

$$b_{i+1} = g(b_i)$$

satisfies that $|a_i - b_i| \le H(\varepsilon)$ for all $i \in \mathbb{N}_0$ for each $i \in \mathbb{N}$ where the positive number $H(\varepsilon)$ converges to zero as $\varepsilon \to 0$.

Proof. Suppose first that $a_0 \in S\left(1+\frac{\tau}{2}\right)$. Then by Proposition 3.3, we have the inequality

$$|b_i - a_i| \leq \frac{1-K^i}{1-K}\,\varepsilon \tag{22}$$

for some $K < 1$. Secondly, assume that $a_0 \in \left(\hat{\mathbb{C}} \setminus S\left(1+\frac{\tau}{2}\right)\right) \setminus \mathcal{R}_g(\infty)$ and $i \leq N$ where N is the escaping time from the region $\left(\hat{\mathbb{C}} \setminus S\left(1+\frac{\tau}{2}\right)\right) \setminus \mathcal{R}_g(\infty)$. Then by Proposition 7.4,

$$|b_i - a_i| \leq \frac{M^i-1}{M-1}\,\varepsilon \tag{23}$$

where $M = \frac{2(k-1)^4}{k^3\delta^2}$. Suppose that $a_0 \in \left(\hat{\mathbb{C}} \setminus S\left(1+\frac{\tau}{2}\right)\right) \setminus \mathcal{R}_g(\infty)$ but $i > N$ for the last case. Then we combine the first and second case as follows

$$|b_i - a_i| \leq \left(\frac{M^N-1}{M-1} + \frac{1-K^{i-N}}{1-K}\right)\varepsilon \tag{24}$$

where K and M are the numbers used in the inequality (22) and (23). □

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