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On the Characteristic Polynomial of Power Graphs

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Abstract. The power graph $\mathcal{P}(G)$ of finite group G is a graph whose vertex set is G and two distinct vertices are adjacent if one is a power of the other. In this paper, we determine the characteristic polynomial of the power graphs of groups of order a product of three primes.

1. Introduction

There are many connections between graph spectra and other aspects of graphs such as diameter, automorphis group, hamiltonicity, etc. Artur Cayley was the first mathematician who defined a graph associated to a finite group. The concept of the directed power graph a semigroup is introduced by Kelarev and Quinnn see [9] as well as [10–12, 14]. The power graph $\mathcal{P}(G)$ of a finite group G is a simple graph where $V(\mathcal{P}(G)) = G$ and two vertices are adjacent if and only if one of them is a power of the other. A survey containing new results of this topic can be found in [1]. The complete power graphs considered first by Chakrabarty et al. in [4] and a formula for the number of edges in a power graph is proposed. Cameron and Ghosh in [2] showed that non-isomorphic finite groups may have isomorphic power graphs, but if G and H are two finite abelian groups, where $\mathcal{P}(G) \cong \mathcal{P}(H)$, then $G \cong H$. Many properties concerning with power graphs are studied in [3–5]. Let Γ be a finite undirected graph without loops or multiple edges with vertices $\{v_1, v_2, \cdots, v_n\}$. If two vertices v_i and v_j are joined by an edge, we say that v_i and v_j are adjacent and write $v_i \sim v_j$. The adjacency matrix A of graph Γ is a $n \times n$ square matrix $A(\Gamma) = a_{ij}$, where

$$a_{ij} = \begin{cases} 1 & if v_i \sim v_j \\ o & otherwise \end{cases}.$$

Hence, A is a symmetric real matrix with zero diagonal and the roots of the characteristic polynomial $P_G(x) = det(xI - A)$ are called the eigenvalues of Γ. The spectrum of Γ which consists of the n eigenvalues of Γ is denoted by $spec(\Gamma)$.

In the next section, we introduce some results that we use in this paper. Section three contains the main results of this paper. This section has two subsections. In the first subsection, we determine the characteristic polynomial of power graph of groups of order p^3, where p is a prime number, and in the second subsection we do this for groups of order a product of three distinct primes .

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2. Definitions and Preliminaries

Here, our notation is standard and mainly taken from [6]. For the sake of completeness, we mention here some results which are crucial throughout this paper.

Theorem 2.1. *[4] Let G be a finite group. Then $\mathcal{P}(G)$ is complete graph if and only if G is a cyclic group of order* 1 *or p^m, for some prime number p and $m \in \mathbf{N}$.*

Theorem 2.2. *[6] The characteristic polynomial of the disjoint union of two graphs Γ_1 and Γ_2 is*

$$P_{\Gamma_1 \cup \Gamma_2}(x) = P_{\Gamma_1}(x) P_{\Gamma_2}(x).$$

It follows that if $\Gamma_1, \Gamma_2, \ldots, \Gamma_s$ are the components of the graph Γ, then

$$P_{\Gamma}(x) = P_{\Gamma_1}(x) P_{\Gamma_2}(x) \ldots P_{\Gamma_s}(x).$$

Suppose $\mathcal{F} = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_s\}$ is a family of graphs. Define $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_s$ to be the join graph of $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_s\}$ with vertex set $V(\Gamma) = \cup_{i=1}^{s} V(\Gamma_i)$ and edge set

$$E(\Gamma) = \cup_{i=1}^{s} E(\Gamma_i) \cup \{(u,v) | u \in V(\Gamma_i), v \in V(\Gamma_j),\ (1 \le i, j \le s)\}.$$

Theorem 2.3. *[6] Let Γ_1, Γ_2 be two graphs on respectively n_1, n_2 vertices. The characteristic polynomial of the join graph $\Gamma_1 + \Gamma_2$ is*

$$\begin{aligned} P_{\Gamma_1+\Gamma_2}(x) &= (-1)^{n_2} P_{\Gamma_1}(x) P_{\bar{\Gamma}_2}(-x-1) + (-1)^{n_1} P_{\Gamma_2}(x) P_{\bar{\Gamma}_1}(-x-1) \\ &- (-1)^{n_1+n_2} P_{\bar{\Gamma}_1}(-x-1) P_{\bar{\Gamma}_2}(-x-1). \end{aligned}$$

Suppose μ_i are the distinct eigenvalues of Γ, then for fixed i, if eigenspaces $V(\mu_i)$ has an orthonormal basis $\{x_1, \ldots, x_d\}$ then $P_i = x_1 x_1^T + \cdots + x_d x_d^T$, otherwise P_i represents the orthogonal projection of $\mathbb{R}^n$ onto $V(\mu_i)$ with respect to the standard orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$. On the other hand, $\mathbf{j}$ is the all-1 vector in $\mathbb{R}^n$ and the numbers $\beta_i = \frac{\|P_i \mathbf{j}\|}{\sqrt{n}}$, $(i = 1, \ldots, m)$ are the main angles of graph Γ; they are the cosines of the angles between eigenspaces and $\mathbf{j}$, see [6]. Note that $\Sigma_{i=1}^{m} \beta_i^2 = 1$, because $\Sigma_{i=1}^{m} P_i \mathbf{j} = \mathbf{j}$. Then we have the following proposition.

Proposition 2.4. *[6] For a given graph Γ, we have*

$$P_{K_1+\Gamma}(x) = P_{\Gamma}(x)(x - \Sigma_{i=1}^{m} \frac{n\beta_i^2}{x - \mu_i}).$$

Theorem 2.5. *[13] The characteristic polynomial of the power graph of the cyclic group $\mathbb{Z}_n$ is*

$$P_{\mathcal{P}(\mathbb{Z}_n)}(x) = P_T(x)(x+1)^{n-t-1},$$

where d_i's $(1 \le i \le t)$, are all non-trivial divisors of n,

$$T = \begin{pmatrix} \varphi(n) & \varphi(d_1) & \varphi(d_2) & \ldots & \varphi(d_t) \\ \varphi(n)+1 & \varphi(d_1)-1 & \alpha_{d_1 d_2} & \ldots & \alpha_{d_1 d_t} \\ \varphi(n)+1 & \alpha_{d_2 d_1} & \varphi(d_2)-1 & \ldots & \alpha_{d_2 d_t} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varphi(n)+1 & \alpha_{d_t d_1} & \alpha_{d_t d_2} & \ldots & \varphi(d_t)-1 \end{pmatrix}$$

and

$$\alpha_{d_i d_j} = \begin{cases} \varphi(d_j) & d_i \mid d_j \text{ or } d_j \mid d_i \\ 0 & \text{otherwise} \end{cases}.$$

The coalescence graph $\Gamma_1.\Gamma_2$ of two graphs Γ_1 and Γ_2 obtained from disjoint union $\Gamma_1 \cup \Gamma_2$ by identifying a vertex u of Γ_1 with a vertex v of Γ_2. In [6] it is proved that

$$P_{\Gamma_1.\Gamma_2}(x) = P_{\Gamma_1}(x)P_{\Gamma_2-v}(x) + P_{\Gamma_1-u}(x)P_{\Gamma_2}(x) - xP_{\Gamma_1-u}(x)P_{\Gamma_2-v}(x).$$

Now, suppose Γ_1 and Γ_2 have respectively subgraphs K, K' where $K \cong K'$ and suppose $\Gamma_1(\Gamma_2)$ has a vertex $u(v)$ adjacent to all vertices of $K(K')$. We can define the generelized coalescence $\Gamma_1 \odot \Gamma_2$ of two graphs Γ_1, Γ_2 by identifying the vertices of subgraph K with the vertices of subgraph K'. Hence, one can see that the adjacency matrix $\Gamma_1 \odot \Gamma_2$ is

$$\begin{pmatrix} A' & C & 0 \\ C^T & 0 & D^T \\ 0 & D & B' \end{pmatrix},$$

where $\begin{pmatrix} A' & C \\ C^T & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & D^T \\ D & B' \end{pmatrix}$ are the adjacency matrices of the graphs Γ_1 and Γ_2, respectively. Also $C(D)$ is the adjacency matrix of subgraph $K(K')$. Now

$$P_{\Gamma_1\odot\Gamma_2}(x) = \begin{vmatrix} xI-A' & -C & 0 \\ -C^T & xI & -D^T \\ 0 & -D & xI-B' \end{vmatrix} = \begin{vmatrix} xI-A' & -C & 0 \\ -C^T & xI & -D^T \\ 0 & 0 & xI-B' \end{vmatrix} + \begin{vmatrix} xI-A' & 0 & 0 \\ -C^T & xI & -D^T \\ 0 & -D & xI-B' \end{vmatrix} - \begin{vmatrix} xI-A' & 0 & 0 \\ -C^T & xI & -D^T \\ 0 & 0 & xI-B' \end{vmatrix}.$$

Thus, we proved the following theorem.

Theorem 2.6. *The characteristic polynomial of generelized coalescence $\Gamma_1 \odot \Gamma_2$ is*

$$P_{\Gamma_1\odot\Gamma_2}(x) = P_{\Gamma_1}(x)P_{\Gamma_2-K}(x) + P_{\Gamma_1-K}(x)P_{\Gamma_2}(x) - P_K(x)P_{\Gamma_1-K}(x)P_{\Gamma_2-K}(x).$$

3. Main Results

Suppose $\mathcal{G}(p,q,r)$ is the class of all groups of order pqr, where p, q and r are three prime numbers. Hölder in [8] investigated the structure of a group in $\mathcal{G}(p,q,r)$. By this notation, in [7] it is shown that if $p = q = r$, then there are five groups of order p^3 as follows:

- $\mathbb{Z}_{p^3}$,
- $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$,
- $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$,
- $\mathbb{Z}_p \rtimes \mathbb{Z}_{p^2}$,
- $\mathbb{Z}_p \rtimes (\mathbb{Z}_p \times \mathbb{Z}_p)$.

If $p > q > r$, then all groups of order pqr are

- $\mathbb{Z}_{pqr}$,
- $\mathbb{Z}_r \times F_{p,q}$ $(q|p-1)$,
- $\mathbb{Z}_q \times F_{p,r}$ $(r|p-1)$,
- $\mathbb{Z}_p \times F_{q,r}$ $(r|q-1)$,
- $F_{p,qr}$ $(qr|p-1)$,
- $G_{i+5} \cong \langle a,b,c : a^p = b^q = c^r = 1, ab = ba, c^{-1}bc = b^u, c^{-1}ac = a^{v^i} \rangle$, where $r|p-1, q-1, o(u) = r$ in $\mathbb{Z}_q^*$ and $o(v) = r$ in $\mathbb{Z}_p^*$ $(1 \le i \le r-1)$.

3.1. The power graph of groups of order p^3

Here, we determine the power graphs of groups of order p^3. In what follows, we determine the characteristic polynomial of $\mathcal{P}(G)$. By using Theorem 2.1, $\mathcal{P}(\mathbb{Z}_{p^3})$ is a complete graph of order p^3.

Theorem 3.1. *Suppose $G \cong \mathbb{Z}_p \times \mathbb{Z}_{p^2} = \langle x, y : x^p = y^{p^2} = 1, xy = yx\rangle$. Then*

$$\mathcal{P}(G) \cong K_1 + (\cup_{i=1}^{p} K_{p-1} \cup (\cup_{i=1}^{p} K_{p^2-p} + K_{p-1})).$$

Proof. We have to partite the vertices of $\mathcal{P}(G)$ to the following subsets:

Subsets A,B. According to Theorem 2.1, the vertices correspond to the elements x^i $(1 \le i \le p-1)$ and y^j $(1 \le j \le p^2 - 1)$ form two cliques of orders $p-1$ and p^2-1, respectively.

Subset C. By Theorem 2.1, the elements correspond to vertices x^iy^j $(1 \le i \le p-1,\ 1 \le j \le p^2-1)$ are of order p^2. Since G is abelian, we can consider two following cases:

Case 1. If $j \ne kp$, then we have $p-1$ cliques of order p^2-p and $(x^iy^j)^{tp} = y^{jtp}$ $(1 \le t \le p-1)$ verify that these vertices are adjacent with vertices of subset B.

Case 2. If $j = kp$, then $(x^iy^{kp})^p = 1$ $(1 \le k \le p-1)$ implies we have $p-1$ cliques of order $p-1$. These new vertices are distinct from the other vertices.

The power graph $\mathcal{P}(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$ is depicted in Figure 1. A bold line between two subsets $\mathcal{X}$ and Y of vertices denotes the join operator, namely $\mathcal{X} + Y$, which means that all vertices of $\mathcal{X}$ are adjacent with all vertices of Y. □

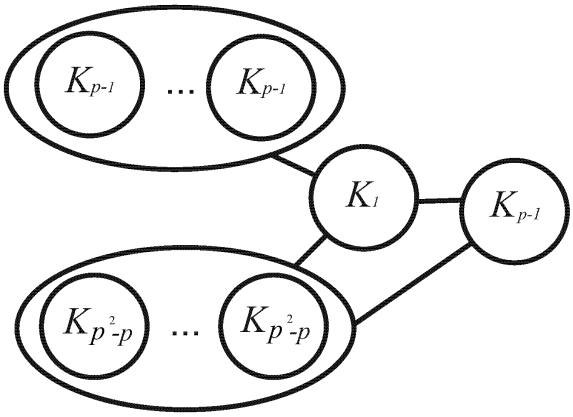

Figure 1: The power graph $\mathcal{P}(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$.

Corollary 3.2. *The characteristic polynomial of $\Gamma = \mathcal{P}(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$ is*

$$\begin{aligned} P_\Gamma(x) &= (x+1)^{p^3-2p-2}(x-p+2)^{p-1}(x-p^2+p+1)^{p-1}(x^4-(p^2+p-5)x^3 \\ &- (p^4-3p^3+6p^2+2p-9)x^2+(p^5-4p^4+8p^3-11p^2+7)x \\ &+ p^6-3p^5+3p^4+p^3-5p^2+p+2). \end{aligned}$$

Proof. By Theorem 2.3, it is sufficent to consider $\Gamma_1 \cong K_1$ and $\Gamma_2 \cong \cup_{i=1}^{p} K_{p-1} \cup (\cup_{i=1}^{p} K_{p^2-p} + K_{p-1})$ and then we have $P_{\Gamma_2}(x) = (x+1)^{p^3-2p-2}(x-(p-2))^p(x-(p^2-p-1))^{p-1}(x^2-(p^2-3)x-(p(p-1)^3+p^2-2))$ and $\bar{\Gamma}_2 \cong K_{p-1,\dots,p-1} + (K_{p^2-p,\dots,p^2-p} \cup \bar{K}_{p-1})$. Thus

$$P_{\bar{\Gamma}_2}(x) \quad = \quad x^{p^3-2p-2}(x+p-1)^{p-1}(x+p(p-1))^{p-1}(x^3-(p^2-1)(p-1)x^2-2p^2(p-1)^2x+p^2(p-1)^4).$$

This completes the proof. □

Suppose $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, then by using the structure of G and Theorem 2.1, the elements correspond to a^i, b^j and c^k $(1 \le i, j, k \le p-1)$ form three cliques of order $p-1$, respectively. For elements a^ib^j, since $o(a^ib^j) = p$, we can derive $p-1$ cliques of order $p-1$. Assume these elements are adjacent with a^i's, then we can see that there exist an integer $1 \le m \le p-1$ such that $(a^ib^j)^m = a^{i'}$ and so $p \mid jm$, a contradiction. By a similar way, we can conclude these vertices are not adjacent with the other vertices. Knowing the elements a^ic^k $(1 \le i, k \le p-1)$ are of order p, we achieve $p-1$ new cliques of order $p-1$. In continuing, the elements b^jc^k $(1 \le j, k \le p-1)$ verify a similar result with the last cases. Finally, for the elements $a^ib^jc^k$ $(1 \le i, j, k \le p-1)$, we know $o(a^ib^jc^k) = p$ which introduce $(p-1)^2$ cliques of order $p-1$. By a similar argument, these vertices are not adjacent with the others. The related graph is depicted in Figure 2. We summarize the above results in the following theorem.

Theorem 3.3. *Let $G = \langle a, b, c : a^p = b^p = c^p = 1, ab = ba, ac = ca, bc = cb\rangle$. Then $\mathcal{P}(G) \cong K_1 + \cup_{i=1}^{p^2+p+1} K_{p-1}$.*

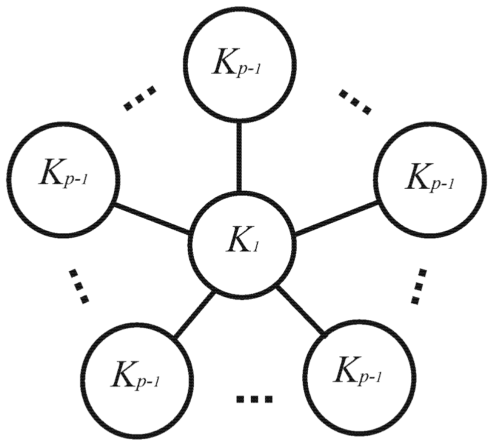

Figure 2: The power graph $\mathcal{P}(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$.

Corollary 3.4. *The characteristic polynomial of $\Gamma = \mathcal{P}(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$ is*

$$P_\Gamma(x) = (x+1)^{(p-2)(p^2+p+1)}(x-(p-2))^{p^2+p}(x^2-(p-2)x-((p-2)(p^2+2)+2)).$$

Proof. Apply Theorem 2.3 to calculate the characteristic polynomial of Γ as follows. Suppose $\Gamma_1 \cong K_1$ and $\Gamma_2 \cong \cup_{i=1}^{p^2+p+1} K_{p-1}$, so we have

$$P_{\Gamma_2}(x) = ((x-(p-2))(x+1)^{p-2})^{p^2+p+1}$$

and $\bar{\Gamma}_2 \cong K_{p-1,\dots,p-1}$. Hence, $P_{\bar{\Gamma}_2}(x) = x^{(p-2)(p^2+p+1)}(x+p-1)^{p(p+1)}(x-p(p-1)^2)$ which completes the proof. □

Theorem 3.5. *Suppose $G \cong \langle x, y : x^{p^2} = y^p = 1, y^{-1}xy = x^{p+1}\rangle$. Then $\mathcal{P}(G) \cong \mathcal{P}(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$ for $p \neq 2$. If $p = 2$, then $\mathcal{P}(G) \cong K_1 + (K_3 \cup (\cup_{i=1}^{4} K_1))$.*

Proof. By the presentation of the group G, we can divide the elements of G to the following three subsets:

First, the vertices correspond to the elements x^i's $(1 \leq i \leq p^2 - 1)$ and y^j's $(1 \leq j \leq p - 1)$ form two cliques of order $p^2 - 1$ and $p - 1$, respectively. We now apply the relation $y^{-1}xy = x^{p+1}$ to conclude that $(y^jx^i)^m = y^{jm}x^{i((p+1)^{j(m-1)}+\cdots+(p+1)^j+1)}$. Let us, distinguish two cases:

Case 1. Assume $p \neq 2$ and $i \neq kp$, then $o(y^jx^i) = p^2$, therefore there are $p-1$ cliques of order $p^2 - p$. The relation $(y^jx^i)^p = x^{kp}$ implies that the correspond vertices are adjacent with x^i's. On the other hand, if $i = kp$, then $o(y^jx^{kp}) = p$ that yields $p - 1$ cliques of order $p - 1$.

Case 2. Suppose $p = 2$, thus $o(y^jx^i) = p$ which means that there are $p + 1$ cliques of order $p - 1$. This graph is depicted in Figure 3. □

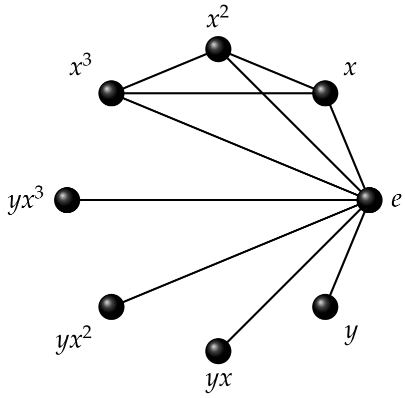

Figure 3: The power graph $\mathcal{P}(\mathbb{Z}_2 \rtimes \mathbb{Z}_4)$.

Corollary 3.6. *The characteristic polynomial of* $\Gamma = \mathcal{P}(\mathbb{Z}_2 \rtimes \mathbb{Z}_4)$ *is*

$$P_\Gamma(x) = x^3(x+1)^2(x^3 - 2x^2 - 7x + 8).$$

Proof. Suppose $\Gamma_1 \cong K_1$ and $\Gamma_2 \cong K_3 \cup (\cup_{i=1}^4 K_1)$, then $P_{\Gamma_2}(x) = (x-2)(x+1)^2x^4$ and $\bar{\Gamma}_2 \cong \cup_{i=1}^3 K_1 + K_4$. So, $P_{\bar{\Gamma}_2}(x) = x^2(x+1)^3(x^2 - 3x - 12)$. This yeilds that $P_{\Gamma_1+\Gamma_2}(x) = x^3(x+1)^2(x^3 - 2x^2 - 7x + 8)$ and we obtain our required. □

Theorem 3.7. *Suppose*

$$G \cong \langle x, y, z : x^p = y^p = z^p = 1, xy = yx, zy = yz, xz = zxy\rangle.$$

Then $\mathcal{P}(G) \cong \mathcal{P}(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p)$ *if* $p \neq 2$ *and* $\mathcal{P}(G) \cong \mathcal{P}(\mathbb{Z}_2 \rtimes \mathbb{Z}_4)$*, if* $p = 2$.

Proof. At first note that each elements of group G is correspond to a vertex of $\mathcal{P}(G)$. If $p \neq 2$, then the elements of group G can be divided to seven subsets as follows:

Subsets A-C. The elements x^i's, y^j's and z^k's $(1 \leq i, j, k \leq p - 1)$ which form three cliques of order $p - 1$.

Subset D,E. Consider the elements x^iy^j' $(1 \leq i, j \leq p - 1)$ and y^jz^k's $(1 \leq j, k \leq p - 1)$ that are of order p. These vertices construct $2(p - 1)$ new disjoint cliques of order $p - 1$.

Subset F. The elements z^kx^i's $(1 \leq i, k \leq p - 1)$ are those lying in $(p - 1)^2$ new cliques of order $p - 1$ and are distinct from the other vertices.

Subset H. The elements $z^kx^iy^j$'s $(1 \leq i, j, k \leq p - 1)$, which are of order p, introduce $p - 1$ cliques of order $p - 1$. By a similar argument, these vertices are distinct from the other vertices. Finally, assume $p = 2$, then $\mathcal{P}(G)$ is as depicted in Figure 4. □

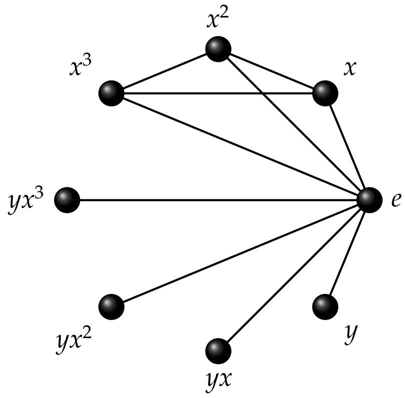

Figure 4: The power graph $\mathcal{P}(\mathbb{Z}_2 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2))$.

3.2. *The power graph of groups of order pqr*

In this section, we apply similar methods as as given in the last section to determine the structure and the characteristic polynomial of $\mathcal{P}(G)$, where $|G| = pqr$ and $p > q > r$ are three prime numbers. Suppose $\Gamma_1, \ldots, \Gamma_n$ are n connected graphs. The graph $C_n[\Gamma_1, \ldots, \Gamma_n]$ is a graph constructed by $\cup_{i=1}^{n}\Gamma_i$ in which every vertex of Γ_i is adjacent with every vertex of Γ_{i+1} $(mod\ n)$ for $1 \leq i \leq n$.

Theorem 3.8. *Let $G \cong \mathbb{Z}_{pqr} = \langle a \rangle$. Then the power graph $\mathcal{P}(G)$ is isomorphic with*

$$K_{\varphi(pqr)+1} + C_6[K_{pq-1}, K_{p-1}, K_{pr-1}, K_{r-1}, K_{qr-1}, K_{q-1}].$$

Proof. We can consider the elements of G as seven subsets which is reported in Table 1. By using the definition of $C_n[\Gamma_1, \ldots, \Gamma_n]$ the proof is complete.

Table 1. The structure of power graph of Group $\mathbb{Z}_{pqr}$.

Elements	# Elements	Adjacent with
a^t $(t, pqr) = 1$	$\varphi(pqr)$	All vertices
a^{ir} $(1 \leq i \leq pq-1)$	$pq - p - q + 1$	a^{iqr} $(1 \leq i \leq p-1)$, a^{jpr} $(1 \leq j \leq q-1)$
a^{jq} $(1 \leq j \leq pr-1)$	$pr - p - r + 1$	a^{iqr} $(1 \leq i \leq p-1)$, a^{kpq} $(1 \leq k \leq r-1)$
a^{kp} $(1 \leq k \leq qr-1)$	$qr - q - r + 1$	a^{kpq} $(1 \leq k \leq r-1)$, a^{jpr} $(1 \leq j \leq q-1)$
a^{iqr} $(1 \leq i \leq p-1)$	$p - 1$	a^{ir} $(1 \leq i \leq pq-1)$, a^{jq} $(1 \leq j \leq pr-1)$
a^{jpr} $(1 \leq j \leq q-1)$	$q - 1$	a^{ir} $(1 \leq i \leq pq-1)$, a^{kp} $(1 \leq k \leq qr-1)$
a^{kpq} $(1 \leq k \leq r-1)$	$r - 1$	a^{kp} $(1 \leq k \leq qr-1)$, a^{jq} $(1 \leq j \leq pr-1)$

□

Corollary 3.9. *The characteristic polynomial of $\mathcal{P}(\mathbb{Z}_{pqr})$ is*

$$P_\Gamma(x) = P_T(x)(x+1)^{pqr-7},$$

where

$$T = \begin{pmatrix} \alpha\beta\gamma & \gamma & \beta & \alpha & \beta\gamma & \alpha\gamma & \alpha\beta \\ \alpha\beta\gamma+1 & \gamma-1 & 0 & 0 & \beta\gamma & \alpha\gamma & 0 \\ \alpha\beta\gamma+1 & 0 & \beta-1 & 0 & \beta\gamma & 0 & \alpha\beta \\ \alpha\beta\gamma+1 & 0 & 0 & \alpha-1 & 0 & \alpha\gamma & \alpha\beta \\ \alpha\beta\gamma+1 & \gamma & \beta & 0 & \beta\gamma-1 & 0 & 0 \\ \alpha\beta\gamma+1 & \gamma & 0 & \alpha & 0 & \alpha\gamma-1 & 0 \\ \alpha\beta\gamma+1 & 0 & \beta & \alpha & 0 & 0 & \alpha\beta-1 \end{pmatrix}$$

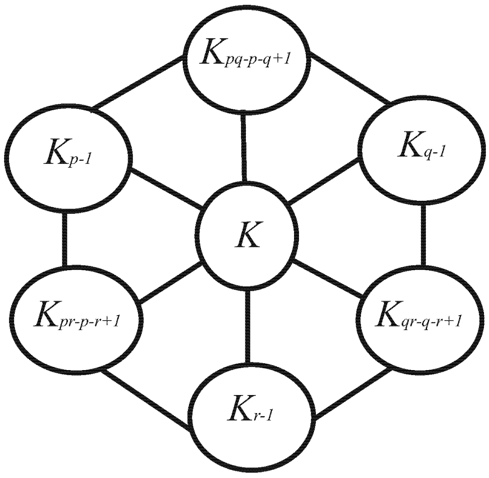

Figure 5: The power graph $\mathcal{P}(\mathbb{Z}_{pqr})$.

$\alpha = p-1, \beta = q-1$ *and* $\gamma = r-1$.

Proof. Use Theorem 2.4. $\square$

Theorem 3.10. *Suposse* $G \cong \langle a,b,c : a^p = b^q = c^r = 1, ac = ca, bc = cb, b^{-1}ab = a^u\rangle$, *where* $u^q \equiv 1$ $(mod\ p)$. *Then* $\mathcal{P}(G) \cong K_1 + (\Gamma_1 \odot \Gamma_2)$, *where* $\Gamma_1 = \Gamma_1' \odot \Gamma_2' \odot \cdots \odot \Gamma_p'$, $\Gamma_i' \cong K_{qr-q-r+1} + (K_{q-1} \cup K_{r-1}))$ *and* $\Gamma_2 \cong K_{pr-p-r+1} + (K_{p-1} \cup K_{r-1})$.

Proof. Consider the vertices of $\mathcal{P}(G)$ as following seven subsets: The vertices correspond to the elements a^is $(1 \le i \le p-1)$, b^j's $(1 \le j \le q-1)$ and c^k's $(1 \le k \le r-1)$ compose three cliques of order respectively, $p-1$, $q-1$ and $r-1$. For elements b^ja^i's $(1 \le i \le p-1,\ 1 \le j \le q-1)$, by using the relations $b^{-1}ab = a^u$ and $(b^ja^i)^m = b^{jm}a^{i(u^{j(m-1)}+\cdots+u^j+1)}$, we obtain $o(b^ja^i) = q$ which yields $p-1$ cliques of order $q-1$. Also these vertices are distinct from the other vertices. Consider now the elements a^ic^k's $(1 \le i \le p-1,\ 1 \le k \le r-1)$. The relation $ac = ca$, yields $o(a^ic^k) = pr$ and then we achieve a clique of order $pr-p-r+1$. On the other hand, we know $(a^ic^k)^p = a^{ip}c^{kp} = c^{kp}$ and $(a^ic^k)^r = a^{ir}c^{kr} = a^{ir}$ that imply these vertices are adjacent with a^i's and c^ks. By the structure of group G, the elements b^jc^k's $(1 \le j \le q-1, 1 \le k \le r-1)$ form a clique of order $qr-q-r+1$ and two relations $(b^jc^k)^r = b^{jr}c^{kr} = b^{jr}$ and $(b^jc^k)^q = b^{jq}c^{kq} = c^{kq}$ verify that these vertices are adjacent with element b^j's and c^k's. The elements $c^kb^ja^i$'s $(1 \le i \le p-1,\ 1 \le j \le q-1,\ 1 \le k \le r-1)$ are of order qr and by using an induction we get that

$$(c^kb^ja^i)^m = c^{km}b^{jm}a^{i(u^{j(m-1)}+\cdots+u^j+1)}.$$

Thus, we have $p-1$ new cliques of order $qr-q-r+1$. Also the relations $(c^kb^ja^i)^q = c^{kq}$ and $(c^kb^ja^i)^r = (b^ja^i)^r$ yield either these vertices are adjacent only with b^ja^i's and c^k's, or one of the following cases hold:

Case 1. There exists an integer $1 \le m \le qr-1$ such that $(c^kb^ja^i)^m = a^{i'}$ and then

$$c^{km}b^{jm}a^{i(u^{j(m-1)}+\cdots+u^j+1)} = a^{i'}$$

which we can conclude $q \mid jm, r \mid km$, a contradiction.

Case 2. There exists an integer $1 \le m \le qr-1$ such that $(c^kb^ja^i)^m = b^{j'}$ and then

$$c^{km}b^{jm}a^{i(u^{j(m-1)}+\cdots+u^j+1)} = b^{j'}.$$

This means $r \mid m,\ p \mid u^{j(m-1)} + \cdots + u^j + 1$ and therefore $q \mid jm$, a contradiction.

Case 3. There exists an integer $1 \le m \le qr-1$ such that $(c^kb^ja^i)^m = c^{k'}a^{i'}$ and then

$$c^{km}b^{jm}a^{i(u^{j(m-1)}+\cdots+u^j+1)} = c^{k'}a^{i'}$$

which yields $q \mid jm$, a contradiction.

Case 4. There exists an integer $1 \leq m \leq qr - 1$ such that $(c^k b^j a^i)^m = c^{k'} b^{j'}$ and then

$$c^{km} b^{jm} a^{i(u^{j(m-1)} + \cdots + u^j + 1)} = c^{k'} b^{j'}$$

which implies that $p \mid i(u^{j(m-1)} + \cdots + u^j + 1)$, a contradiction. The power graph $\mathcal{P}(G)$ is depicted in Figure 6. □

Corollary 3.11. *The characteristic polynomial of* $\Gamma_1 \odot \Gamma_2$ *is*

$$P_{\Gamma_1 \odot \Gamma_2}(x) = (x+1)^{p(qr-1)-4}(x - (qr - r - 1))^{p-1}(x^4 - ax^3 - bx^2 + cx - d)$$

where $a = r(p+q-1)-5$, $b = p(q-1)(r-1)^2 + (p+1)(r+1) - 7 - (qr-r-1)(pr-4)$, $c = (qr-r-1)(pr+p+r-6) + ((p-1)^2 + p(pr-r+q-3)(q-1))(r-1)^2 - p - r + 3$ *and* $d = p(pr-r-1)(q-2)(q-1)(r-1)^2 + (qr-r-1)((p-1)^2(r-1)^2 - p - r + 3)$.

Proof. First assume $\Gamma = \Gamma'_1 \odot \Gamma'_2 \odot \cdots \odot \Gamma'_p$ then by using Theorem 2.5. we have

$$P_\Gamma(x) = pP_{\Gamma'_i}(x)(P_{(\Gamma'_i - K_{r-1})}(x))^{p-1} - (p-1)P_{K_{r-1}}(x)(P_{(\Gamma'_i - K_{r-1})}(x))^p.$$

We know

$$P_{\Gamma'_i}(x) = (x+1)^{qr-4}(x^3 - (qr-4)x^2 - ((q+1)(r+1) - 7)x + (q-1)^2(r-1)^2 - q - r + 3)$$

and $\Gamma'_i - K_{r-1} = K_{qr-r}$. Hence

$$\begin{aligned} P_\Gamma(x) &= (x+1)^{p(qr-r-1)+r-3}(x - (qr - r - 1))^{p-1}(x^3 - (qr-4)x^2 \\ &- ((p-1)(q-1)(r-1)^2 + (q+1)(r+1) - 7)x \\ &+ p(q-2)(q-1)(r-1)^2 + (r-2)(qr - r - 1)). \end{aligned}$$

On the other hand, we can see that

$$\begin{aligned} P_{\Gamma_2}(x) &= (x+1)^{pr-4}(x^3 - (pr-4)x^2 - ((p+1)(r+1) - 7)x \\ &+ (p-1)(r-1)(pr - p - r) + (p-2)(r-2)), \end{aligned}$$

and $\Gamma_2 - K_{r-1} = K_{qr-r}$, $(\Gamma'_1 \odot \Gamma'_2 \odot \cdots \odot \Gamma'_p) - K_{r-1} = \cup_{i=1}^p K_{qr-r}$ which completes the proof. □

The characteristic polynomial of $K_1 + (\Gamma_1 \odot \Gamma_2)$ follows immediately from the Proposition 2.1 and Corollary 2.5.

Apply Theorem 2.1 to determine the structure of power graphs of groups $G \cong \mathbb{Z}_q \times F_{p,r}(r|p-1)$ and $G \cong \mathbb{Z}_p \times F_{q,r}(r|q-1)$ as given in Theorem 3.5.

Theorem 3.12. *Let* $G \cong F_{p,qr}$ $(qr|p-1)$. *Then one of the following occurs:*

i) If $r = 3$ *or* $q = 3$, *then* $\mathcal{P}(G) \cong K_1 + (\Gamma_1 \odot \Gamma_2)$, *where* $\Gamma_1 \cong \cup_{i=1}^p K_{qr-r} + K_{r-1}$ *and* $\Gamma_2 \cong K_{pr-p-r+1} + (K_{p-1} \cup K_{r-1})$.

ii) If $r, q \neq 3$, *then* $\mathcal{P}(G) \cong K_1 + (\cup_{i=1}^p K_{qr-1} \cup K_{p-1})$.

Proof. *i)* In this case we can partite the vertices of $\mathcal{P}(G)$ to three subsets as follows: The vertices corresponding to the elements x^i's $(1 \leq i \leq p-1)$, y^j's $(1 \leq j \leq qr-1)$ form two cliques of orders $p-1$ and $qr-1$, respectively. Now, consider the element $y^j x^i$ $(1 \leq j \leq qr-1,\ 1 \leq i \leq p-1)$ in which $(y^j x^i)^m = y^{jm} x^{i(u^{j(m-1)} + \cdots + u^j + 1)}$. We distinguish two cases:

Case 1. Assume $j \neq kq$, then $o(y^j x^i) = qr$ which implies there are $p-1$ cliques of order $qr - r$. We can obtain $(y^j x^i)^q = y^{jq}$ that yields these vertices are adjacent with the elements y^j's.

Case 2. If $j = kq$, then $o(y^j x^i) = pr$, $(y^j x^i)^r = x^{ir}$ and $(y^j x^i)^p = y^{tq}$ $(1 \leq t \leq r-1)$. Therefore, we have a clique of order $pr - p - r + 1$ whose vertices are adjacent with the elements x^i's and y^j's $(j = kq)$.

ii) In this case, by a similar discussion as given in the last case, we have $o(y^j x^i) = qr$ where $(1 \leq j \leq qr-1)$ and these vertices form $p-1$ cliques of order $qr-1$. On the other hand, the structure of the group G indicates that these vertices are distinct from the other vertices. The power graph $\mathcal{P}(G)$ is depicted in Figures 7, 8, respectively. □

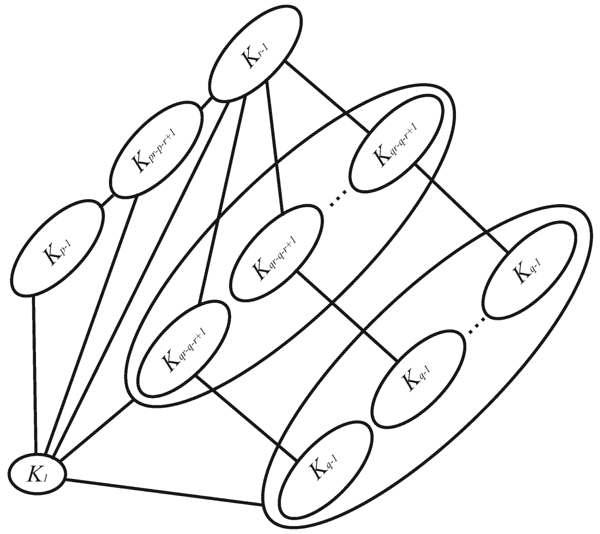

Figure 6: The power graph $\mathcal{P}(\mathbb{Z}_r \times F_{p,q})$.

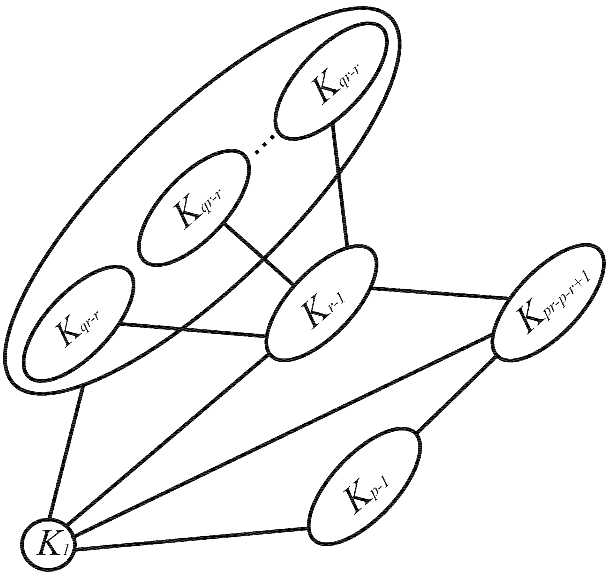

Figure 7: The power graph $\mathcal{P}(F_{p,qr})$ $r = 3$ *or* $q = 3$.

Corollary 3.13. *Suppose* $r = 3$ *or* $q = 3$*, then the characteristic polynomial of graph* $\mathcal{P}(F_{p,qr})$ *is*

$$
\begin{aligned}
P_{\mathcal{P}(F_{p,qr})} &= (x+1)^{pqr-p-4}(x-(qr-r-1))^{p-1}(x^5-(r(p+q-1)-7)x^4\\
&+ (r(q-1)(p-6)+pr(q-3)+13)x^3+((p-1)(r-1)(pr-p-r)\\
&+ (q-1)(r-1)(pr(pr-r-1)-3)\\
&+ r(q-1)((p-1)(5-pr)+r-5)-pr-qr+3q+10)x^2\\
&+ ((p-1)r(q-1)(pr(r-3)-(p-1)(r-1)+3)\\
&+ (p+r-3)((2r-1)(q-1)-2)\\
&- (p-1)^2(r-1)^2(qr-q-2)+(q-1)(r-1)(pr(pr-r-2)-3))x\\
&+ (p-1)^2(r-1)^2(r(p-1)(q-1)+1)-(p-1)r(q-1)(pr-1)\\
&+ pr(q-1)(r-1)(pr-r-1)+(qr-r-1)(p+r-3)\\
&+ (p-1)(r-1)+1).
\end{aligned}
$$

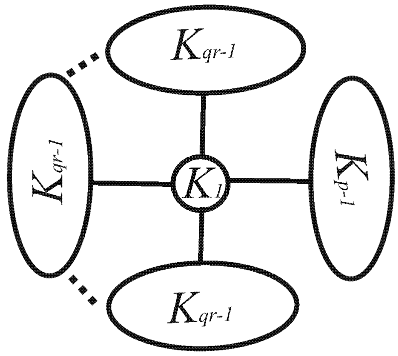

Figure 8: The power graph $\mathcal{P}(F_{p,qr})$.

In addition, if $r, q \neq 3$, *then*

$$\begin{aligned} P_{\mathcal{P}(F_{p,qr})}(x) &= (x+1)^{pqr-p-2}(x-(qr-2))^{p-1}(x^3-(qr+p-4)x^2 \\ &- (2p+2qr-5)x+p^2(qr-1)-qr(p+1)+2). \end{aligned}$$

Proof. If $r = 3$ or $q = 3$, then we apply Theorem 2.3 to conclude that

$$\begin{aligned} P_{\Gamma_1}(x) &= (x+1)^{p(qr-r-1)+r-2}(x-(qr-r-1))^{p-1}(x^2-(qr-3)x \\ &- r(q-1)(p-1)(r-1)-qr+2) \end{aligned}$$

and

$$\begin{aligned} P_{\Gamma_2}(x) &= (x+1)^{pr-4}(x^3-(pr-4)x^2-((p+1)(r+1)-7)x \\ &+ (p-1)(r-1)(pr-p-r)+(p-2)(r-2)). \end{aligned}$$

On the other hand,

$$P_{\Gamma_1-K_{r-1}}(x) = (x+1)^{p(qr-r-1)}(x-(qr-r-1))^p$$

and

$$P_{\Gamma_2-K_{r-1}}(x) = (x+1)^{pr-r-1}(x-(pr-r-1)).$$

Now from Theorem 2.5, we can see that

$$\begin{aligned} P_{\Gamma_1\odot\Gamma_2}(x) &= (x+1)^{p(qr-1)-4}(x-(qr-r-1))^{p-1}(x^4-(r(p+q-1)-5)x^3 \\ &+ (pr(q-2)-4(qr-r-1))x^2+((p-1)^2(r-1)^2+(q-1)(r-1) \\ &\times (pr(pr-r-1)-3)+(p+r-3)(qr-q-r))x+pr(q-1)(r-1) \\ &\times (pr-r-1)+(qr-r-1)((p+r-3)-(p-1)^2(r-1)^2)). \end{aligned}$$

In continuing, we put $\Gamma = \Gamma_1 \odot \Gamma_2$, therefore $\bar{\Gamma} = \Gamma_1' \odot \Gamma_2'$, where $\Gamma_1' \cong K_{qr-r,\ldots,qr-r} + (\bar{K}_{pr-p-r+1} \cup \bar{K}_{p-1})$ and $\Gamma_2' \cong K_{p-1,r-1}$. Hence, by using Theorem 2.3, we can see

$$P_{\Gamma_1'}(x) = x^{pqr-p-r-1}(x+qr-r)^{p-1}(x^2-r(p-1)(q-1)x-pr^2(p-1)(q-1))$$

and

$$P_{\Gamma_1'-\bar{K}_{p-1}}(x) = x^{pqr-2p-r}(x+qr-r)^{p-1}(x^2-r(p-1)(q-1)x-pr(p-1)(r-1)(q-1)).$$

Also, $P_{\Gamma_2'}(x) = x^{p+r-4}(x^2-(p-1)(r-1))$ and $P_{\Gamma_2'-\bar{K}_{p-1}}(x) = x^{r-1}$. We now substituate these results in Theorem 2.5 to get

$$\begin{aligned} P_{\bar{\Gamma}}(x) &= x^{pqr-p-4}(x+qr-r)^{p-1}(x^4-r(p-1)(q-1)x^3+(p-1)(-pr^2q+pr^2+r-1)x^2 \\ &+ r(p-1)^2(r-1)(q-1)x+pr(p-1)^2(r-1)^2(q-1)). \end{aligned}$$

Finally, suppose that $q, r \neq 3$ and $\Gamma = \cup_{i=1}^{p} K_{qr-1} \cup K_{p-1}$, then we have

$$P_{\Gamma}(x) = (x+1)^{pqr-p-2}(x-(qr-2))^p(x-(p-2)),$$

$$P_{\bar{\Gamma}}(x) = x^{pqr-p-2}(x+qr-1)^{p-1}(x^2-(p-1)(qr-1)(x+p)).$$

□

Theorem 3.14. *Let $G \cong G_{i+5} \cong \langle a,b,c : a^p = b^q = c^r = 1, ab = ba, c^{-1}bc = b^u, c^{-1}ac = a^{v^i}\rangle$, where $r|p-1, q-1$, $o(u) = r$ in $\mathbb{Z}_q^*$ and $o(v) = r$ in $\mathbb{Z}_p^*$ $(1 \leq i \leq r-1)$. Then $\mathcal{P}(G) \cong K_1 + (\cup_{i=1}^{pq} K_{r-1} \cup (K_{pq-p-q+1} + (K_{p-1} \cup K_{q-1})))$.*

Proof. We cliam that $\mathcal{P}(G)$ has $pq+3$ cliques. First, we prove that different cliques have at most one vertex in common. We can partite the vertices of $\mathcal{P}(G)$ to seven subsets:

Subsets A-C. The vertices correspond to the elements a^t's $(1 \leq t \leq p-1)$, b^j's $(1 \leq j \leq q-1)$ and c^k's $(1 \leq k \leq r-1)$ introduce three cliques of orders $p-1$, $q-1$ and $r-1$ respectively.

Subset D. We can see that the vertex a^tb^j $(1 \leq t \leq p-1,\ 1 \leq j \leq q-1)$ is of order pq which yields a clique of order $pq-p-q+1$. Apply the relations $(a^tb^j)^p = b^{jp}$ and $(a^tb^j)^q = a^{tq}$ to determine their adjacency with the vertices of Subsets 1,2.

Subset E. Consider the vertex c^ka^t $(1 \leq i \leq p-1,\ 1 \leq k \leq r-1)$. The relation $c^{-1}ac = a^{v^i}$, where $v^r \equiv 1\ (mod\ p)$ yields $o(c^ka^t) = r$. Hence, we achieve $p-1$ cliques of order $r-1$ that are distinct from the other vertices.

Subset F. We know the vertices c^kb^j's $(1 \leq j \leq q-1,\ 1 \leq k \leq r-1)$ are of order r and form $q-1$ cliques of order $r-1$.

Subset H. Finally, by the presentation of the group G, the vertices $c^kb^ja^t$'s $(1 \leq t \leq p-1,\ 1 \leq j \leq q-1,\ 1 \leq k \leq r-1)$ form $(p-1)(q-1)$ cliques of order $r-1$. The power graph $\mathcal{P}(G)$ is dipected in Figure 9. □

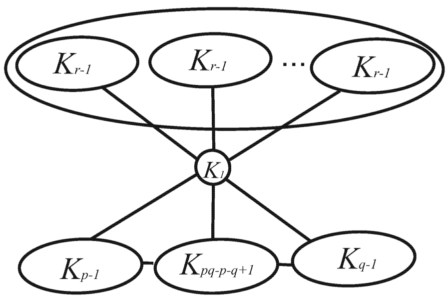

Figure 9: The power graph $\mathcal{P}(G_{i+5})$.

Corollary 3.15. *For $1 \leq i \leq r-1$, the characteristic polynomial of $\mathcal{P}(G_{i+5})$ is*

$$\begin{aligned} P_{\mathcal{P}(G_{i+5})}(x) &= (x+1)^{pq(r-1)-4}(x-r+2)^{pq-1}(x^5-(pq+r-6)x^4 \\ &- (3pq+p+q+4r-15)x^3+((p-1)^2(q-1)^2+(p-2)(q-2) \\ &+ (r-1)pq(pq-3)+(r-3)((p+1)(q+1)-8)-(pq-8))x^2 \\ &+ ((r-1)((pq-1)^2-3(p-1)(q-1)+pq(p+q)+4) \\ &- (r-3)(p-1)^2(q-1)^2+(p-5)(q-5)-10r-5)x \\ &+ (p-1)^2(q-1)^2((r-1)(pq-1)+1)+(r-1)(-pq(p+q-3)+1) \\ &- (p-2)(q-2)(r-2)-1. \end{aligned}$$

Proof. Assume $\Gamma_1 = \cup_{i=1}^{pq} K_{r-1} \cup (K_{pq-p-q+1} + (K_{p-1} \cup K_{q-1}))$, then by applying Theorem 2.3, we have:

$$\begin{aligned} P_{\Gamma_2}(x) &= (x+1)^{pq-4}(x^3-(pq-4)x^2-((p+1)(q+1)-7)x \\ &+ (p-1)(q-1)(pq-p-q)+(p-2)(q-2)), \end{aligned}$$

where $\Gamma_2 = K_{pq-p-q+1} + (K_{p-1} \cup K_{q-1})$ and

$$\begin{aligned} P_{\Gamma_1}(x) &= (x+1)^{pq(r-1)-4}(x-(r-2))^{pq}(x^3-(pq-4)x^2-((p+1)(q+1)-7)x \\ &+ (p-1)(q-1)(pq-p-q)+(p-2)(q-2)). \end{aligned}$$

In continuing, suppose $\bar{\Gamma}_1 = K_{r-1,\ldots,r-1} + (\bar{K}_{pq-p-q+1} \cup K_{p-1,q-1})$, then by using Theorem 2.3, we have

$$\begin{aligned} P_{\bar{\Gamma}_1}(x) &= x^{pq(r-1)-4}(x+r-1)^{pq-1}(x^4-(pq-1)(r-1)x^3-(pq(pq-1)(r-1) \\ &+ (p-1)(q-1))x^2-(p-1)(q-1)(r-1)(pq+1)x \\ &- pq(r-1)((p-1)^2(q-1)^2-2(p+q-3))). \end{aligned}$$

□

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