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Ricci Tensors on Trans-Sasakian 3-manifolds

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Abstract. In this paper, it is proved that a trans-Sasakian 3-manifold is locally symmetric if and only if it is locally isometric to the sphere space $\mathbb{S}^3(c^2)$, the hyperbolic space $\mathbb{H}^3(-c^2)$, the Euclidean space $\mathbb{R}^3$, the product space $\mathbb{R}\times\mathbb{S}^2(c^2)$ or $\mathbb{R}\times\mathbb{H}^2(-c^2)$, where c is a nonzero constant. Some examples are constructed to illustrate main results. We also give some new conditions for a compact trans-Sasakian 3-manifold to be proper.

1. Introduction

In geometry of almost contact manifolds, trans-Sasakian manifolds are an important class of almost contact metric manifolds because they include Sasakian, Kenmotsu and cosymplectic manifolds as their special cases. A connected trans-Sasakian manifold of type (α,β) of dimension greater than three satisfies either $\alpha=0$ or $\beta=0$ ([19]). For the later case, we even have that α is a constant. In view of this, an interesting question has been a topic for recent ten years, namely on what conditions a trans-Sasakian 3-manifold is proper? Where by a proper trans-Sasakian 3-manifold we mean that it satisfies either $\alpha=0$ or $\beta=0$.

In order to give answers of the above question, some authors studied trans-Sasakian 3-manifolds from various points of view (see De et al. [6–11] and Deshmukh et al. [12–14]). However, in these papers, a very important geometric condition, namely local symmetry, was neglected. In this paper, we study this condition on trans-Sasakian 3-manifolds for the first time and obtain a complete classification result (see Theorem 3.12). Consequently, one observes easily that under the local symmetry condition a trans-Sasakian manifold must be proper. Some examples illustrating our main results are also constructed. From these examples, we see that local symmetry can not be replaced by certain weaker conditions.

Before closing the paper, we consider infinitesimal harmonic transformation on trans-Sasakian 3-manifolds. We also show that on a compact trans-Sasakian 3-manifold satisfying the above condition, the manifold is proper.

2. Trans-Sasakian Manifolds

An almost contact metric structure (see [3]) defined on a smooth manifold M of dimension $2n+1$ is a (ϕ,ξ,η,g)-structure satisfying

$$\begin{aligned}&\phi^2=-\,\mathrm{id}+\eta\otimes\xi,\ \ \eta(\xi)=1,\\ &\quad\phi^*g=g-\eta\otimes\eta,\end{aligned}\tag{1}$$

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where ϕ is a $(1,1)$-type tensor field, ξ is a vector field called the characteristic or the Reeb vector field and η is a 1-form called the almost contact 1-form. A Riemannian manifold M furnished with an almost contact metric structure is said to be an *almost contact metric manifold*, denoted by (M, ϕ, ξ, η, g).

Let M be an almost contact metric manifold of dimension $2n+1$. On the product $M \times \mathbb{R}$ there exists an almost complex structure J defined by

$$J\left(X, f\frac{\mathrm{d}}{\mathrm{d}t}\right) = \left(\phi X - f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}\right),$$

where X denotes a vector field tangent to M^{2n+1}, t is the coordinate of $\mathbb{R}$ and f is a C^∞-function on $M^{2n+1} \times \mathbb{R}$. An almost contact metric manifold is said to be *normal* if the above almost complex structure J is integrable and this is equivalent to $[\phi, \phi] = -2\mathrm{d}\eta \otimes \xi$, where $[\phi, \phi]$ denotes the Nijenhuis tensor of ϕ.

A normal almost contact metric manifold is called a *trans-Sasakian manifold* (see [19]) if it satisfies $\mathrm{d}\eta = \alpha\Phi$ and $\mathrm{d}\Phi = 2\beta\eta \wedge \Phi$, where $\alpha = \frac{1}{2n}\mathrm{tr}(\phi\nabla\xi)$, $\beta = \frac{1}{2n}\mathrm{div}\xi$ and $\Phi(\cdot,\cdot) = g(\cdot, \phi\cdot)$. It is known that (see [4]) an almost contact metric manifold M is trans-Sasakian if and only if there exist two smooth functions α and β satisfying

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2}$$

for any vector fields X and Y. In view of (2), a trans-Sasakian manifold is denoted by $(M, \phi, \xi, \eta, \alpha, \beta)$ and called a trans-Sasakian manifold of type (α, β).

A normal almost contact metric manifold is called an *α-Sasakian manifold* if it satisfies $\mathrm{d}\eta = \alpha\Phi$ and $\mathrm{d}\Phi = 0$, where α is a nonzero constant (see [17]). An α-Sasakian manifold becomes a Sasakian manifold when $\alpha = 1$.

A normal almost contact metric manifold is called a *β-Kenmotsu manifold* if it satisfies $\mathrm{d}\eta = 0$ and $\mathrm{d}\Phi = 2\beta\eta \wedge \Phi$, where β is a nonzero constant (see [17]). A β-Kenmotsu manifold becomes a Kenmotsu manifold when $\beta = 1$.

A normal almost contact metric manifold is called a *cosymplectic manifold* if it satisfies $\mathrm{d}\eta = 0$ and $\mathrm{d}\Phi = 0$.

From the definition of tran-Sasakian manifolds, substituting $Y = \xi$ in (2) and using (1) we have

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi) \tag{3}$$

for any vector field X.

In this paper, all manifolds are assumed to be connected and smooth.

3. Ricci Tensors on Trans-Sasakian 3-manifolds

A trans-Sasakian manifold of type (α, β) is said to be of C_6-class if $\beta = 0$ (see [5]). As seen in [19], α on a trans-Sasakian manifold of C_6-class of dimension greater than three is a constant. Then, a trans-Sasakian manifold of C_6-class of dimension greater than 3 is just an α-Sasakian manifold (see [17]). However, α on a trans-Sasakian 3-manifold of C_6-class is not necessarily a constant (see [19]).

A trans-Sasakian manifold of type (α, β) is said to be of C_5-class if $\alpha = 0$ (see [5]). On such manifolds of dimension greater than three there holds naturally $\mathrm{d}\beta \wedge \eta = 0$. However, the above equation does not necessarily hold for dimension three. The set of all trans-Sasakian manifolds of C_5-class contains the set of all β-Kenmotsu manifolds as its proper subset. Some non-trivial examples are given in Section 3.

In the present paper, a trans-Sasakian 3-manifold is said to be proper if it is of either C_5 or C_6-class. Throughout the paper, we denote by ∇f the gradient of a function f.

Before stating our main results, we collect some useful lemmas as follows.

Lemma 3.1 ([10]). *On a trans-Sasakian 3-manifold of type (α, β) we have*

$$\xi(\alpha) + 2\alpha\beta = 0. \tag{4}$$

Lemma 3.2 ([10]). *On a trans-Sasakian* 3*-manifold of type* (α, β)*, the Ricci operator is given by*

$$\begin{aligned} Q = &\left(\frac{r}{2} + \xi(\beta) - \alpha^2 + \beta^2\right)\mathrm{id} - \left(\frac{r}{2} + \xi(\beta) - 3\alpha^2 + 3\beta^2\right)\eta \otimes \xi \\ &+ \eta \otimes (\phi(\nabla\alpha) - \nabla\beta) + g(\phi(\nabla\alpha) - \nabla\beta, \cdot) \otimes \xi. \end{aligned} \tag{5}$$

On an n-dimensional Riemannian manifold (M, g), the rough Laplacian operator $\bar{\Delta}$ acting on a smooth vector field X is defined by (see [16])

$$\bar{\Delta}X = \sum_{i=1}^{n}(\nabla_{\nabla_{e_i}e_i}X - \nabla_{e_i}\nabla_{e_i}X),$$

where $\{e_i : i = 1 \cdots n\}$ is a local orthonormal frame for the tangent space at each point of the manifold. This operator is self adjoint elliptic operator. Using the above symbol, from [13, Lemma 2.3] we have

Lemma 3.3 ([13]). *On a trans-Sasakian* 3*-manifold of type* (α, β) *we have*

$$\bar{\Delta}\xi = \phi(\nabla\alpha) - \nabla\beta + (2(\alpha^2 + \beta^2) + \xi(\beta))\xi. \tag{6}$$

First, we discuss the constancy of scalar curvatures of α-Sasakian, β-Kenmotsu and cosymplectic 3-manifolds, respectively.

Proposition 3.4. *A* β*-Kenmotsu* 3*-manifold is of constant scalar curvature if and only if it is locally isometric to the hyperbolic space* $\mathbb{H}^3(-\beta^2)$.

Proof. For a β-Kenmotsu 3-manifold M we have from Lemma 3.2 that

$$QX = \left(\frac{r}{2} + \beta^2\right)X - \left(\frac{r}{2} + 3\beta^3\right)\eta(X)\xi$$

for any vector field X. Taking the covariant derivative of the above relation we have

$$(\nabla_Y Q)X = -\left(\frac{r}{2} + 3\beta^2\right)g(\nabla_Y\xi, X)\xi - \left(\frac{r}{2} + 3\beta^2\right)\eta(X)\nabla_Y\xi$$

for any vector fields X, Y. Applying the well known formula $\mathrm{div}Q = \frac{1}{2}\nabla r$ on the above equation we have

$$X(r) = -\left(r + 6\beta^2\right)\eta(X)\mathrm{div}\xi.$$

From (3) we have $\mathrm{div}\xi = 2\beta \in \mathbb{R}^*$. When the scalar curvature of M is a constant, using $\mathrm{div}\xi = 2\beta$ and considering $X = \xi$ on the above equation we have $r = -6\beta^2$. Thus, it is clear that M is Einstein, i.e. $Q = -2\beta^2\mathrm{id}$. Since M is of dimension three, then it is of constant sectional curvature $-\beta^2$. The converse is easy to check. This completes the proof. □

From proof of Proposition 3.4, we have

Corollary 3.5. *On a* β*-Kenmotsu* 3*-manifold, the scalar curvature is invariant along the distribution* $\{\xi\}^\perp$.

Cosymplectic 3-manifolds with constant scalar curvatures were classified by the first present author in [25].

Proposition 3.6 ([25]). *A cosymplectic* 3*-manifold is of constant scalar curvature if and only if it is locally isometric to the Euclidean space* $\mathbb{R}^3$*, the product space* $\mathbb{R} \times \mathbb{S}^2(c^2)$ *or* $\mathbb{R} \times \mathbb{H}^2(-c^2)$*, where c is a nonzero constant.*

In fact, from proof of [25, Proposition 4.2] we have

Remark 3.7. *On any cosymplectic* 3*-manifold, the scalar curvature is invariant along the Reeb vector field* ξ*.*

Given an almost contact metric manifold (M, ϕ, ξ, η, g). Let X be a vector field on M orthogonal to ξ. The plane section $\{X, \phi X\}$ spanned by X and ϕX is called a ϕ-section and $\frac{g(R(X,\phi X)\phi X, X)}{g(X,X)^2}$ is called its ϕ-sectional curvature.

A cosymplectic manifold M^{2n+1} with constant ϕ-sectional curvature c is said to be a *cosymplectic space from*, denoted by $M^{2n+1}(c)$, whose curvature tensor is given (see [1]) as the following

$$\begin{aligned} R(X,Y)Z =& \frac{c}{4}(g(\phi Y, \phi Z)X - g(\phi X, \phi Z)Y + \eta(Y)g(X,Z)\xi \\ & - \eta(X)g(Y,Z)\xi + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z) \end{aligned} \tag{7}$$

for any vector fields X, Y, Z. From Proposition 3.6, one can check that the ϕ-sectional curvature of a cosymplectic 3-manifold of constant scalar curvature is constant. Thus, we have

Proposition 3.8. *A cosymplectic* 3*-manifold is of constant scalar curvature if and only if it is a cosymplectic space form.*

An almost contact metric manifold (M, ϕ, ξ, η, g) is called *locally ϕ-symmetric* (see [28]) if there holds

$$\phi^2(\nabla_W R)(X,Y)Z = 0 \tag{8}$$

for any vector fields X, Y, Z, W orthogonal to the Reeb vector field ξ. Obviously, any locally symmetric space is also locally ϕ-symmetric but the converse is not necessarily true. In this paper, a locally ϕ-symmetric space is said to be *strictly locally ϕ-symmetric* if it is not locally symmetric.

Proposition 3.9. *An α-Sasakian* 3*-manifold is of constant scalar curvature if and only if it is locally ϕ-symmetric.*

Proof. For an α-Sasakian 3-manifold M, from [10, Corollary 4.2], we know that the curvature tensor of M is given by

$$\begin{aligned} R(X,Y)Z =& \left(\frac{r}{2} - 2\alpha^2\right)[g(Y,Z)X - g(X,Z)Y] \\ & - \left(\frac{r}{2} - 3\alpha^2\right)\eta(X)g(Y,Z)\xi + \left(\frac{r}{2} - 3\alpha^2\right)\eta(Y)g(X,Z)\xi \\ & - \left(\frac{r}{2} - 3\alpha^2\right)\eta(Y)\eta(Z)X + \left(\frac{r}{2} - 3\alpha^2\right)\eta(X)\eta(Z)Y \end{aligned} \tag{9}$$

for any vector fields X, Y, Z. From the above relation and (1) we have

$$\begin{aligned} (\nabla_W R)(X,Y)Z =& \frac{1}{2}W(r)(g(Y,Z) - \eta(Y)\eta(Z))X + \frac{1}{2}W(r)(\eta(X)\eta(Z) - g(X,Z))Y \\ & + \frac{1}{2}W(r)(g(X,Z)\eta(Y) - g(Y,Z)\eta(X))\xi \\ & - \left(\frac{r}{2} - 3\alpha^2\right)((\nabla_W \eta)X\,\xi + \eta(X)\nabla_W \xi)g(Y,Z) \\ & + \left(\frac{r}{2} - 3\alpha^2\right)((\nabla_W \eta)Y\,\xi + \eta(Y)\nabla_W \xi)g(X,Z) \\ & - \left(\frac{r}{2} - 3\alpha^2\right)(\eta(Z)(\nabla_W \eta)Y + \eta(Y)(\nabla_W \eta)Z)X \\ & + \left(\frac{r}{2} - 3\alpha^2\right)(\eta(Z)(\nabla_W \eta)X + \eta(X)(\nabla_W \eta)Z)Y \end{aligned} \tag{10}$$

for any vector fields X, Y, Z, W. Because of (1), it follows directly that

$$\phi^2(\nabla_W R)(X,Y)Z = -\frac{1}{2}W(r)(g(Y,Z)X - g(X,Z)Y) \tag{11}$$

for any vector fields X, Y, Z, W orthogonal to ξ. If M is locally ϕ-symmetric, from (8) and (11), we observe that the scalar curvature r is invariant along the distribution $\{\xi\}^\perp$, or equivalently, $\mathrm{d}r = \xi(r)\eta$. Taking the exterior differentiation of this relation we have

$$\mathrm{d}\xi(r) \wedge \eta + \xi(r)\Phi = 0. \tag{12}$$

Let e be a unit vector field orthogonal to ξ. The action of (12) on $(e, \phi e)$ gives $\xi(r) = 0$. This implies that r is a constant. The converse follows directly from (11). This completes the proof. □

By a similar proof as that of Corollary 3.5, we have

Corollary 3.10. *On an α-Sasakian 3-manifold, the scalar curvature is invariant along the Reeb vector field ξ.*

Unlike Kenmotsu case, the constancy of the scalar curvature of an α-Sasakian 3-manifold does not necessarily imply that the manifold is of constant sectional curvature.

An α-Sasakian manifold having constant ϕ-sectional curvature c is said to be an *α-Sasakian space form* and is denoted by $M^{2n+1}(\alpha, c)$. It becomes the well known Sasakian space form when $\alpha = 1$. The curvature tensor of α-Sasakian space form is given by (see [1])

$$\begin{aligned} R(X, Y)Z =& \frac{c + 3\alpha^2}{4}(g(Y, Z)X - g(X, Z)Y) \\ &+ \frac{c - \alpha^2}{4}(g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z) \\ &+ \frac{c - \alpha^2}{4}(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \eta(Y)g(X, Z)\xi - \eta(X)g(Y, Z)\xi) \end{aligned} \tag{13}$$

for any vector fields X, Y, Z, where c and α are both constants. One can check that $M(\alpha, c)$ is an α-Sasakian manifold with constant scalar curvature whose sectional curvature is not a constant unless $c = \alpha^2$.

From (9) and (13), we have the following result which is similar as Proposition 3.8.

Proposition 3.11. *An α-Sasakian 3-manifold is of constant scalar curvature if and only if it is an α-Sasakian space from.*

Now are ready to give our main results which generalize Propositions 3.4, 3.8 and 3.11.

Theorem 3.12. *A trans-Sasakian 3-manifold is locally symmetric if and only if it is locally isometric to the sphere space $\mathbb{S}^3(c^2)$, the hyperbolic space $\mathbb{H}^3(-c^2)$, the Euclidean space $\mathbb{R}^3$, the product space $\mathbb{R} \times \mathbb{S}^2(c^2)$ or $\mathbb{R} \times \mathbb{H}^2(-c^2)$, where c is a nonzero constant.*

Proof. Let M be a trans-Sasakian 3-manifold. Taking the covariant derivative of (5) we have

$$\begin{aligned} (\nabla_Y Q)X =& Y\left(\frac{r}{2} + \xi(\beta) - \alpha^2 + \beta^2\right)X - Y\left(\frac{r}{2} + \xi(\beta) - 3\alpha^2 + 3\beta^2\right)\eta(X)\xi \\ &- \left(\frac{r}{2} + \xi(\beta) - 3\alpha^2 + 3\beta^2\right)(g(\nabla_Y\xi, X)\xi + \eta(X)\nabla_Y\xi) \\ &+ g(\nabla_Y\xi, X)(\phi\nabla\alpha - \nabla\beta) + \eta(X)(\nabla_Y(\phi\nabla\alpha) - \nabla_Y\nabla\beta) \\ &+ g(\nabla_Y(\phi\nabla\alpha) - \nabla_Y\nabla\beta, X)\xi + g(\phi\nabla\alpha - \nabla\beta, X)\nabla_Y\xi \end{aligned} \tag{14}$$

for any vector fields X, Y, Z.

Let e be an arbitrary unit vector field orthogonal to the Reeb vector field ξ on M. Then, $\{\xi, e, \phi e\}$ forms a local orthonormal frame for tangent space at each point of M. Note that the scalar curvature of a locally symmetric space is always a constant. Also, a Reimannian 3-manifold is locally symmetric if and only if its Ricci tensor is parallel. From now on, we assume that M is locally symmetric.

Putting $X = Y = e$ in (14) we see that $(\nabla_e Q)e = 0$ if and only if

$$\begin{aligned} e(\xi(\beta) - \alpha^2 + \beta^2) - 2\beta(\phi e(\alpha) + e(\beta)) &= 0, \\ \beta(e(\alpha) - \phi e(\beta)) + \alpha(\phi e(\alpha) + e(\beta)) &= 0, \\ g(\nabla_e(\phi\nabla\alpha - \nabla\beta), e) - \beta(\frac{r}{2} + 2\xi(\beta) - 3\alpha^2 + 3\beta^2) &= 0. \end{aligned} \tag{15}$$

Similarly, putting $X = Y = \phi e$ in (14), $(\nabla_{\phi e} Q)\phi e = 0$ if and only if

$$\begin{aligned} \phi e(\xi(\beta) - \alpha^2 + \beta^2) + 2\beta(e(\alpha) - \phi e(\beta)) &= 0, \\ -\beta(\phi e(\alpha) + e(\beta)) + \alpha(e(\alpha) - \phi e(\beta)) &= 0, \\ g(\nabla_{\phi e}(\phi\nabla\alpha - \nabla\beta), \phi e) - \beta(\frac{r}{2} + 2\xi(\beta) - 3\alpha^2 + 3\beta^2) &= 0. \end{aligned} \tag{16}$$

Similarly, putting $X = e$ and $Y = \phi e$ in (14), $(\nabla_e Q)\phi e = 0$ if and only if

$$\begin{aligned} \phi e(\xi(\beta) - \alpha^2 + \beta^2) - 2\alpha(\phi e(\alpha) + e(\beta)) &= 0, \\ -\beta(\phi e(\alpha) + e(\beta)) + \alpha(e(\alpha) - \phi e(\beta)) &= 0, \\ g(\nabla_{\phi e}(\phi\nabla\alpha - \nabla\beta), e) - \alpha(\frac{r}{2} + 2\xi(\beta) - 3\alpha^2 + 3\beta^2) &= 0. \end{aligned} \tag{17}$$

Similarly, putting $X = \phi e$ and $Y = e$ in (14), $(\nabla_{\phi e} Q)e = 0$ if and only if

$$\begin{aligned} e(\xi(\beta) - \alpha^2 + \beta^2) - 2\alpha(e(\alpha) - \phi e(\beta)) &= 0, \\ \beta(e(\alpha) - \phi e(\beta)) + \alpha(\phi e(\alpha) + e(\beta)) &= 0, \\ g(\nabla_e(\phi\nabla\alpha - \nabla\beta), \phi e) + \alpha(\frac{r}{2} + 2\xi(\beta) - 3\alpha^2 + 3\beta^2) &= 0. \end{aligned} \tag{18}$$

Combining the second terms of (17) and (18) we get

$$(\alpha^2 + \beta^2)(e(\alpha) - \phi e(\beta)) = (\alpha^2 + \beta^2)(\phi e(\alpha) + e(\beta)) = 0. \tag{19}$$

In view of (19), we consider the fist possible case, namely $\alpha^2 + \beta^2 = 0$. Obviously, this implies that M is a cosymplectic manifold. Since the scalar curvature is a constant, then from Proposition 3.6 we know M is locally isometric to the Euclidean space $\mathbb{R}^3$, the product $\mathbb{R} \times \mathbb{S}^2(c^2)$ or $\mathbb{R} \times \mathbb{H}^2(-c^2)$.

Otherwise, when $\alpha^2 + \beta^2 \neq 0$, from (19) we have

$$e(\alpha) - \phi e(\beta) = \phi e(\alpha) + e(\beta) = 0.$$

The second equality of the above relation is equivalent to $g(e, \phi\nabla\alpha - \nabla\beta) = 0$. In view of e an arbitrary unit vector field orthogonal to ξ, the above relation is also equivalent to $\phi\nabla\alpha - \nabla\beta = \eta(\phi\nabla\alpha - \nabla\beta)\xi$, or equivalently,

$$\phi\nabla\alpha - \nabla\beta + \xi(\beta)\xi = 0. \tag{20}$$

The Levi-Civita connection ∇ of M can be written as the following (see [15]):

$$\begin{aligned} \nabla_\xi \xi &= 0,\ \nabla_\xi e = \lambda\phi e,\ \nabla_\xi \phi e = -\lambda e, \\ \nabla_e \xi &= \beta e - \alpha\phi e,\ \nabla_e e = -\beta\xi + \gamma\phi e,\ \nabla_e \phi e = \alpha\xi - \gamma e, \\ \nabla_{\phi e} \xi &= \alpha e + \beta\phi e,\ \nabla_{\phi e} e = -\alpha\xi - \delta\phi e,\ \nabla_{\phi e} = -\beta\xi + \delta e, \end{aligned}$$

where λ, γ and δ are smooth functions on some open subset of the manifold. Therefore, making use of the previous relations and substituting (20) into the last term of (15) or (16) we obtain

$$\beta(r - 6\alpha^2 + 6\beta^2 + 6\xi(\beta)) = 0. \tag{21}$$

If $\beta = 0$, from (20) we obtain that $\nabla\alpha = \xi(\alpha)\xi$. In view of Lemma 3.1 we obtain $\nabla\alpha = 0$ and hence α is a nonzero constant. Because the scalar curvature is a constant, from Proposition 3.11, M is an α-Sasakian space form. From (13) we have

$$Q = \frac{1}{4}(7\alpha^2 + c)\mathrm{id} + \frac{1}{4}(\alpha^2 - c)\eta \otimes \xi.$$

By (3), one can check that the Ricci tensor of M is parallel if and only if $c = \alpha^2$ and in this case M is of constant sectional curvature $\alpha^2 > 0$.

Finally, from (21) we next consider the last case, namely $\beta \neq 0$ and hence $r - 6\alpha^2 + 6\beta^2 + 6\xi(\beta) = 0$. Substituting this and (20) into (5) we observe that M is Einstein, i.e., $Q = \frac{r}{3}\mathrm{id}$. Since M is of dimension three, then M is of constant sectional curvature. In particular, when $\alpha = 0$ and β is a nonzero constant, by Proposition 3.4 we see that M is locally isometric to the hyperbolic space $\mathbb{H}^3(-\beta^2)$. The converse is easy to check. This competes the proof. □

Example 3.13. *Let x, y, z be the standard coordinates in $\mathbb{R}^3$. On $\mathbb{R}^3$ there exists Sasakian structure (ϕ, ξ, η, g) (see [3]) given as the following:*

$$\xi = 2\frac{\partial}{\partial z},\ \eta = \frac{1}{2}(dz - ydx),\ g = \eta \otimes \eta + \frac{1}{4}(dx \otimes dx + dy \otimes dy),$$

$$\phi\frac{\partial}{\partial x} = -\frac{\partial}{\partial y},\ \phi\frac{\partial}{\partial y} = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z},\ \phi\frac{\partial}{\partial z} = 0.$$

One can check that $(\mathbb{R}^3, \phi, \xi, \eta, g)$ is a Sasakian space form with constant ϕ-sectional curvature -3.

By Proposition 3.4, local symmetry and constancy of the scalar curvature are equivalent for β-Kenmotsu 3-manifolds. However, from proof of Theorem 3.12, Examples 3.13, 3.17 and relation (13) we have

Remark 3.14. *If local symmetry assumption is replaced by a weaker condition, namely the scalar curvature is a constant, then conclusion of Theorem 3.12 is not necessarily true.*

Example 3.15. *Let x, y, z be the standard coordinates in $\mathbb{R}^3$. Let $M := \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$. On M we consider three linearly independent vector fields e_1, e_2 and e_3 defined as the following*

$$e_1 = z^2\frac{\partial}{\partial x},\ e_2 = z^2\frac{\partial}{\partial y},\ e_3 = \frac{\partial}{\partial z}.$$

Let g be a Riemannian metric such that $\{e_1, e_2, e_3\}$ is a orthonormal frame. On M we defined an almost contact metric structure $\{\phi, \xi, \eta\}$ as the following

$$\phi e_1 = e_2,\ \phi e_2 = -e_1,\ \phi e_3 = 0,$$

$$\xi := e_3,\ \eta = g(e_3, \cdot).$$

One can check that M is a trans-Sasakian 3-manifold of type $(0, -\frac{2}{z})$ (see [29]). Moreover, the Levi-Civita connection of M is given by

$$\nabla_{e_1}e_1 = \frac{2}{z}e_3,\ \nabla_{e_1}e_2 = 0,\ \nabla_{e_1}e_3 = -\frac{2}{z}e_1,$$

$$\nabla_{e_2}e_1 = 0,\ \nabla_{e_2}e_2 = \frac{2}{z}e_3,\ \nabla_{e_2}e_3 = -\frac{2}{z}e_2,$$

$$\nabla_{e_3}e_1 = 0,\ \nabla_{e_3}e_2 = 0,\ \nabla_{e_3}e_3 = 0.$$

By a direction calculation, the Ricci operator Q is given as the following

$$Qe_1 = -\frac{10}{z^2}e_1,\ Qe_2 = -\frac{10}{z^2}e_2,\ Qe_3 = -\frac{12}{z^2}e_3.$$

Applying the above relations, we have

$$(\nabla_{e_1}Q)e_2 = (\nabla_{e_2}Q)e_1 = 0,$$
$$(\nabla_{e_1}Q)e_1 = (\nabla_{e_2}Q)e_2 = \frac{4}{z^3}e_3.$$

An almost contact metric manifold (M,ϕ,ξ,η,g) is said to be *Ricci η-parallel* if its Ricci tensor satisfies $(\nabla_{\phi X}S)(\phi Y,\phi Z) = 0$ for any vector fields X, Y, Z. Obviously, Ricci parallelism implies the above condition but the converse is not necessarily true.

Note that an almost contact metric 3-manifold is normal if and only if it is trans-Sasakian (see [22]). It was proved in [11] that a trans-Sasakian 3-manifold satisfies $dr = \xi(r)\eta$ if it is locally ϕ-symmetric or Ricci η-parallel. Thus, from Example 3.15 and relation (8), by a direct calculation, we have

Remark 3.16. *If local symmetry assumption is replaced by some weaker conditions such as Ricci η-parallelism or local ϕ-symmetry, then conclusion of Theorem 3.12 not necessarily holds.*

Let N be a Kähler manifold of dimension 2 and t the coordinate of $\mathbb{R}$. It was proved in [1, pp.179] that the warped product $\mathbb{R}\times_f N$ admits a trans-Sasakian structure of type $(0,\frac{f'}{f})$, where $f = f(t)$ is the positive warping function defined on the base $\mathbb{R}$ with coordinate t. Let σ be the natural projection from $\mathbb{R}\times_f N$ onto the fiber N. Moreover, from [21, Corollary 43] we know that the scalar curvature r of the warped product $\mathbb{R}\times_f N$ is given by

$$r = \frac{r^N}{f^2} - 4\frac{\Delta f}{f} - 2\frac{\|\nabla f\|^2}{f^2}, \tag{22}$$

where r^N, Δf and ∇f denote the pullback of the scalar curvature of the fiber N by projection σ, the Laplacian and the gradient of the warping function f, respectively.

Applying the above statement and (22), we can construct a trans-Sasakian 3-manifolds of type $(0,\beta)$ with constant scalar curvature.

Example 3.17. *Let N be the complex projective space $\mathbb{C}P^1$ of dimension two with standard Kähler structure whose holomorphic sectional curvature is equal to 1. Let t be the global coordinate of $\mathbb{R}$. A simple calculation gives that $r^N = 2$, $\Delta f = 0$ and $\nabla f = \frac{\partial}{\partial t}$. Then, the warped product $\mathbb{R}\times_t N$ is a trans-Sasakian 3-manifold of type $(0,\frac{1}{t})$ with constant scalar curvature equal to zero.*

It was proved in [2, Proposition 4.2] that a warped product manifold $B\times_f F$ is locally symmetric if and only if B is locally symmetric and F is of constant curvature K such that $\nabla_X\left(\frac{\mathrm{Hess}_f}{f}\right) = 0$ for any vector field X on B and $\kappa f^2 + \|\mathrm{grad} f\|^2 = K$ for certain constant κ. Applying such proposition, we know that the warped product in Example 3.17 can not be locally symmetric. Therefore, we have

Remark 3.18. *Example 3.17 shows that conclusions of Propositions 3.4 and 3.6 are not necessarily true for trans-Sasakian 3-manifolds of type $(0,\beta)$.*

Remark 3.19. *As far as we know, Example 3.17 is the first trans-Sasakian 3-manifold having constant scalar curvature which is of neither the type $(\alpha,0)$ with $\alpha\in\mathbb{R}^*$ nor the type $(0,\beta)$ with $\beta\in\mathbb{R}^*$.*

Before closing this paper, next we give a new condition under which a compact trans-Sasakian 3-manifold is proper.

A vector field V on a Riemannian manifold (M,g) is called an *infinitesimal harmonic transformation* if the local 1-parameter group of infinitesimal point transformations generated by V forms a group of harmonic transformations (see [20, 24]), or equivalently, $\mathrm{trace}_g(L_V\nabla) = 0$, where L denotes the Lie derivative.

It has been proved in [24, Theorem 2.1] that a vector field V on M generates an infinitesimal harmonic transformation if and only if $\Delta V = 2QV$, where Δ is known as the Laplacian and is determined by the

Weitzenböck formula $\Delta V = \nabla^*\nabla V + QV$, and ∇^* is the formal adjoint of ∇ and Q denotes the usual Ricci operator. The rough Laplacian $\bar{\Delta}$ defined in page 3 can be expressed as $\bar{\Delta}V = -\mathrm{trace}\nabla^2 V = \nabla^*\nabla V$, and therefore $\Delta V = \bar{\Delta}V + QV$. Thus, we say that V generates an infinitesimal harmonic transformation if and only if

$$\bar{\Delta}V = QV.$$

Theorem 3.20. *If the Reeb vector field of a compact trans-Sasakian 3-manifold of type (α, β) generates an infinitesimal harmonic transformation, then β vanishes.*

Proof. From Lemmas 3.2, we obtain immediately the following relation

$$Q\xi = \phi(\nabla\alpha) - \nabla\beta + (2(\alpha^2 - \beta^2) - \xi(\beta))\xi. \tag{23}$$

Obviously, from (23) and Lemma 3.3, the Reeb vector field ξ generates an infinitesimal harmonic transformation if and only if

$$2\beta^2 + \xi(\beta) = 0. \tag{24}$$

Applying (3), we compute the divergence of $\beta^3\xi$ as the following:

$$\mathrm{div}\beta^3\xi = 3\beta^2\xi(\beta) + \beta^3\mathrm{div}\xi = -4\beta^4,$$

where we have used (24). Since the manifold is assumed to compact, applying the divergence theorem on the above relation we obtain $\beta = 0$. This completes the proof. □

Note that α on a trans-Sasakian 3-manifold of type $(\alpha, 0)$ is not necessarily a constant even if the manifold is compact (for such example we refer the reader to [26, 27]).

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