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Multiple Positive Solutions of Integral Boundary Value Problem for a Class of Nonlinear Fractional-order Differential Coupling System with Eigenvalue Argument and (p_1, p_2)-Laplacian

Kaihong Zhao[a]

[a]*Department of Applied Mathematics, Kunming University of Science and Technology, Yunnan, Kunming, 650093, P. R. China*

Abstract. This paper is concerned with the integral boundary value problem for a class of nonlinear fractional order differential coupling system with eigenvalue argument and (p_1, p_2)-Laplacian. Some sufficient criteria have been established to guarantee the existence and multiplicity of positive solution by the fixed point index theorem in cones. Meanwhile, we obtain the range of eigenvalue parameter. As an application, one example is also provided to illustrate the validity of our main results.

1. Introduction

In this paper, we mainly study the integral boundary value problem for a class of nonlinear fractional order differential coupling system with eigenvalue argument and (p_1, p_2)-Laplacian as follows:

$$\begin{cases} D_{0+}^{\beta_1}\left((\phi_{p_1}\left(D_{0+}^{\alpha_1}u_1(t)\right)\right) = \lambda_1 f_1\Big(t, u_1(t), u_2(t), u_1'(t), u_2'(t)\Big), & t \in J, \\ D_{0+}^{\beta_2}\left((\phi_{p_2}\left(D_{0+}^{\alpha_2}u_2(t)\right)\right) = \lambda_2 f_2\Big(t, u_1(t), u_2(t), u_1'(t), u_2'(t)\Big), & t \in J, \end{cases} \tag{1}$$

with the boundary value conditions

$$\begin{cases} a_1u_1(0) - b_1u_1'(0) = \int_0^1 \varphi_1(t)u_1(t)dt, \quad u_1'(1) = u_1''(0) = 0, \\ a_2u_2(0) - b_2u_2'(0) = \int_0^1 \varphi_2(t)u_2(t)dt, \quad u_2'(1) = u_2''(0) = 0, \\ \phi_{p_1}\left(D_{0+}^{\alpha_1}u_1(1)\right) = \int_0^1 \phi_{p_1}\left(D_{0+}^{\alpha_1}u_1(s)\right)dA_1(s), \quad \left(\phi_{p_1}\left(D_{0+}^{\alpha_1}u_1(0)\right)\right)' = 0, \\ \phi_{p_2}\left(D_{0+}^{\alpha_2}u_2(1)\right) = \int_0^1 \phi_{p_2}\left(D_{0+}^{\alpha_2}u_2(s)\right)dA_2(s), \quad \left(\phi_{p_2}\left(D_{0+}^{\alpha_2}u_2(0)\right)\right)' = 0, \end{cases} \tag{2}$$

where $J = [0, 1]$, $1 < \beta_i \le 2 < \alpha_i \le 3$, $\lambda_i > 0$, $a_i \ge 0$ and $b_i \ge 0 (i = 1, 2)$. $D_{0+}^{\alpha_i}$ and $D_{0+}^{\beta_i}(i = 1, 2)$ are the standard Caputo fractional derivative. $\phi_{p_i}(x) = |x|^{p_i-2}x(i = 1, 2)$ is p_i-Laplacian. We know that $\phi_{q_i} = \phi_{p_i}^{-1}$, $p_i > 1$ and

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Email address: zhaokaihongs@126.com (Kaihong Zhao)

$\frac{1}{p_i}+\frac{1}{q_i}=1$. $A_i\in C(J,\mathbb{R}_+)$ are nondecreasing functions of bounded variation, and the integrals in (2) are Riemann-Stieltjes integrals. $\varphi_i\in L^1(J,\mathbb{R}_+^0)(i=1,2)$, $f_i\in C(J\times(\mathbb{R}_+^0)^4,\mathbb{R}_+)(i=1,2)$, $\mathbb{R}_+=(0,\infty)$, $\mathbb{R}_+^0=[0,\infty)$.

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, Bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, and so forth. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional differential equations in comparison with classical integer-order models. In consequence, the subject of fractional differential equations is gaining much importance and attention. Especially, many people pay attention to the existence and multiplicity of solutions or positive solutions for the boundary value problems of nonlinear fractional differential equations (see [1, 3, 5, 6, 10, 12, 14–20]). Moreover, the boundary value problems with p-Laplacian have been discussed extensively in the past few decades (see [4, 8, 9, 13, 21, 22]). However, there is relatively rare paper dealing with the Riemann-Stieltjes integral boundary problems for high-order nonlinear fractional differential coupling system. Therefore, it is worth to study the existence and multiplicity of positive solutions for the high-order nonlinear fractional differential coupling system (1).

The rest of this paper is organized as follows. In Section 2, we recall some useful definitions and properties, and present the properties of the Green's functions. In Section 3, we give some sufficient conditions for the existence and multiplicity of positive solution for BVP (1)-(2). As applications, an example is also provided to illustrate the validity of our main results in Section 4. Finally, the conclusion is given to simply recall our studied contents and obtained results in Section 5.

2. Preliminaries

For the convenience of the reader, we introduce the definitions and lemmas of fractional calculus theory.

Definition 2.1. (*see* [7, 11]) *The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $f:(0,\infty)\to\mathbb{R}$ is given by*

$$I_{0+}^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)ds,$$

provided that the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2.2. (*see* [7, 11]) *The Caputo fractional derivative of order $\alpha>0$ of a continuous function $f:(0,\infty)\to\mathbb{R}$ is given by*

$$D_{0+}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}ds,$$

where $n-1<\alpha\le n$,provided that the right-hand side is pointwise defined on $(0,\infty)$.

Lemma 2.3. (*see* [7]) *Assume that $u\in C(0,1)\cap L(0,1)$ with a Caputo fractional derivative of order $\alpha>0$ that belongs to $u\in C^n[0,1]$, then*

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t)=u(t)+c_0+c_1t+\cdots+c_{n-1}t^{n-1},$$

for some $c_i\in\mathbb{R}$, $i=0,1,\dots,n-1$, where n is the smallest integer greater than or equal to α.

Lemma 2.4. *The p-Laplacian operator have the following properties:*

(1) *If $x\ge0$, then $\phi_p(x)=x^{p-1}$, $\phi_p(x)$ is increasing with respect to x.*

(2) *If $x,y\ge0$, then $\phi_p(xy)=\phi_p(x)\phi_p(y)$*

(3) *If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, then $\phi_q(\phi_p(x))=\phi_p(\phi_q(x))=x$.*

(4) *If $0\le x\le y$, then $x\le y\Longleftrightarrow\phi_q(x)\le\phi_q(y)$.*

(5) *If* $0 \le x \le \phi_q^{-1}(y)$*, then* $x \le \phi_q^{-1}(y) \Longleftrightarrow \phi_q(x) \le y$.

(6) *If* $q \ge 2$ *and* $0 \le x, y \le M$*, then* $|\phi_q(x) - \phi_q(y)| \le (q-1)M^{q-2}|x-y|$.

(7) *If* $1 < q < 2$ *and* $x, y \ge m \ge 0$*, then* $|\phi_q(x) - \phi_q(y)| \le (q-1)m^{q-2}|x-y|$.

Lemma 2.5. (*see* [2]) *Let* E *be a Banach spaces and* $K \subset E$ *be a cone in* E*. Let* $r > 0$ *and* $\Omega_r = \{x \in K : \|x\| < r\}$*. Assume that* $S : \overline{\Omega}_r \to K$ *is a completely continuous operator such that* $Sx \neq x$ *for* $x \in \partial\Omega_r$.

(i) *If* $\|Sx\| \le \|x\|$ *for* $x \in \partial\Omega_r$*, then* $i(S, \Omega_r, K) = 1$.

(ii) *If* $\|Sx\| \ge \|x\|$ *for* $x \in \partial\Omega_r$*, then* $i(S, \Omega_r, K) = 0$.

For convenience, we introduce the following notations. Let

$$\mu_i = \int_0^1 \varphi_i(t)dt,\ \Delta_i = \int_0^1 dA_i(s),\ \Delta_i' = \int_0^1 \tau dA_i(\tau),\ \gamma = \min\left\{\frac{b_1}{a_1+b_1}, \frac{b_2}{a_2+b_2}\right\},\ N_i = \frac{1}{(1-\Delta_i)\Gamma(\beta_i)},$$

$$M_i = \frac{a_i\phi_{q_i}(N_i)}{a_i-\mu_i}\int_0^1 G_{\alpha_i}(1,s)ds,\ N_i' = \frac{\Delta_i - \Delta_i'}{(1-\Delta_i)\Gamma(\beta_i - 1)},\ M_i' = \frac{a_ib_i\phi_{q_i}(N_i')}{(a_i+b_i)(a_i-\mu_i)}\int_0^1 G_{\alpha_i}(1,s)ds,$$

$$f_i^0 = \lim_{x_1+x_2+x_3+x_4\to 0}\frac{f_i(t,x_1,x_2,x_3,x_4)}{\phi_{p_i}\left(\sum_{j=1}^4 x_j\right)} \quad \text{uniformly for } t \in [0,1],$$

$$f_i^\infty = \lim_{x_1+x_2+x_3+x_4\to \infty}\frac{f_i(t,x_1,x_2,x_3,x_4)}{\phi_{p_i}\left(\sum_{j=1}^4 x_j\right)} \quad \text{uniformly for } t \in [0,1],$$

where $i = 1, 2$, $\Gamma(\cdot)$ is the Gamma function.

Now we present the Green's functions associated with BVP (1)-(2).

Lemma 2.6. *If* $\varphi_i, v_i \in L^1(J)$ *and* $\mu_i \neq a_i$, $q_i > 1$ $(i = 1, 2)$*, then the boundary value problem*

$$\begin{cases} -D_{0+}^{\alpha_i}u_i(t) = \phi_{q_i}(v_i(t)),\ t \in J, \\ a_iu_i(0) - b_iu_i'(0) = \int_0^1 \varphi_i(t)u_i(t)dt, \\ u_i'(1) = u_i''(0) = 0 \end{cases} \tag{3}$$

has a unique solution

$$u_i(t) = \int_0^1 H_i(t,s)\phi_{q_i}(v_i(s))ds, \tag{4}$$

where

$$H_i(t,s) = G_{\alpha_i}(t,s) + \frac{1}{a_i-\mu_i}\int_0^1 \varphi_i(\varsigma)G_{\alpha_i}(\varsigma,s)d\varsigma, \tag{5}$$

and

$$G_{\alpha_i}(t,s) = \begin{cases} \frac{(a_it+b_i)(\alpha_i-1)(1-s)^{\alpha_i-2} - a_i(t-s)^{\alpha_i-1}}{a_i\Gamma(\alpha_i)}, & 0 \le s \le t \le 1, \\ \frac{(a_it+b_i)(\alpha_i-1)(1-s)^{\alpha_i-2}}{a_i\Gamma(\alpha_i)}, & 0 \le t \le s \le 1. \end{cases} \tag{6}$$

Proof. By applying Lemma (2.3), we have

$$u_i(t) = -I_{0+}^{\alpha_i}\left(\phi_{q_i}(v_i(t))\right) + c_{i0} + c_{i1}t + c_{i2}t^2. \tag{7}$$

$u_i'(1) = u_i''(0) = 0$ yields that $c_{i2} = 0$, $c_{i1} = I_{0+}^{\alpha_i-1}\left(\phi_{q_i}(v_i(1))\right)$. According to the second equation of (3), we have

$$a_ic_{i0} - b_ic_{i1} = \int_0^1 \varphi_i(t)u_i(t)dt \Rightarrow c_{i0} = \frac{1}{a_i}\left[b_iI_{0+}^{\alpha_i-1}\left(\phi_{q_i}(v_i(1))\right) + \int_0^1 \varphi_i(t)u_i(t)dt\right].$$

By (7), we have

$$\begin{aligned} u_i(t) =& - I_{0+}^{\alpha_i}\left(\phi_{q_i}(v_i(t))\right) + \frac{1}{a_i}\left[b_iI_{0+}^{\alpha_i-1}\left(\phi_{q_i}(v_i(1))\right) + \int_0^1 \varphi_i(t)u_i(t)dt\right] + I_{0+}^{\alpha_i-1}\left(\phi_{q_i}(v_i(1))\right)t, \\ =& \frac{1}{a_i\Gamma(\alpha_i)}\Big[- a_i\int_0^t (t-s)^{\alpha_i-1}\phi_{q_i}(v_i(s))\,ds + (a_it+b_i)\int_0^t(\alpha_i-1)(1-s)^{\alpha_i-2}\phi_{q_i}(v_i(s))ds \\ &+ (a_it+b_i)\int_t^1(\alpha_i-1)(1-s)^{\alpha_i-2}\phi_{q_i}(v_i(s))ds\Big] + \frac{1}{a_i}\int_0^1 \varphi_i(t)u_i(t)dt \\ =& \int_0^1 G_{\alpha_i}(t,s)\phi_{q_i}(v_i(s))\,ds + \frac{1}{a_i}\int_0^1 \varphi_i(t)u_i(t)dt. \end{aligned} \tag{8}$$

Thus,

$$\begin{aligned} \int_0^1 \varphi_i(t)u_i(t)dt =& \int_0^1 \varphi_i(t)\left[\int_0^1 G_{\alpha_i}(t,s)\phi_{q_i}(v_i(s))\,ds + \frac{1}{a_i}\int_0^1 \varphi_i(t)u_i(t)dt\right]dt \\ =& \int_0^1 \varphi_i(t)\left(\int_0^1 G_{\alpha_i}(t,s)\phi_{q_i}(v_i(s))\,ds\right)dt + \frac{1}{a_i}\int_0^1 \varphi_i(t)dt\int_0^1 \varphi_i(t)u_i(t)dt \end{aligned}$$

implies that

$$\int_0^1 \varphi_i(t)u_i(t)dt = \frac{a_i}{a_i-\mu_i}\int_0^1 \varphi_i(t)\left(\int_0^1 G_{\alpha_i}(t,s)\phi_{q_i}(v_i(s))\,ds\right)dt. \tag{9}$$

Substituting (9) into (8), we obtain

$$\begin{aligned} u_i(t) =& \int_0^1 G_{\alpha_i}(t,s)\phi_{q_i}(v_i(s))\,ds + \frac{1}{a_i-\mu_i}\int_0^1 \varphi_i(t)\left(\int_0^1 G_{\alpha_i}(t,s)\phi_{q_i}(v_i(s))\,ds\right)dt \\ =& \int_0^1\left(G_{\alpha_i}(t,s) + \frac{1}{a_i-\mu_i}\int_0^1 \varphi_i(\varsigma)G_{\alpha_i}(\varsigma,s)d\varsigma\right)\phi_{q_i}(v_i(s))\,ds = \int_0^1 H_i(t,s)\phi_{q_i}(v_i(s))\,ds, \end{aligned}$$

where $G_{\alpha_i}(t,s)$ and $H_i(t,s)$ are defined by (5) and (6), respectively.

Now, we will prove the uniqueness of solution for BVP (3). In fact, let $u_i(t)$, $U_i(t)$ are any two solutions of (3). Denote $W_i(t) = U_i(t) - u_i(t)$, then (3) be changed into the following system:

$$\begin{cases} -D_{0+}^{\alpha_i}W_i(t) = 0, & t\in J, \\ a_iW_i(0) - b_iW_i'(0) = \int_0^1 \varphi_i(t)W_i(t)dt, \\ W_i'(1) = W_i''(0) = 0. \end{cases}$$

Similar to above arguement, we get $W_i(t) = 0$, that is $U_i(t) = u_i(t)$, which mean that the solution for BVPs (3) is unique. The proof is completed. □

Similar to Lemma 2.6, we obtain the following Lemma.

Lemma 2.7. *If $A_i : [0,1] \to \mathbb{R}$ is a function of bounded variation with $\Delta_i \neq 1$, $1 < \beta_i \leq 2$ and $w_i \in L^1(J)$ $(i = 1,2)$, then the boundary value problem*

$$\begin{cases} -D_{0+}^{\beta_i} v_i(t) = w_i(t), \ t \in J, \\ v_i'(0) = 0, \quad v_i(1) = \int_0^1 v_i(s)dA_i(s) \end{cases} \tag{10}$$

has a unique solution

$$v_i(t) = \int_0^1 K_i(t,s)w_i(s)ds, \tag{11}$$

where

$$K_i(t,s) = G_{\beta_i}(t,s) + \frac{1}{1-\Delta_i}\int_0^1 G_{\beta_i}(\tau,s)dA_i(\tau), \tag{12}$$

and

$$G_{\beta_i}(t,s) = \begin{cases} \frac{(1-s)^{\beta_i-1}-(t-s)^{\beta_i-1}}{\Gamma(\beta_i)}, & 0 \leq s \leq t \leq 1, \\ \frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{13}$$

Throughout this paper, we need the following assumptions.

(H_1) $a_i \leq (\alpha_i - 1)b_i$, $i = 1,2$.

(H_2) $0 \leq \mu_i < a_i$, $0 \leq \Delta_i < 1$, $\Delta_i' \leq \Delta_i$, $i = 1,2$.

(H_3) $\varphi_i \in L^1(J, \mathbb{R}_+^0)$, $f_i \in C(J \times (\mathbb{R}_+^0)^4, \mathbb{R}_+)$ and $A_i \in C(J, \mathbb{R}_+)$ are nondecreasing functions of bounded variation, and the integrals in (2) are Riemann-Stieltjes integrals, $i = 1,2$.

Lemma 2.8. *If (H_1) holds, then the functions $G_{\alpha_i}(t,s)$ and $G_{\beta_i}(t,s) (i = 1,2)$ defined by (6) and (13) satisfy the following*

(1) $G_{\alpha_i}(t,s) \in C([0,1]\times[0,1])$, $G_{\alpha_i}(t,s) > 0$ *for all* $t,s \in (0,1)$.

(2) $\frac{b_i G_{\alpha_i}(1,s)}{a_i+b_i} \leq G_{\alpha_i}(0,s) \leq G_{\alpha_i}(t,s) \leq G_{\alpha_i}(1,s)$ *for all* $t,s \in [0,1]$.

(3) $\frac{\partial G_{\alpha_i}(t,s)}{\partial t} \in C([0,1]\times[0,1])$, $\frac{\partial G_{\alpha_i}(t,s)}{\partial t} > 0$ *for all* $t,s \in (0,1)$.

(4) $\frac{\partial G_{\alpha_i}(t,s)}{\partial t} \leq \frac{a_i G_{\alpha_i}(0,s)}{b_i}$ *for all* $t,s \in [0,1]$.

(5) $\frac{a_i G_{\alpha_i}(0,s)}{b_i} \leq G_{\alpha_i}(1,s)$ *for all* $t,s \in [0,1]$.

(6) $G_{\beta_i}(t,s) \in C([0,1]\times[0,1])$, $G_{\beta_i}(t,s) > 0$ *for all* $t,s \in (0,1)$.

(7) $(\beta_i - 1)(1-t)G_{\beta_i}(s,s) \leq G_{\beta_i}(t,s) \leq G_{\beta_i}(s,s)$ *for all* $t,s \in [0,1]$.

Proof. Now we shall prove (1)-(5). In fact, it follows from the definition of G_{α_i} and G_{β_i} that G_{α_i} and G_{β_i} are continuous on $[0,1]\times[0,1]$. For $2 < \alpha_i \leq 3$, since

$$\frac{\partial G_{\alpha_i}(t,s)}{\partial t} = \begin{cases} \frac{a_i(\alpha_i-1)(1-s)^{\alpha_i-2}-a_i(\alpha_i-1)(t-s)^{\alpha_i-2}}{a_i\Gamma(\alpha_i)} \geq 0, & 0 \leq s \leq t \leq 1, \\ \frac{a_i(\alpha_i-1)(1-s)^{\alpha_i-2}}{a_i\Gamma(\alpha_i)} \geq 0, & 0 \leq t \leq s \leq 1, \end{cases}$$

and

$$\begin{aligned}\frac{G_{\alpha_i}(0,s)}{G_{\alpha_i}(1,s)} &= \frac{b_i(\alpha_i-1)(1-s)^{\alpha_i-2}}{(a_i+b_i)(\alpha_i-1)(1-s)^{\alpha_i-2}-a_i(1-s)^{\alpha_i-1}}\\ &= \frac{b_i(\alpha_i-1)}{(a_i+b_i)(\alpha_i-1)-a_i(1-s)} \geq \frac{b_i(\alpha_i-1)}{(a_i+b_i)(\alpha_i-1)} = \frac{b_i}{a_i+b_i}, \quad s\in J.\end{aligned}$$

It is clear that $\frac{\partial G_{\alpha_i}(t,s)}{\partial t}$ is continuous on $[0,1]\times[0,1]$ and $G_{\alpha_i}(t,s)$ is increasing on $t\in[0,1]$. So $\frac{b_iG_{\alpha_i}(1,s)}{a_i+b_i} \leq G_{\alpha_i}(0,s)\leq G_{\alpha_i}(t,s)\leq G_{\alpha_i}(1,s)$ for $(t,s)\in[0,1]\times[0,1]$, and $G_{\alpha_i}(t,s)\geq G_{\alpha_i}(0,s)=\frac{b_i(\alpha_i-1)(1-s)^{\alpha_i-2}}{a_i\Gamma(\alpha_i)}>0$ for $t,s\in(0,1)$. Since

$$\frac{\partial^2 G_{\alpha_i}(t,s)}{\partial t^2} = \begin{cases} \frac{-a_i(\alpha_i-1)(\alpha_i-2)(t-s)^{\alpha_i-3}}{a_i\Gamma(\alpha_i)}\leq 0, & 0\leq s<t\leq 1,\\ 0, & 0\leq t<s\leq 1.\end{cases}$$

It is clear that $\frac{\partial G_{\alpha_i}(t,s)}{\partial t}$ is decreasing on $t\in[0,1]$. So $\frac{\partial G_{\alpha_i}(t,s)}{\partial t}\leq\frac{\partial G_{\alpha_i}(0,s)}{\partial t}=\frac{a_iG_{\alpha_i}(0,s)}{b_i}$ for $(t,s)\in[0,1]\times[0,1]$, and $\frac{\partial G_{\alpha_i}(t,s)}{\partial t}\geq\frac{\partial G_{\alpha_i}(1,s)}{\partial t}>0$ for $(t,s)\in(0,1)\times(0,1)$. In addition, for $a_i\leq(\alpha_i-1)b_i$, $s\in[0,1)$, $i=1,2$, we have

$$\begin{aligned}\frac{b_iG_{\alpha_i}(1,s)}{a_iG_{\alpha_i}(0,s)} &= \frac{b_i\left[(a_i+b_i)(\alpha_i-1)(1-s)^{\alpha_i-2}-a_i(1-s)^{\alpha_i-1}\right]}{a_ib_i(\alpha_i-1)(1-s)^{\alpha_i-2}}\\ &= \frac{a_i(\alpha_i+s-2)+b_i(\alpha_i-1)}{a_i(\alpha_i-1)} \geq \frac{a_i(\alpha_i+s-2)+a_i}{a_i(\alpha_i-1)} \geq \frac{a_i(\alpha_i-2)+a_i}{a_i(\alpha_i-1)} = 1.\end{aligned}$$

Notice that $G_{\alpha_i}(0,1)=G_{\alpha_i}(1,1)=0$. So, $\frac{a_iG_{\alpha_i}(0,s)}{b_i}\leq G_{\alpha_i}(1,s)$ holds for $s\in[0,1]$.

In what follows, we show that (6) and (7) hold. Indeed, since

$$\frac{\partial G_{\beta_i}(t,s)}{\partial t} = \begin{cases} -\frac{(\beta_i-1)(t-s)^{\beta_i-2}}{\Gamma(\beta_i)}\leq 0, & 0\leq s\leq t\leq 1,\\ 0, & 0\leq t\leq s\leq 1\end{cases}$$

that is, $G_{\beta_i}(t,s)$ is decreasing with respect to $t\in[0,1]$. So $G_{\beta_i}(t,s)\geq G_{\beta_i}(1,s)=0$ and $G_{\beta_i}(t,s)\neq 0$ for $t,s\in(0,1)$. Therefore, $G_{\beta_i}(t,s)>0$ for $t,s\in(0,1)$. When $0\leq t\leq s\leq 1$, $G_{\beta_i}(t,s)=G_{\beta_i}(s,s)=\frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)}$. When $0\leq s\leq t\leq 1$, $G_{\beta_i}(t,s)=\frac{(1-s)^{\beta_i-1}-(t-s)^{\beta_i-1}}{\Gamma(\beta_i)}\leq\frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)}=G_{\beta_i}(s,s)$. Therefore, $G_{\beta_i}(t,s)\leq G_{\beta_i}(s,s)$ for $t,s\in[0,1]$. In addition, when $0\leq s\leq t<1$, there exists $\varepsilon\in[t-s,1-s]$ such that

$$\begin{aligned}(1-s)^{\beta_i-1}-(t-s)^{\beta_i-1} &= (\beta_i-1)\int_{t-s}^{1-s} r^{\beta_i-2}dr = (\beta_i-1)\varepsilon^{\beta_i-2}(1-s-t+s)\\ &\geq (\beta_i-1)(1-t)(1-s)^{\beta_i-2} > (\beta_i-1)(1-t)(1-s)^{\beta_i-1}>0,\end{aligned}$$

which implies that $G_{\beta_i}(t,s)>(\beta_i-1)(1-t)G_{\beta_i}(s,s)$. When $0\leq s\leq t=1$, $G_{\beta_i}(t,s)=G_{\beta_i}(1,s)=0$ and $(\beta_i-1)(1-t)G_{\beta_i}(s,s)=0$. When $0\leq t\leq s\leq 1$, $G_{\beta_i}(t,s)=G_{\beta_i}(s,s)=\frac{(1-s)^{\beta_i-1}}{\Gamma(\beta_i)}$. So we know that $(\beta_i-1)(1-t)G_{\beta_i}(s,s)\leq G_{\beta_i}(t,s)$ for $t,s\in[0,1]$. Thus, the proof of Lemma 2.8 is completed. □

Lemma 2.9. *Assume that* (H_1) *and* (H_2) *hold. Then the functions* $H_i(t,s)$ *and* $K_i(t,s)$ $(i=1,2)$ *defined by* (5) *and* (12) *satisfy the following properties:*

(1) $H_i(t,s)\in C([0,1]\times[0,1])$, $H_i(t,s)>0$ *for all* t, $s\in(0,1)$.

(2) $\frac{a_ib_iG_{\alpha_i}(1,s)}{(a_i+b_i)(a_i-\mu_i)}\leq H_i(t,s)\leq\frac{a_iG_{\alpha_i}(1,s)}{a_i-\mu_i}$ *for all* t, $s\in[0,1]$.

(3) $\frac{\partial H_i(t,s)}{\partial t}\in C([0,1]\times[0,1])$, $\frac{\partial H_i(t,s)}{\partial t}>0$ *for all* t, $s\in(0,1)$.

(4) $\frac{\partial H_i(t,s)}{\partial t} \le \frac{a_i G_{\alpha_i}(0,s)}{b_i}$ *for all* $t,\ s \in [0,1]$.

(5) $\frac{(a_i-\mu_i)G_{\alpha_i}(0,s)}{b_i} \le G_{\alpha_i}(1,s)$ *for all* $t,\ s \in [0,1]$.

(6) $K_i(t,s) \in C([0,1]\times[0,1])$, $K_i(t,s) > 0$ *for all* $t,\ s \in (0,1)$.

(7) $\frac{(\Delta_i-\Delta_i')(\beta_i-1)G_{\beta_i}(s,s)}{1-\Delta_i} \le K_i(t,s) \le \frac{G_{\beta_i}(s,s)}{1-\Delta_i}$ *for all* $t,\ s \in [0,1]$.

Proof. By Lemma 2.8 and $0 \le \mu_i < a_i$, we have

$$\begin{aligned} H_i(t,s) &= G_{\alpha_i}(t,s) + \frac{1}{a_i-\mu_i}\int_0^1 \varphi_i(\varsigma)G_{\alpha_i}(\varsigma,s)d\varsigma \ge G_{\alpha_i}(0,s) + \frac{G_{\alpha_i}(0,s)}{a_i-\mu_i}\int_0^1 \varphi_i(\varsigma)d\varsigma \\ &= G_{\alpha_i}(0,s)\left(1+\frac{\mu_i}{a_i-\mu_i}\right) = \frac{a_i}{a_i-\mu_i}G_{\alpha_i}(0,s) \ge \frac{a_i b_i G_{\alpha_i}(1,s)}{(a_i+b_i)(a_i-\mu_i)}, \end{aligned}$$

and

$$\begin{aligned} H_i(t,s) &= G_{\alpha_i}(t,s) + \frac{1}{a_i-\mu_i}\int_0^1 \varphi_i(\varsigma)G_{\alpha_i}(\varsigma,s)d\varsigma \le G_{\alpha_i}(1,s) + \frac{G_{\alpha_i}(1,s)}{a_i-\mu_i}\int_0^1 \varphi_i(\varsigma)d\varsigma \\ &= G_{\alpha_i}(1,s)\left(1+\frac{\mu_i}{a_i-\mu_i}\right) = \frac{a_i}{a_i-\mu_i}G_{\alpha_i}(1,s), \end{aligned}$$

which imply that (1)-(2) hold. $\frac{\partial H_i(t,s)}{\partial t} = \frac{\partial G_{\alpha_i}(t,s)}{\partial t}$ and Lemma 2.8 yield that (3)-(5) hold.

Next, we prove that (6) and (7) hold. Indeed, by Lemma 2.8 and $0 \le \Delta_i < 1$, we get

$$\begin{aligned} K_i(t,s) &= G_{\beta_i}(t,s) + \frac{1}{1-\Delta_i}\int_0^1 G_{\beta_i}(\tau,s)dA_i(\tau) \\ &\ge (\beta_i-1)(1-t)G_{\beta_i}(s,s) + \frac{(\beta_i-1)G_{\beta_i}(s,s)}{1-\Delta_i}\int_0^1 (1-\tau)dA_i(\tau) \\ &= (\beta_i-1)G_{\beta_i}(s,s)\left(1-t+\frac{1}{1-\Delta_i}\int_0^1 (1-\tau)dA_i(\tau)\right) \\ &\ge (\beta_i-1)G_{\beta_i}(s,s)\left(\frac{\Delta_i}{1-\Delta_i} - \frac{1}{1-\Delta_i}\int_0^1 \tau dA_i(\tau)\right) = \frac{(\Delta_i-\Delta_i')(\beta_i-1)G_{\beta_i}(s,s)}{1-\Delta_i}, \end{aligned}$$

and

$$\begin{aligned} K_i(t,s) &= G_{\beta_i}(t,s) + \frac{1}{1-\Delta_i}\int_0^1 G_{\beta_i}(\tau,s)dA_i(\tau) \le G_{\beta_i}(s,s) + \frac{G_{\beta_i}(s,s)}{1-\Delta_i}\int_0^1 dA_i(\tau) \\ &= G_{\beta_i}(s,s)\left(1+\frac{\Delta_i}{1-\Delta_i}\right) = \frac{G_{\beta_i}(s,s)}{1-\Delta_i}. \end{aligned}$$

The proof is completed. □

Corollary 2.10. *If* $0 \le \Delta_i < 1$ *and* $\Delta_i' \le \Delta_i$, *then*

$$N_i' \le K_i(t,s) \le N_i, \quad i = 1,\ 2. \tag{14}$$

Proof. From Lemma 2.9, we have $K_i(t,s) \le \frac{G_{\beta_i}(s,s)}{1-\Delta_i} \le \frac{1}{\Gamma(\beta_i)(1-\Delta_i)}$. Let $\omega(s) = \frac{(\Delta_i-\Delta_i')(\beta_i-1)G_{\beta_i}(s,s)}{(1-s)(1-\Delta_i)}$, for $1 < \beta_i \le 2$, we have

$$\omega'(s) = \frac{(\Delta_i-\Delta_i')(\beta_i-1)(2-\beta_i)(1-s)^{\beta_i-3}}{(1-\Delta_i)\Gamma(\beta_i)} \ge 0, \quad s \in [0,1),\ i = 1,2.$$

Therefore, for $s \in [0,1]$, we have $K_i(t,s) \ge \omega(s) \ge \omega(0) = \frac{\Delta_i-\Delta_i'}{(1-\Delta_i)\Gamma(\beta_i-1)}$, $i = 1,\ 2$. The proof is completed. □

3. Main Results

In this section, we will discuss the existence and multiplicity of positive solutions to the BVP (1)-(2).

$C[0,1]$ with the norm $\|u\|_\infty = \max_{t\in J}|u(t)|$ is the Banach space. Let E be the Banach space of $C^1[0,1]$ endowed with the norm $\|u\|_* = \max\{\|u\|_\infty, \|u'\|_\infty\}$. Thus, $X = E\times E$ is a Banach with the norm defined by $\|(u_1,u_2)\| = \max\{\|u_1\|_*, \|u_2\|_*\}$, for $(u_1,u_2)\in X$.

From Lemma 2.6 and Lemma 2.7, replacing $w_i(t)$ with $\lambda_i f_i(t,u_1(t),u_2(t),u_1'(t),u_2'(t))$, $i=1,2$, we obtain the following lemma.

Lemma 3.1. *Assume that $\mu_i \neq a_i$ (i = 1, 2). Then BVP* (1)-(2) *have a pair of solutions $(u_1,u_2)\in X$ if and only if $(u_1,u_2)\in X$ is a pair of solutions to the integral system*

$$u_i(t) = \int_0^1 H_i(t,s)\phi_{q_i}\left(\lambda_i\int_0^1 K_i(s,\tau)f_i\left(\tau,u_1(\tau),u_2(\tau),u_1'(\tau),u_2'(\tau)\right)d\tau\right)ds,\quad i = 1,\, 2.$$

Define two scalar operators $T_1 : X\to E$ and $T_2 : X\to E$ by

$$T_1(u_1,u_2)(t) = \int_0^1 H_1(t,s)\phi_{q_1}\left(\lambda_1\int_0^1 K_1(s,\tau)f_1\left(\tau,u_1(\tau),u_2(\tau),u_1'(\tau),u_2'(\tau)\right)d\tau\right)ds \tag{15}$$

and

$$T_2(u_1,u_2)(t) = \int_0^1 H_2(t,s)\phi_{q_2}\left(\lambda_2\int_0^1 K_2(s,\tau)f_2\left(\tau,u_1(\tau),u_2(\tau),u_1'(\tau),u_2'(\tau)\right)d\tau\right)ds. \tag{16}$$

Now, we define an vector operator $S : X\to X$ by

$$S(u_1,u_2)(t) = \Big(T_1(u_1,u_2)(t),\ T_2(u_1,u_2)(t)\Big), \tag{17}$$

where $T_1(u_1,u_2)(t)$ and $T_2(u_1,u_2)(t)$ are defined as (15) and (16), respectively.

We claim that whenever $(u_1,\,u_2)\in X$ is a fixed point of the operator S defined in (17), it follows that $(u_1(t),\,u_2(t))$ is a pair of solutions for BVP (1)-(2). That is, a pair of functions $(u_1,u_2)\in X$ is a pair of solutions to BVP (1)-(2) if and only if (u_1,u_2) is a fixed point of the operator S defined in (17).

Define the cone $K\subset X$ by

$$K = \Big\{(u_1,u_2)\in X : u_i,\ u_i'\geq 0,\ u_1(t)+u_2(t)\geq\gamma\|(u_1,u_2)\|,\ t\in J,\ i = 1,\ 2\Big\}. \tag{18}$$

Lemma 3.2. *If* (H_1) *holds, then* $\|u_i'\|_\infty\leq\|u_i\|_\infty$ *and* $\|u_i\|_* = \|u_i\|_\infty$, i = 1, 2.

Proof. In view of (2) and (5) of Lemma 2.9, we have

$$\begin{aligned}
u_i(t) &= \int_0^1 H_i(t,s)\phi_{q_i}\left(\lambda_i\int_0^1 K_i(s,\tau)f_i\left(\tau,u_1(\tau),u_2(\tau),u_1'(\tau),u_2'(\tau)\right)d\tau\right)ds\\
&\geq\frac{a_ib_i}{(a_i+b_i)(a_i-\mu_i)}\int_0^1 G_{\alpha_i}(1,s)\phi_{q_i}\left(\lambda_i\int_0^1 K_i(s,\tau)f_i\left(\tau,u_1(\tau),u_2(\tau),u_1'(\tau),u_2'(\tau)\right)d\tau\right)ds\\
&\geq\frac{a_ib_i}{(a_i+b_i)(a_i-\mu_i)}\int_0^1 \frac{a_i-\mu_i}{b_i}G_{\alpha_i}(0,s)\phi_{q_i}\left(\lambda_i\int_0^1 K_i(s,\tau)f_i\left(\tau,u_1(\tau),u_2(\tau),u_1'(\tau),u_2'(\tau)\right)d\tau\right)ds\\
&=\frac{a_i}{a_i+b_i}\int_0^1 G_{\alpha_i}(0,s)\phi_{q_i}\left(\lambda_i\int_0^1 K_i(s,\tau)f_i\left(\tau,u_1(\tau),u_2(\tau),u_1'(\tau),u_2'(\tau)\right)d\tau\right)ds,\ t\in[0,1].
\end{aligned}$$

By (4) of Lemma 2.9, we get

$$\begin{aligned} u_i'(t) = & \int_0^1 \frac{\partial H_i(t,s)}{\partial t}\phi_{q_i}\left(\lambda_i\int_0^1 K_i(s,\tau)f_i\left(\tau,u_1(\tau),u_2(\tau),u_1'(\tau),u_2'(\tau)\right)d\tau\right)ds \\ \le & \frac{a_i}{b_i}\int_0^1 G_{\alpha_i}(0,s)\phi_{q_i}\left(\lambda_i\int_0^1 K_i(s,\tau)f_i\left(\tau,u_1(\tau),u_2(\tau),u_1'(\tau),u_2'(\tau)\right)d\tau\right)ds,\ t\in[0,1]. \end{aligned}$$

From above two inequalities, we obtain

$$u_i(t) \ge \frac{b_i}{a_i+b_i}u_i'(t) \ge u_i'(t),\ t\in[0,1],$$

which implies that $\|u_i'\|_\infty \le \|u_i\|_\infty$. Thus, $\|u_i\|_* = \max\{\|u_i\|_\infty, \|u_i'\|_\infty\} = \|u_i\|_\infty$. The proof is completed. □

Lemma 3.3. *Assume that* (H_1)-(H_3) *hold, then the operator* $S: K\to K$ *defined by* (17) *is completely continuous.*

Proof. According to (H_1)-(H_3), we know that $T_i(u_1,u_2)(t)$, $T_i'(u_1,u_2)(t) = \frac{d}{dt}[T_i(u_1,u_2)(t)] \ge 0$ for each $t\in J$, $i=1,2$. For $t\in J$, by Lemma 2.9 and Lemma 3.2, we have

$$\begin{aligned} & T_1(u_1,u_2)(t)+T_2(u_1,u_2)(t) \\ = & \sum_{i=1}^2\int_0^1 H_i(t,s)\phi_{q_i}\left(\lambda_i\int_0^1 K_i(s,\tau)f_i\left(\tau,u_1(\tau),u_2(\tau),u_1'(\tau),u_2'(\tau)\right)d\tau\right)ds \\ \ge & \sum_{i=1}^2\frac{a_ib_i}{(a_i+b_i)(a_i-\mu_i)}\int_0^1 G_{\alpha_i}(1,s)\phi_{q_i}\left(\lambda_i\int_0^1 K_i(s,\tau)f_i\left(\tau,u_1(\tau),u_2(\tau),u_1'(\tau),u_2'(\tau)\right)d\tau\right)ds \\ \ge & \sum_{i=1}^2\frac{b_i}{a_i+b_i}\int_0^1 H_i(1,s)\phi_{q_i}\left(\lambda_i\int_0^1 K_i(s,\tau)f_i\left(\tau,u_1(\tau),u_2(\tau),u_1'(\tau),u_2'(\tau)\right)d\tau\right)ds \\ = & \sum_{i=1}^2\frac{b_i}{a_i+b_i}\sup_{t\in J}\left|\int_0^1 H_i(t,s)\phi_{q_i}\left(\lambda_i\int_0^1 K_i(s,\tau)f_i\left(\tau,u_1(\tau),u_2(\tau),u_1'(\tau),u_2'(\tau)\right)d\tau\right)ds\right| \\ = & \sum_{i=1}^2\frac{b_i}{a_i+b_i}\|T_i(u_1,u_2)\|_\infty = \sum_{i=1}^2\frac{b_i}{a_i+b_i}\|T_i(u_1,u_2)\|_* \\ \ge & \gamma\sum_{i=1}^2\|T_i(u_1,u_2)\|_* \ge \gamma\max\{\|T_1(u_1,u_2)\|_*, \|T_2(u_1,u_2)\|_*\} \\ = & \gamma\|(T_1(u_1,u_2),T_2(u_1,u_2))\| = \gamma\|S(u_1,u_2)\|, \end{aligned}$$

which implies that $S(K)\subset K$. Let B be any bounded subset in X. By virtue of the Ascoli-Arzela theorem, we need to show that $S(B)$ is uniformly bounded in X and $S: K\to K$ is equicontinuous. In fact, if B is bounded, then $\exists\, R>0$, such that for all $(u_1, u_2)\in B$, we have $\|(u_1,u_2)\|\le R$, that is $\|u_1\|_*\le R$, $\|u_2\|_*\le R$. Since $f_i\in C([0,1]\times\mathbb{R}_+^4,\mathbb{R}_+)$, for some $\varphi_i\in L^1[0,1]$ such that $f_i(t,u_1,u_2,u_1',u_2')\le\varphi_i(t)$ for $t\in[0,1]$, we derive from Corollary 2.10 that

$$\begin{aligned} 0\le & T_i(u_1,u_2)(t) = \int_0^1 H_i(t,s)\phi_{q_i}\left(\lambda_i\int_0^1 K_i(s,\tau)f_i\left(\tau,u_1(\tau),u_2(\tau),u_1'(\tau),u_2'(\tau)\right)d\tau\right)ds \\ \le & \frac{a_i}{a_i-\mu_i}\int_0^1 G_{\alpha_1}(1,s)\phi_{q_i}\left(\lambda_i\int_0^1\frac{\varphi_i(\tau)}{(1-\Delta_i)\Gamma(\beta_i)}d\tau\right)ds \\ \le & \frac{a_i}{a_i-\mu_i}\left[\frac{\lambda_i\|\varphi_i\|_1}{(1-\Delta_i)\Gamma(\beta_i)}\right]^{\frac{1}{p_i-1}}\int_0^1 G_{\alpha_i}(1,s)ds = \frac{a_i[(\alpha_i-1)a_i+\alpha_ib_i]}{\alpha_i(a_i-\mu_i)\Gamma(\alpha_i+1)}\left[\frac{\lambda_i\|\varphi_i\|_1}{(1-\Delta_i)\Gamma(\beta_i)}\right]^{\frac{1}{p_i-1}}, \end{aligned}$$

which means that $T_i(B)$ is uniformly bounded in X. So $S(B)$ is also uniformly bounded in X.

In what follows, we show that $S: K \to K$ is equicontinuous. Indeed, from Lemma 2.9, we get $H_i(s,t)$ and $\frac{\partial H_i(t,s)}{\partial t}$ are uniformly continuous on $[0,1]$. Therefore, for all $\epsilon > 0, \exists\, \delta = \delta(\epsilon) > 0$ such that for all $t_1,\ t_2,\ s \in [0,1]$ and $|t_1 - t_2| < \delta$, we have $|H_i(t_1,s) - H_i(t_2,s)| < \frac{\epsilon}{L_i}$ and $\left|\frac{\partial H_i(t_1,s)}{\partial t} - \frac{\partial H_i(t_2,s)}{\partial t}\right| < \frac{\epsilon}{L_i}$, where $L_i = \left[\frac{\lambda_i\|\varphi_i\|_1}{(1-\Delta_i)\Gamma(\beta_i)}\right]^{\frac{1}{p_i-1}}$. For all $(u_1,u_2) \in B$, we have

$$\begin{aligned}\left|T_i(u_1,u_2)(t_1) - T_i(u_1,u_2)(t_2)\right| &\leq \int_0^1 \left|H_i(t_1,s) - H_i(t_2,s)\right|\phi_{q_i}\left(\lambda_i\int_0^1 K_i(s,\tau)f_i\left(\tau,u_1(\tau),u_2(\tau),u_1'(\tau),u_2'(\tau)\right)d\tau\right)ds\\ &\leq \int_0^1 \frac{\epsilon}{L_i}\phi_{q_i}\left(\lambda_i\int_0^1 \frac{\varphi_i(\tau)}{(1-\Delta_i)\Gamma(\beta_i)}d\tau\right)ds < \frac{\epsilon}{L_i}\left[\frac{\lambda_i\|\varphi_i\|_1}{(1-\Delta_i)\Gamma(\beta_i)}\right]^{\frac{1}{p_i-1}} = \epsilon.\end{aligned}$$

Similarly, we get $\left|T_i'(u_1,u_2)(t_1) - T_i'(u_1,u_2)(t_2)\right| < \epsilon$. This means that $T_i : X \to E$ is equicontinuous on $[0,1]$. Therefore, $S : X \to X$ is equicontinuous on $[0,1]$. The proof of Lemma 3.3 is completed. □

Theorem 3.4. *Assume that* (H_1)-(H_3) *hold. Assume further that the following conditions are satisfied:*

(B_1) *There exist nonnegative functions* $a_i(t) \in L^1(J)(i=1,2)$ *and nonnegative nondecreasing functions* $\psi_i(i=1,2)$ *with respect to each variable* x_j $(j=1,2,3,4)$ *such that*

$$|f_i(t,x_1,x_2,x_3,x_4) - f_i(t,y_1,y_2,y_3,y_4)| \leq a_i(t)\psi_i(|x_1-y_1|,|x_2-y_2|,|x_3-y_3|,|x_4-y_4|), \quad i=1,\ 2;$$

(B_2) *There exist nonnegative functions* $b_i(t), c_i(t) \in L^1(J)(i=1,2)$ *such that*

$$b_i(t) \leq f_i(t,x,y,z,w) \leq c_i(t), \quad (t,x,y,z,w) \in (0,1)\times\mathbb{R}_+^4,\ i=1,\ 2;$$

(B_3) *For any* $s>0$, $\psi_i(s,s,s,s) \leq s,\ i=1,\ 2.$

Then we have drawn two conclusions as follows:

(i) *If* $p_i > 2,\ i=1,2$, *then the BVP* (1)-(2) *have a pair of unique solutions for* $\lambda_i \in (0,\ \lambda_i^*)$, *where*

$$\lambda_i^* = \left(\frac{a_i - \mu_i}{a_iN_i^*\int_0^1 G_{\alpha_i}(1,s)ds}\right)^{\frac{1}{q_i-1}}, \quad i=1,2,$$

$$N_i^* = (q_i-1)N_i\int_0^1 a_i(\tau)d\tau\left(N_i'\int_0^1 b_i(\tau)d\tau\right)^{q_i-2}, \quad i=1,2.$$

(ii) *If* $1 < p_i \leq 2,\ i=1,2$, *then the BVP* (1)-(2) *have a pair of unique solutions for* $\lambda_i \in (0,\ \lambda_i^{**})$, *where*

$$\lambda_i^{**} = \left(\frac{a_i - \mu_i}{a_iN_i^{**}\int_0^1 G_{\alpha_i}(1,s)ds}\right)^{\frac{1}{q_i-1}}, \quad i=1,2,$$

$$N_i^{**} = (q_i-1)N_i\int_0^1 a_i(\tau)d\tau\left(N_i\int_0^1 c_i(\tau)d\tau\right)^{q_i-2}, \quad i=1,2.$$

Proof. Now we prove that (i) holds. In this case $p_1, p_2 > 2$, we have $1 < q_1, q_2 < 2$. Defining the operator $S : X \to X$ as (17), we assert that the operator $S : X \to X$ is contraction. In fact, let $(x_1,y_1),(x_2,y_2) \in X$, by (14) and (B_2), we have

$$\int_0^1 K_i(s,\tau)f_i\left(\tau,x_1(\tau),y_1(\tau),x_1'(\tau),y_1'(\tau)\right)d\tau \geq N_i'\int_0^1 b_i(\tau)d\tau.$$

Thus, (B_1) and (B_3) associated with (7) of Lemma 2.4 yield

$$\begin{aligned}
&\left|\phi_{q_i}\left(\int_0^1 K_i(s,\tau)f_i\Big(\tau,x_1(\tau),y_1(\tau),x_1'(\tau),y_1'(\tau)\Big)d\tau\right)-\phi_{q_i}\left(\int_0^1 K_i(s,\tau)f_i\Big(\tau,x_2(\tau),y_2(\tau),x_2'(\tau),y_2'(\tau)\Big)d\tau\right)\right|\\
\le&(q_i-1)\left(N_i'\int_0^1 b_i(\tau)d\tau\right)^{q_i-2}\int_0^1 K_i(s,\tau)\left|f_i\Big(\tau,x_1(\tau),y_1(\tau),x_1'(\tau),y_1'(\tau)\Big)-f_i\Big(\tau,x_2(\tau),y_2(\tau),x_2'(\tau),y_2'(\tau)\Big)\right|d\tau\\
\le&(q_i-1)\left(N_i'\int_0^1 b_i(\tau)d\tau\right)^{q_i-2}N_i\int_0^1\left|f_i\Big(\tau,x_1(\tau),y_1(\tau),x_1'(\tau),y_1'(\tau)\Big)-f_i\Big(\tau,x_2(\tau),y_2(\tau),x_2'(\tau),y_2'(\tau)\Big)\right|d\tau\\
\le&(q_i-1)\left(N_i'\int_0^1 b_i(\tau)d\tau\right)^{q_i-2}N_i\int_0^1 a_i(\tau)\psi_i\Big(|x_1(\tau)-x_2(\tau)|,|y_1(\tau)-y_2(\tau)|,|x_1'(\tau)-x_2'(\tau)|,|y_1'(\tau)-y_2'(\tau)|\Big)d\tau\\
\le&(q_i-1)\left(N_i'\int_0^1 b_i(\tau)d\tau\right)^{q_i-2}N_i\int_0^1 a_i(\tau)\psi_i\Big(\|x_1-x_2\|_\infty,\|y_1-y_2\|_\infty,\|x_1'-x_2'\|_\infty,\|y_1'-y_2'\|_\infty\Big)d\tau\\
\le&(q_i-1)\left(N_i'\int_0^1 b_i(\tau)d\tau\right)^{q_i-2}N_i\int_0^1 a_i(\tau)\psi_i\left(\|x_1-x_2\|_*,\|y_1-y_2\|_*,\|x_1-x_2\|_*,\|y_1-y_2\|_*\right)d\tau\\
\le&(q_i-1)\left(N_i'\int_0^1 b_i(\tau)d\tau\right)^{q_i-2}N_i\int_0^1 a_i(\tau)d\tau\max\{\|x_1-x_2\|_*,\|y_1-y_2\|_*\}\\
=&(q_i-1)\left(N_i'\int_0^1 b_i(\tau)d\tau\right)^{q_i-2}N_i\int_0^1 a_i(\tau)d\tau\|(x_1-x_2,y_1-y_2)\|\\
=&N_i^*\|(x_1-x_2,y_1-y_2)\|.
\end{aligned}$$

So it follows from above inequality and Lemma 2.9 that

$$\begin{aligned}
&|T_i(x_1,y_1)(t)-T_i(x_2,y_2)(t)|\\
=&\left|\lambda_i^{q_i-1}\int_0^1 H_i(t,s)\Big[\phi_{q_i}\left(\int_0^1 K_i(s,\tau)f_i\Big(\tau,x_1(\tau),y_1(\tau),x_1'(\tau),y_1'(\tau)\Big)d\tau\right)\right.\\
&\left.-\phi_{q_i}\left(\int_0^1 K_i(s,\tau)f_i\Big(\tau,x_2(\tau),y_2(\tau),x_2'(\tau),y_2'(\tau)\Big)d\tau\right)\Big]ds\right|\\
\le&\frac{a_i\lambda_i^{q_i-1}N_i^*}{a_i-\mu_i}\int_0^1 G_{\alpha_i}(1,s)ds\|(x_1-x_2,y_1-y_2)\|\\
<&\frac{a_iN_i^*}{a_i-\mu_i}\int_0^1 G_{\alpha_i}(1,s)ds\times(\lambda_i^*)^{q_i-1}\|(x_1-x_2,y_1-y_2)\|=\|(x_1-x_2,y_1-y_2)\|,\ i=1,2.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|S(x_1,y_1)-S(x_2,y_2)\|=\left\|\Big(T_1(x_1,y_1)-T_1(x_2,y_2),T_2(x_1,y_1)-T_2(x_2,y_2)\Big)\right\|\\
=&\max\{\|T_1(x_1,y_1)-T_1(x_2,y_2)\|_*,\|T_2(x_1,y_1)-T_2(x_2,y_2)\|_*\}\\
<&\|(x_1-x_2,y_1-y_2)\|=\|(x_1,y_1)-(x_2,y_2)\|.
\end{aligned}$$

Therefore, we know that $S:X\to X$ is a contraction mapping. By Banach contraction principle, for $\lambda_i<\lambda_i^*$ ($i=1,2$), $S:X\to X$ has a unique fixed point $(u_1^*,u_2^*)\in X$, which is a pair of solutions of BVP (1)-(2). Similarly, we can easily show that (ii) holds. So we omit it. The proof is completed. □

Theorem 3.5. *Assume that* (H_1)-(H_3) *hold. If* $f_i^0=f_i^\infty=+\infty$ *and there exist constants* $\rho_1,r_1,r_2>0$ *such that*

$$f_i(t,x_1,x_2,x_3,x_4)<\phi_{p_i}(r_i\rho_1),\quad (t,x_1,x_2,x_3,x_4)\in[0,1]\times[0,\rho_1]^4,\ i=1,2,\tag{19}$$

then BVP (1)-(2) *have at least two pairs of positive solutions* (u_1^0, u_2^0) *and* (u_1^*, u_2^*) *satisfying* $0 < \|(u_1^0, u_2^0)\| < \rho_1 < \|(u_1^*, u_2^*)\|$ *for any* $\lambda_i \in \left(\underline{\Lambda}_i, \overline{\Lambda}_i\right)$*, where* $\underline{\Lambda}_i = (\gamma M_i')^{1-p_i}$*,* $\overline{\Lambda}_i = (r_i M_i)^{1-p_i}$*.*

Proof. Define the cone $K \subset X$ and the operator $S : X \to X$ as (18) and (17), respectively. From Lemma 3.3, we know that $S : K \to K$ is completely continuous. In view of $f_i^0 = +\infty$, there exist $\sigma_1 \in (0, \rho_1)$, $L_1 > 1$ such that $0 < u_1 + u_2 + u_1' + u_2' \le \sigma_1$ and

$$f_i(t, u_1, u_2, u_1', u_2') \ge L_1 \phi_{p_i}\left(u_1 + u_2 + u_1' + u_2'\right) \ge \phi_{p_i}(u_1 + u_2),\ \ 0 \le t \le 1. \tag{20}$$

Let $\Omega_{\sigma_1} = \{(u_1, u_2) \in K : \|(u_1, u_2)\| < \sigma_1\}$. Then for each $(u_1, u_2) \in \partial\Omega_{\sigma_1}$, by Lemma 2.9, (14) and (20), we have

$$\begin{aligned} T_i(u_1, u_2)(t) &= \int_0^1 H_i(t, s)\phi_{q_i}\left(\lambda_i \int_0^1 K_i(s, \tau) f_i\left(\tau, u_1(\tau), u_2(\tau), u_1'(\tau), u_2'(\tau)\right) d\tau\right) ds \\ &\ge \frac{a_i b_i \phi_{q_i}(\lambda_i)}{(a_i + b_i)(a_i - \mu_i)} \int_0^1 G_{\alpha_i}(1, s)\phi_{q_i}\left(N_i' \int_0^1 \phi_{p_i}(u_1(\tau) + u_2(\tau)) d\tau\right) ds \\ &= \frac{a_i b_i \phi_{q_i}(N_i')\phi_{q_i}(\lambda_i)}{(a_i + b_i)(a_i - \mu_i)} \int_0^1 G_{\alpha_i}(1, s)\phi_{q_i}\left(\int_0^1 \phi_{p_i}(u_1(\tau) + u_2(\tau)) d\tau\right) ds \\ &\ge \frac{a_i b_i \phi_{q_i}(N_i')\phi_{q_i}(\lambda_i)}{(a_i + b_i)(a_i - \mu_i)} \int_0^1 G_{\alpha_i}(1, s)\phi_{q_i}\left(\int_0^1 \phi_{p_i}(\gamma\|(u_1, u_2)\|)) d\tau\right) ds \\ &= \lambda_i^{q_i - 1}\gamma M_i'\|(u_1, u_2)\| > \underline{\Lambda}_i^{q_i - 1}\gamma M_i'\|(u_1, u_2)\| = \|(u_1, u_2)\|, \quad i = 1, 2. \end{aligned}$$

So $\|T_i(u_1, u_2)\|_* > \|(u_1, u_2)\|$ for each $(u_1, u_2) \in \partial\Omega_{\sigma_1}$, $i = 1, 2$. In consequence,

$$\|S(u_1, u_2)\| = \|(T_1(u_1, u_2), T_2(u_1, u_2))\| = \max\{\|T_1(u_1, u_2)\|_*, \|T_2(u_1, u_2)\|_*\} > \|(u_1, u_2)\|,$$

which implies $\|S(u_1, u_2)\| > \|(u_1, u_2)\|$ for $(u_1, u_2) \in \partial\Omega_{\sigma_1}$. Hence, according to Lemma 2.5, we obtain

$$i(S, \Omega_{\sigma_1}, K) = 0. \tag{21}$$

On the other hand, since $f_i^\infty = +\infty$, there exist $\sigma_3 > \rho_1, L_2 > 1$ such that $u_1 + u_2 + u_1' + u_2' \ge \sigma_3$ and

$$f_i(t, u_1, u_2, u_1', u_2') \ge L_2 \phi_{p_i}\left(u_1 + u_2 + u_1' + u_2'\right) \ge \phi_{p_i}(u_1 + u_2),\ \ 0 \le t \le 1.$$

Let $\sigma_2 \ge \frac{2\sigma_3}{\gamma}$ and $\Omega_{\sigma_2} = \{(u_1, u_2) \in K : \|(u_1, u_2)\| < \sigma_2\}$. Then for each $(u_1, u_2) \in \partial\Omega_{\sigma_2}$, we have $u_1(t) + u_2(t) \ge \gamma\|(u_1, u_2)\| = \gamma\sigma_2 \ge 2\sigma_3$. Similar to Lemma 3.2, we get $T_i(u_1, u_2)(t) > \|(u_1, u_2)\|$, $i = 1, 2$. So $\|T_i(u_1, u_2)\|_* > \|(u_1, u_2)\|$ for each $(u_1, u_2) \in \partial\Omega_{\sigma_2}$, which implies $\|S(u_1, u_2)\| > \|(u_1, u_2)\|$ for $(u_1, u_2) \in \partial\Omega_{\sigma_2}$. Hence, applying Lemma 2.5, we also get

$$i(S, \Omega_{\sigma_2}, K) = 0. \tag{22}$$

Let $\Omega_{\rho_1} = \{(u_1, u_2) \in K : \|(u_1, u_2)\| < \rho_1\}$. Then, for any $(u_1, u_2) \in \partial\Omega_{\rho_1}$, by Lemma 2.9, (14) and (19), we have

$$\begin{aligned} T_i(u_1, u_2)(t) &= \phi_{q_i}(\lambda_i) \int_0^1 H_i(t, s)\phi_{q_i}\left(\int_0^1 K_i(s, \tau) f_i\left(\tau, u_1(\tau), u_2(\tau), u_1'(\tau), u_2'(\tau)\right) d\tau\right) ds \\ &\le \frac{a_i \phi_{q_i}(\lambda_i)}{a_i - \mu_i} \int_0^1 G_{\alpha_i}(1, s)\phi_{q_i}\left(N_i \int_0^1 \phi_{p_i}(r_i\rho_1) d\tau\right) ds \\ &= \frac{a_i \phi_{q_i}(\lambda_i)}{a_i - \mu_i} \int_0^1 G_{\alpha_i}(1, s)\phi_{q_i}\left(N_i \phi_{p_i}(r_i\rho_1)\right) ds = \phi_{q_i}(\lambda_i) M_i r \rho_1 \\ &= \lambda_i^{q_i - 1} r_i M_i\|(u_1, u_2)\| < \overline{\Lambda}_i^{q_i - 1} r_i M_i\|(u_1, u_2)\| = \|(u_1, u_2)\|. \end{aligned}$$

Consequently, $\|S(u_1,u_2)\| < \|(u_1,u_2)\|$ for $(u_1,u_2) \in \partial\Omega_{\rho_1}$. Hence, in the light of Lemma 2.5, we also obtain

$$i(S,\Omega_{\rho_1},K) = 1. \tag{23}$$

By (21)-(23) and noting that $\sigma_1 < \rho_1 < \sigma_2$, we obtain

$$\begin{aligned} i(S,\Omega_{\rho_1}\backslash\overline{\Omega}_{\sigma_1},K) &= i(S,\Omega_{\rho_1},K) - i(S,\Omega_{\sigma_1},K) = 1,\\ i(S,\Omega_{\sigma_2}\backslash\overline{\Omega}_{\rho_1},K) &= i(S,\Omega_{\sigma_2},K) - i(S,\Omega_{\rho_1},K) = -1. \end{aligned}$$

Therefore, $S : X \to X$ have both nonzero fixed points $(u_1^0,u_2^0) \in \Omega_{\rho_1}\backslash\overline{\Omega}_{\sigma_1}$ and $(u_1^*,u_2^*) \in \Omega_{\sigma_2} \setminus \overline{\Omega}_{\rho_1}$. They are two pairs of positive solutions of BVP (1)-(2) with $0 < \|(u_1^0,u_2^0)\| < \rho_1 < \|(u_1^*,u_2^*)\|$. The proof of Theorem 3.5 is completed. □

Theorem 3.6. *Assume that* (H_1)-(H_3) *hold. If* $f_i^0 = f_i^\infty = 0$ *and there exist constants* $\rho_2, m_1, m_2 > 0$ *such that*

$$f_i(t,x_1,x_2,x_3,x_4) \geq \phi_{p_i}(m_i\rho_2), \quad (t,x_1,x_2,x_3,x_4) \in [0,1]\times[0,\rho_2]^4,\ i = 1,2, \tag{24}$$

then BVP (1)-(2) *have at least two pairs of positive solutions* (u_1^0,u_2^0) *and* (u_1^*,u_2^*) *satisfying* $0 < \|(u_1^0,u_2^0)\| < \rho_2 < \|(u_1^*,u_2^*)\|$ *for any* $\lambda_i \in (\underline{\Theta}_i,\overline{\Theta}_i)$, *where* $\underline{\Theta}_i = (m_iM_i)^{1-p_i}$, $\overline{\Theta}_i = (4M_i)^{1-p_i}$, $i = 1,2$.

Proof. Define the cone $K \subset X$ and the operator $S : X \to X$ as (18) and (17), respectively. From Lemma 3.3, we know that $S : K \to K$ is completely continuous. In view of $f_i^0 = 0$, there exist $\delta_1 \in (0,\rho_2)$, $0 < \varepsilon_1 \leq 1$ such that for $0 < u_1 + u_2 + u_1' + u_2' \leq \delta_1$ and

$$\begin{aligned} f_i(t,u_1,u_2,u_1',u_2') &\leq \varepsilon_1\phi_{p_i}\left(u_1+u_2+u_1'+u_2'\right) \leq \phi_{p_i}(\|u_1\|_\infty + \|u_2\|_\infty + \|u_1'\|_\infty + \|u_2'\|_\infty)\\ &\leq \phi_{p_i}\Big(2(\|u_1\|_\infty + \|u_2\|_\infty)\Big) \leq \phi_{p_i}\Big(4\|(u_1,u_2)\|\Big),\ 0 \leq t \leq 1. \end{aligned} \tag{25}$$

Let $\Omega_{\delta_1} = \{(u_1,u_2) \in K : \|(u_1,u_2)\| < \delta_1\}$. Then for each $(u_1,u_2) \in \partial\Omega_{\delta_1}$, by Lemma 2.9, (14) and (25), we have

$$\begin{aligned} T_i(u_1,u_2)(t) &= \phi_{q_i}(\lambda_i)\int_0^1 H_i(t,s)\phi_{q_i}\left(\int_0^1 K_i(s,\tau)f_i\left(\tau,u_1(\tau),u_2(\tau),u_1'(\tau),u_2'(\tau)\right)d\tau\right)ds\\ &\leq \frac{a_i\phi_{q_i}(\lambda_i)}{a_i-\mu_i}\int_0^1 G_{\alpha_i}(1,s)\phi_{q_i}\left(N_i\int_0^1 \phi_{p_i}\Big(4\|(u_1,u_2)\|\Big)d\tau\right)ds\\ &= 4M_i\phi_{q_i}(\lambda_i)\|(u_1,u_2)\| < 4M_i\overline{\Theta}_i^{q_i-1}\|(u_1,u_2)\| = \|(u_1,u_2)\|, \quad i = 1,2. \end{aligned}$$

So $\|T_i(u_1,u_2)\|_* < \|(u_1,u_2)\|$ for each $(u_1,u_2) \in \partial\Omega_{\delta_1}, i = 1,2$. Consequently,

$$\|S(u_1,u_2)\| = \|(T_1(u_1,u_2),T_2(u_1,u_2))\| = \max\{\|T_1(u_1,u_2)\|_*, \|T_2(u_1,u_2)\|_*\} < \|(u_1,u_2)\|,$$

which implies $\|S(u_1,u_2)\| < \|(u_1,u_2)\|$ for $(u_1,u_2) \in \partial\Omega_{\delta_1}$. Hence, according to Lemma 2.5, we have

$$i(S,\Omega_{\sigma_1},K) = 1. \tag{26}$$

On the other hand, since $f_i^\infty = 0$, there exist $\delta_3 > \rho_2$, $0 < \varepsilon_2 \leq 1$ such that $u_1 + u_2 + u_1' + u_2' \geq \delta_3$ and

$$f_i(t,u_1,u_2,u_1',u_2') \leq \varepsilon_2\phi_{p_i}\left(u_1+u_2+u_1'+u_2'\right) \leq \phi_{p_i}\Big(4(\|(u_1,u_2)\|)\Big),\ 0 \leq t \leq 1.$$

Next the discussion is divided into two cases.

Case 1. Suppose that f_i are bounded. In view of $f_i : [0,1]\times\Omega \to [0,\infty)$ are continuous, there exists $M_i^* >$

$0(i=1,2)$ such that $f_i(t,u_1,u_2,u_1',u_2') \le \phi_{p_i}(M_i^*)$ for all $(t,u_1,u_2,u_1',u_2') \in [0,1]\times[0,\infty)^4$. Take $\delta_4 \ge \max\{\frac{M_i^*}{4},\delta_3\}$. Then, for $(u_1,u_2)\in K$ with $\|(u_1,u_2)\| = \delta_4$, we get

$$\begin{aligned} T_i(u_1,u_2)(t) &= \int_0^1 H_i(t,s)\phi_{q_i}\left(\lambda_i\int_0^1 K_i(s,\tau)f_i\left(\tau,u_1(\tau),u_2(\tau),u_1'(\tau),u_2'(\tau)\right)d\tau\right)ds\\ &\le \frac{a_i\phi_{q_i}(\lambda_i)}{a_i-\mu_i}\int_0^1 G_{\alpha_i}(1,s)\phi_{q_i}\Big(N_i\phi_{p_i}(M_i^*)\Big)ds\\ &= M_iM_i^*\phi_{q_i}(\lambda_i) < \frac{M_i^*}{4} \le \delta_4 = \|(u_1,u_2)\|, \quad i=1,2. \end{aligned}$$

Case 2. Suppose that f_i are unbounded. In view of f_i are continuous, there exist $t'\in[0,1]$ and $\delta_5 \ge \max\{\delta_3,\rho_2\}$ such that $f_i(t,u_1,u_2,u_1',u_2') \le f_i(t',\delta_5,\delta_5,\delta_5,\delta_5)$ for all $(t,u_1,u_2,u_1',u_2')\in[0,1]\times[0,\delta_5]^4$, $i=1,2$. Then, for $(u_1,u_2)\in K$ with $\|(u_1,u_2)\| = \delta_5$, we get

$$\begin{aligned} T_i(u_1,u_2)(t) &= \phi_{q_i}(\lambda_i)\int_0^1 H_i(t,s)\phi_{q_i}\left(\int_0^1 K_i(s,\tau)f_i\left(\tau,u_1(\tau),u_2(\tau),u_1'(\tau),u_2'(\tau)\right)d\tau\right)ds\\ &\le \phi_{q_i}(\lambda_i)\int_0^1 H_i(t,s)\phi_{q_i}\left(\int_0^1 K_i(s,\tau)f_i\left(t',\delta_5,\delta_5,\delta_5,\delta_5\right)d\tau\right)ds\\ &\le \frac{a_i\phi_{q_i}(\lambda_i)}{a_i-\mu_i}\int_0^1 G_{\alpha_i}(1,s)\phi_{q_i}\left(N_i\int_0^1\phi_{p_i}\Big(4\|(u_1,u_2)\|\Big)d\tau\right)ds\\ &= 4M_i\phi_{q_i}(\lambda_i)\|(u_1,u_2)\| < 4M_i\overline{\Theta}_i^{q_i-1}\|(u_1,u_2)\| = \|(u_1,u_2)\|, \quad i=1,2. \end{aligned}$$

So, if we always choose $\Omega_{\delta_2} = \{(u_1,u_2)\in K : \|(u_1,u_2)\| < \delta_2 = \max\{\delta_4,\delta_5\}$, then we have $\|S(u_1,u_2)\| < \|(u_1,u_2)\|$ for $(u_1,u_2)\in\partial\Omega_{\delta_2}$. Therefore, by Lemma 2.5, we also get

$$i(S,\Omega_{\sigma_2},K) = 1. \tag{27}$$

Let $\Omega_{\rho_2} = \{(u_1,u_2)\in K : \|(u_1,u_2)\| < \rho_2\}$. Then, for any $(u_1,u_2)\in\partial\Omega_{\rho_2}$, similar to the above arguments, we obtain

$$\begin{aligned} T_i(u_1,u_2)(t) &= \int_0^1 H_i(t,s)\phi_{q_i}\left(\lambda_i\int_0^1 K_i(s,\tau)f_i\left(\tau,u_1(\tau),u_2(\tau),u_1'(\tau),u_2'(\tau)\right)d\tau\right)ds\\ &\ge \frac{a_ib_i\phi_{q_i}(\lambda_i)}{(a_i+b_i)(a_i-\mu_i)}\int_0^1 G_{\alpha_i}(1,s)\phi_{q_i}\left(N_i'\int_0^1\phi_{p_i}(m_i\rho_2)d\tau\right)ds\\ &= \lambda_i^{q_i-1}m_iM_i'\|(u_1,u_2)\| > \underline{\Theta}_i^{q_i-1}m_iM_i'\|(u_1,u_2)\| = \|(u_1,u_2)\|, \quad i=1,2. \end{aligned}$$

Consequently, $\|S(u_1,u_2)\| > \|(u_1,u_2)\|$ for $(u_1,u_2)\in\partial\Omega_{\rho_2}$. Applying Lemma 2.5 again, we derive

$$i(S,\Omega_{\rho_2},K) = 0. \tag{28}$$

By (22)-(24) and noticing that $\delta_1 < \rho_2 < \delta_2$, we have

$$\begin{aligned} i(S,\Omega_{\rho_2}\setminus\overline{\Omega}_{\delta_1},K) &= i(S,\Omega_{\rho_2},K) - i(S,\Omega_{\delta_1},K) = -1,\\ i(S,\Omega_{\delta_2}\setminus\overline{\Omega}_{\rho_2},K) &= i(S,\Omega_{\delta_2},K) - i(S,\Omega_{\rho_2},K) = 1. \end{aligned}$$

Therefore, S have both nonzero fixed points $(u_1^0,u_2^0)\in\Omega_{\rho_2}\setminus\overline{\Omega}_{\delta_1}$ and $(u_1^*,u_2^*)\in\Omega_{\delta_2}\setminus\overline{\Omega}_{\rho_2}$. They are two pairs of positive solutions of BVP (1)-(2) with $0 < \|(u_1^0,u_2^0)\| < \rho_2 < \|(u_1^*,u_2^*)\|$. The proof of theorem 3.6 is completed. □

4. Illustrative Example

Consider the following BVP with p-Laplacian, for $t \in J = [0,1]$:

$$\begin{cases} D_{0+}^{\frac{3}{2}}\left((\phi_{p_1}\left(D_{0+}^{\frac{5}{2}}u_1(t)\right)\right) = \lambda_1 f_1\big(t, u_1(t), u_2(t), u_1'(t), u_2'(t)\big), \\ D_{0+}^{\frac{4}{3}}\left((\phi_{p_2}\left(D_{0+}^{\frac{7}{3}}u_2(t)\right)\right) = \lambda_2 f_2\big(t, u_1(t), u_2(t), u_1'(t), u_2'(t)\big), \end{cases} \tag{29}$$

with the boundary value conditions

$$\begin{cases} u_1(0) - 2u_1'(0) = \int_0^1 e^{-t}u_1(t)dt, \quad u_1'(1) = u_1''(0) = 0, \\ 2u_2(0) - 3u_2'(0) = \int_0^1 (\cos t + 1)u_2(t)dt, \quad u_2'(1) = u_2''(0) = 0, \\ \phi_{p_1}\left(D_{0+}^{\frac{5}{2}}u_1(1)\right) = \int_0^1 \phi_{p_1}\left(D_{0+}^{\frac{5}{2}}u_1(s)\right)dA_1(s), \quad \left(\phi_{p_1}\left(D_{0+}^{\frac{5}{2}}u_1(0)\right)\right)' = 0, \\ \phi_{p_2}\left(D_{0+}^{\frac{7}{3}}u_2(1)\right) = \int_0^1 \phi_{p_2}\left(D_{0+}^{\frac{7}{3}}u_2(s)\right)dA_2(s), \quad \left(\phi_{p_2}\left(D_{0+}^{\frac{7}{3}}u_2(0)\right)\right)' = 0, \end{cases} \tag{30}$$

where $\alpha_1 = \frac{5}{2}$, $\alpha_2 = \frac{7}{3}$, $\beta_1 = \frac{3}{2}$, $\beta_2 = \frac{4}{3}$, $p_1 = \frac{3}{2}$, $p_2 = 2$, $a_1 = 1$, $a_2 = b_1 = 2$, $b_2 = 3$, $\varphi_1(t) = e^{-t}$, $\varphi_2(t) = \cos t + 1$ and $A_1(t) = A_2(t) = \frac{e^t+1}{2}$. In addition, we set

$$f_1(t, x_1, x_2, x_3, x_4) = \frac{e^{-2t}}{3200}\left[(x_1 + x_2 + x_3 + x_4)^{\frac{1}{4}} + (x_1 + x_2 + x_3 + x_4)^2\right],$$
$$f_2(t, x_1, x_2, x_3, x_4) = \frac{e^{-3t}}{20000}\left[(x_1 + x_2 + x_3 + x_4)^{\frac{1}{2}} + (x_1 + x_2 + x_3 + x_4)^3\right].$$

Clearly, for $t \in [0,1]$ and $x_i > 0$ $(i = 1,2,3,4)$, $f_1(t, x_1, x_2, x_3, x_4) > 0$, $f_2(t, x_1, x_2, x_3, x_4) > 0$. By the simple calculation, we get $0 \le \mu_1 = 1 - \frac{1}{e} < 1 < a_1 = 1$, $0 \le \mu_2 = 1 - \sin 1 < a_2 = 2$, $0 \le \Delta_i = \frac{e-1}{2} < 1$ $(i = 1,2)$, $f_i^0 = f_i^\infty = \infty$ $(i = 1,2)$. Let $\rho_1 = 1$, because $f_i(t, x_1, x_2, x_3, x_4)$ is monotone increasing function for each $x_1, x_2, x_3, x_4 \ge 0$, taking $r_1 = \frac{1}{1600}$, $r_2 = \frac{1}{250}$, for $(t, x_1, x_2, x_3, x_4) \in [0,1] \times [0,1]^4$, we have

$$f_1(t, x_1, x_2, x_3, x_4) \le \frac{1}{3200}\left(4^{\frac{1}{4}} + 4^2\right) \approx 0.0109 < \phi_{p_1}(r_1\rho_1) = \left(\frac{1}{1600}\right)^{\frac{1}{2}} = 0.0250,$$
$$f_2(t, x_1, x_2, x_3, x_4) \le \frac{1}{20000}\left(4^{\frac{1}{2}} + 4^3\right) = 0.0033 < \phi_{p_2}(r_2\rho_1) = \frac{1}{250} = 0.0040,$$
$$M_1 = \frac{a_1\phi_{q_1}(N_1)}{a_1 - \mu_1}\int_0^1 G_{\alpha_1}(1,s)ds \approx 341.1709, \quad M_1' = \frac{a_1b_1\phi_{q_1}(N_1')}{(a_1 + b_1)(a_1 - \mu_1)}\int_0^1 G_{\alpha_1}(1,s)ds \approx 7.3342,$$
$$M_2 = \frac{a_2\phi_{q_2}(N_2)}{a_2 - \mu_2}\int_0^1 G_{\alpha_2}(1,s)ds \approx 174.4957, \quad M_2' = \frac{a_2b_2\phi_{q_2}(N_2')}{(a_2 + b_2)(a_2 - \mu_2)}\int_0^1 G_{\alpha_2}(1,s)ds \approx 12.5337.$$

Therefore,

$$\underline{\Lambda}_1 = (\gamma M_1')^{1-p_1} \approx 0.4767, \quad \overline{\Lambda}_1 = (r_1M_1)^{1-p_1} \approx 2.1656,$$
$$\underline{\Lambda}_2 = (\gamma M_2')^{1-p_2} \approx 0.1330, \quad \overline{\Lambda}_2 = (r_2M_2)^{1-p_2} \approx 1.4327.$$

Thus, all conditions of Theorem 3.5 hold. According to Theorem 3.5, when $\lambda_1 \in (0.4767, 2.1656)$ and $\lambda_2 \in (0.1330, 1.4327)$, BVP (29)-(30) have at least two pairs of positive solutions (u_1^0, u_2^0) and (u_1^*, u_2^*) satisfying $0 < \|(u_1^0, u_2^0)\| < 1 < \|(u_1^*, u_2^*)\|$.

5. Conclusions

In this paper, we study the integral boundary value problem for a class of nonlinear fractional order differential coupling system with eigenvalue argument and p-Laplacian. By the fixed point index theorem, some sufficient conditions have been obtained to ensure the existence and multiplicity of positive solution. We also give the range of eigenvalue parameter. Compared with the single fractional differential equation, the study of fractional order coupling system is more complicated and challenging since it is difficult to find the Green function of system (1). Our results are new and interesting. Our methods can be used to study the existence of positive solutions for the high order or multiple-point boundary value problems of nonlinear fractional differential coupling system. However, there exist some difficulties and complexities to address the structure of the Green's function for these boundary value problems.

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