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Hartwig's Triple Reverse Order Law in C*-algebras

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Abstract. In this paper Hartwig's triple reverse order law for the Moore-Penrose inverse is proved for C*-algebras. A very simple algebraic proof for Hartwig's triple reverse order law for operators on Hilbert spaces is given.

1. Introduction

Let $\mathcal{A}$ be a complex unital C*-algebra. An element $a \in \mathcal{A}$ is said to be *regular* (in the sense of von Neumann) if there exists $b \in \mathcal{A}$ for which $aba = a$; any such b is called an *inner inverse* of a.
An element $x \in \mathcal{A}$ which satisfies the four Penrose equations [7], [1],

$$(1)\ axa = a, \quad (2)\ xax = x, \quad (3)\ (ax)^* = ax, \quad (4)\ (xa)^* = xa,$$

if it exists, is called the Moore-Penrose inverse of a and is denoted by $a^\dagger$. From the definition of the Moore-Penrose inverse, we conclude that both $a^\dagger a$ and $aa^\dagger$ are projections, where by a projection we mean an element $e \in \mathcal{A}$ which is a hermitian idempotent, i.e. such that $e^2 = e = e^*$. A Moore-Penrose inverse is unique if it exists, and this is the case exactly when $a \in \mathcal{A}$ is regular (see [6]):

$$a \text{ is regular } \Leftrightarrow a\mathcal{A} \text{ is closed } \Leftrightarrow a^\dagger \text{ exists.}$$

An element $a \in \mathcal{A}$ is EP if there exists $a^\dagger$ and $aa^\dagger = a^\dagger a$. (See [10].) For $K \subseteq \{1,2,3,4\}$, we shall call $x \in \mathcal{A}$ a K-*inverse* of $a \in \mathcal{A}$ if it satisfies the Penrose equation (j) for each $j \in K$. We shall write aK for the collection of all K-inverses of $a \in \mathcal{A}$, and a^K for an unspecified element $x \in aK$.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and let $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ denote the set of all bounded linear operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. For $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, $\mathcal{R}(A)$ denote the range of A. It is well- known that for $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, the Moore-Penrose inverse of A exists if and only if $\mathcal{R}(A)$ is closed.

The reverse order law for the Moore-Penrose inverse seems first to have been studied by Greville In [8] , in the '60s , giving a necessary and sufficient condition for the reverse order law

$$(AB)^\dagger = B^\dagger A^\dagger,$$

for matrices A and B. This has been followed by Hartwig [3], who studied the reverse order law for the Moore-Penrose inverse of products of three matrices. Suppose A, B and C are complex matrices for which ABC can be defined. We use notations

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$$P = A^{\dagger}ABCC^{\dagger}, \qquad Q = CC^{\dagger}B^{\dagger}A^{\dagger}A. \tag{1}$$

Theorem 1.1. [3] *The following conditions are equivalent:*

(i) $(ABC)^{\dagger} = C^{\dagger}B^{\dagger}A^{\dagger}$;

(ii) $Q \in P\{1,2\}$ *and both of* A^*APQ *and* $QPCC^*$ *are hermitian;*

(iii) $Q \in P\{1,2\}$ *and both of* A^*APQ *and* $QPCC^*$ *are EP;*

(iv) $Q \in P\{1\}$, $\mathcal{R}(A^*AP) = \mathcal{R}(Q^*)$ *and* $\mathcal{R}(CC^*P^*) = \mathcal{R}(Q)$;

(v) $PQ = PQPQ$, $\mathcal{R}(A^*AP) = \mathcal{R}(Q^*)$ *and* $\mathcal{R}(CC^*P^*) = \mathcal{R}(Q)$.

Hartwig's proof of this result is valid, with some comments, for the operators on infinite dimensional Hilbert spaces except the proof of implication (v) $\Rightarrow$ (ii) witch use matrix rang. In this paper we will present a very simple algebraic proof of Hartwig's result for the regular elements in C*-algebra. Notice that one generalization on Hartwig's result is given in [4] for the case of closed-range bounded linear operators on infinite dimensional Hilbert spaces based on operator matrices.
For huge number of different reverse order laws see [2]. Also, some interesting results on the reverse order law can be founded in the following papers [11–18].

2. Result

For regular elements a, b and c of C*-algebra $\mathcal{A}$ we use notations

$$p = a^{\dagger}abcc^{\dagger}, \qquad q = cc^{\dagger}b^{\dagger}a^{\dagger}a,$$

analogously to (1).

Theorem 2.1. *Let* $\mathcal{A}$ *be a complex unital C*-algebra and let* $a, b, c \in \mathcal{A}$ *be such that* a, b, c *and* abc *are regular. Then the following conditions are equivalent:*

(i) $(abc)^{\dagger} = c^{\dagger}b^{\dagger}a^{\dagger}$;

(ii) $q \in p\{1,2\}$ *and both of* a^*apq *and* $qpcc^*$ *are hermitian;*

(iii) $q \in p\{1,2\}$ *and both of* a^*apq *and* $qpcc^*$ *are EP;*

(iv) $q \in p\{1\}$, $a^*ap\mathcal{A} = q^*\mathcal{A}$ *and* $cc^*p^*\mathcal{A} = q\mathcal{A}$;

(v) $pq = pqpq$, $a^*ap\mathcal{A} = q^*\mathcal{A}$ *and* $cc^*p^*\mathcal{A} = q\mathcal{A}$.

Proof. (i) $\Leftrightarrow$ (ii) : This can be showed exactly as in [3].

(ii) $\Rightarrow$ (iii) : This will follows if we show that a^*apq and $qpcc^*$ are regular. Indeed, we can check that $a^{\dagger}(a^{\dagger})^* \in (a^*apq)\{1\}$ and $(c^{\dagger})^*c^{\dagger} \in (qpcc^*)\{1\}$:

$$a^*apqa^{\dagger}(a^{\dagger})^*a^*apq = a^*apqa^{\dagger}apq = a^*apqpq = a^*apq,$$

$$qpcc^*(c^{\dagger})^*c^{\dagger}qpcc^* = qpcc^{\dagger}qpcc^* = qpqpcc^* = qpcc^*.$$

(iii) $\Rightarrow$ (iv) : Since $a^*ap\mathcal{A} = q^*\mathcal{A}$ is equivalent with the facts that $a^*ap \in q^*\mathcal{A}$ and $q^* \in a^*ap\mathcal{A}$, we have

$$a^*ap = a^*apqp = a^*apq(a^*apq)^\dagger a^*apqp = (a^*apq)^\dagger a^*apqa^*ap = q^*p^*a^*a((a^*apq)^\dagger)^*a^*ap \in q^*\mathcal{A},$$

and

$$\begin{aligned} q^* &= q^*p^*q^* = q^*p^*a^\dagger aq^* = q^*p^*a^*(a^\dagger)^*q^* = q^*p^*a^*aa^\dagger(a^\dagger)^*q^* = (a^*apq(a^*apq)^\dagger a^*apq)^*a^\dagger(a^\dagger)^*q^* \\ &= a^*apq(a^*apq)^\dagger q^* \in a^*ap\mathcal{A}. \end{aligned}$$

Similarly, $cc^*p^*\mathcal{A} = q\mathcal{A}$ is equivalent with the facts that $cc^*p^* \in q\mathcal{A}$ and $q \in cc^*p^*\mathcal{A}$, so we have

$$cc^*p^* = cc^*p^*q^*p^* = (qpcc^*(qpcc^*)^\dagger qpcc^*)^*p^* = qpcc^*(qpcc^*)^\dagger cc^*p^* \in q\mathcal{A},$$

and

$$q = qpq = qpcc^\dagger q = qpcc^*(c^\dagger)^*c^\dagger q = qpcc^*(qpcc^*)^\dagger qpcc^*(c^\dagger)^*c^\dagger q = cc^*p^*q^*((qpcc^*)^\dagger)^*q \in cc^*p^*\mathcal{A}.$$

$(iv) \Rightarrow (v)$: Trivial.

$(v) \Rightarrow (ii)$: First we will show that pc and $qa^\dagger$ are regular. Indeed, $pc = a^\dagger abc$ and $a^\dagger abc(abc)^\dagger aa^\dagger abc = a^\dagger abc$. Also, $cc^*p^*((pc)^\dagger)^*c^\dagger cc^*p^* = cc^*p^*$, so cc^*p^* is regular and then, since $qa^\dagger \in q\mathcal{A} = cc^*p^*\mathcal{A}$ and $cc^*p^*(cc^*p^*)^\dagger \in cc^*p^*\mathcal{A} = q\mathcal{A}$ we have $qa^\dagger = cc^*p^*x = cc^*p^*(cc^*p^*)^\dagger cc^*p^*x = qycc^*p^*x = qyqa^\dagger = qa^\dagger ayqa^\dagger$. Hence $qa^\dagger$ is regular. Now, analogously using $cc^*p^*\mathcal{A} = q\mathcal{A}$, we get

$$p = pcc^\dagger = pc(pc)^\dagger pcc^\dagger = pcc^*p^*((pc)^\dagger)^*c^\dagger = pqu,$$

and consequently $pqp = pqpqu = pqu = p$. This shows that $q \in p\{1\}$ and $qpqp = qp$. Also, using $a^*ap\mathcal{A} = q^*\mathcal{A}$, we get

$$q = qa^\dagger a = qa^\dagger(qa^\dagger)^\dagger qa^\dagger a = qa^\dagger(a^\dagger)^*q^*((qa^\dagger)^\dagger)^*a = qa^\dagger(a^\dagger)^*a^*apv = qa^\dagger apv = qpv,$$

which gives $qpq = qpqpv = qpv = q$. To complete the proof notice that, by $a^*ap\mathcal{A} = q^*\mathcal{A}$ and $cc^*p^*\mathcal{A} = q\mathcal{A}$,

$$q^*p^*a^*apq = q^*p^*q^*t = q^*t = a^*apq$$

and

$$qpcc^*p^*q^* = qpqz = qz = cc^*p^*q^*$$

imply that a^*apq and $qpcc^*$ are hermitian. □

Remark 2.1. *In the case when A, B and C are bounded linear operators on Hilbert space $\mathcal{H}$ can be seen by Theorem* 1. *from [5] that condition* (iv) $((v))$ *from Theorems* 1.1 *and* 2.1 *are equivalent.*

Remark 2.2. *Let $\mathcal{H}_i, i = \overline{1,4}$ be arbitrary Hilbert spaces, $C \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2), B \in \mathcal{B}(H_2, \mathcal{H}_3)$ and $A \in \mathcal{B}(\mathcal{H}_3, \mathcal{H}_4)$ bounded operators with closed ranges such that ABC has closed range. Hartwig's proof of Theorem* 1.1 *can be improved for the case of closed range operators with pure algebraic technique similarly as in proof of Theorem* 2.1. *Namely, the regularity of elements A^*APQ and $QPCC^*$ can be shown as in the proof of Theorem* 2.1. *Now, as we said, the proof given by Hartwig's stay valid except for the implication* $(v) \Rightarrow (ii)$. *The regularity of element PC can be verified as in the proof of Theorem* 2.1, *and now as in [3] we get that $PQP = P$ and consequently $QPQP = QP$. To see that the element $QA^\dagger$ is regular notice that $\mathcal{R}(PCC^*) = \mathcal{R}(PC)$ is closed and consequently $\mathcal{R}(QA^\dagger) = \mathcal{R}(Q) = \mathcal{R}(CC^*P^*)$ is closed. Now, using $\mathcal{R}(Q^*) = \mathcal{R}(A^*AP)$, follows*

$$\mathcal{R}(Q) = \mathcal{R}(QA^\dagger) = \mathcal{R}(QA^\dagger(QA^\dagger)^*) = \mathcal{R}(QA^\dagger(A^\dagger)^*Q^*) = \mathcal{R}(QA^\dagger(A^\dagger)^*A^*AP) = \mathcal{R}(QA^\dagger AP) = \mathcal{R}(QP)$$

and now, since QP is idempotent with range $\mathcal{R}(Q)$ then $QPQ = Q$. The rest of the proof is as in [3].

Remark 2.3. *Let us mention for some special cases when triple reverse order low for the Moore-Penrose inverse of products of three regular elements a, b and c of C^* algebra $\mathcal{A}$ holds.*
If a is unitary we get that

$$(abc)^\dagger = c^\dagger b^\dagger a^\dagger \Leftrightarrow (bc)^\dagger = c^\dagger b^\dagger.$$

Similarly, if c is unitary

$$(abc)^\dagger = c^\dagger b^\dagger a^\dagger \Leftrightarrow (ab)^\dagger = b^\dagger a^\dagger.$$

The case when b is unitary is not trivial as previous two, but can be deduce easily from known result. For elements $x, y \in \mathcal{A}$ set $[x, y] = xy - yx$. In an analogical manner as in Theorem 3. from [9] can be shown:

Theorem 2.2. *Let $\mathcal{A}$ be a complex unital C^*-algebra, let $a, b, c \in \mathcal{A}$ be regular elements and let b be unitary. Then the following conditions are equivalent:*

(*i*) *abc is regular and $(abc)^\dagger = c^\dagger b^\dagger a^\dagger$,*

(*ii*) $[bcc^\dagger b^\dagger, a^*a] = 0$ *and* $[b^\dagger a^\dagger ab, cc^*] = 0$.

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