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Existence of Ψ-bounded Solutions for Sylvester Matrix Dynamical Systems on Time Scales

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Abstract. In this paper, we study the existence criteria for Ψ-bounded solutions of Sylvester matrix dynamical systems on time scales. The advantage of studying this system is it unifies continuous and discrete systems. First, we prove a necessary and sufficient condition for the existence of atleast one Ψ-bounded solution for Sylvester matrix dynamical systems on time scales, for every Lebesgue Ψ- deltaintegrable function F, on time scale $\mathbb{T}^+$. Further, we obtain a result relating to asymptotic behavior of Ψ-bounded solutions of this equation. The results are illustrated with suitable examples.

1. Introduction

Sylvester matrix and Lyapunov matrix differential equations arise in a number of areas of applied mathematics such as control systems, dynamic programming, optimal filters, quantum mechanics and systems engineering. The aim of this paper is to give necessary and sufficient condition so that linear Sylvester matrix dynamical system

$$X^{\Delta}(t) = A(t)X(t) + X(t)B(t) + \mu(t)A(t)X(t)B(t) + F(t), \tag{1.1}$$

has at least one Ψ-bounded solution for every Lebesgue Ψ- deltaintegrable function F, on time scale $\mathbb{T}^+$. Here Ψ is regressive and rd-continuous matrix function. The calculus of time scales was initiated by Stefan Hilger(1988) in order to create a theory that can unify discrete and continuous analysis. The study of dynamic equations on time scales, is an area of mathematics that has recently received a lot of attention and sheds new light on the discrepancies between continuous differential equations and discrete difference equations. It also prevents one from proving a result twice, once for differential equations and once for difference equations. The general idea, which is the main goal of Bohner and Peterson's excellent introductory text [5], is to prove a result for a dynamic equation where the domain of the unknown function is so-called time scale.

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$ and the system (1.1) become Sylvester matrix differential system,

$$X^{'}(t) = A(t)X(t) + X(t)B(t) + F(t). \tag{1.2}$$

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If $\mathbb{T} = \mathbb{Z}$, then $\mu(t) = 1$ and the system (1.1) become Sylvester matrix delta difference system,

$$\Delta X(t) = A(t)X(t) + X(t)B(t) + A(t)X(t)B(t) + F(t). \tag{1.3}$$

In the above system (1.3), if we put $A(t) = A_1(t) - I_n$ and $B(t) = B_1(t) - I_n$ then the system becomes the following matrix difference system

$$X(t + 1) = A_1(t)X(t)B_1(t). \tag{1.4}$$

Therefore, study of behavior of Ψ-bounded solutions of Sylvester matrix dynamical system (1.1) unify the study of (1.2), (1.3), (1.4) and extended to matrix dynamical systems on time scales. The analytical, numerical solutions and control aspects of Sylvester matrix differential equations was studied by Fausset [14]. The existence and uniqueness of matrix difference system (1.3) was studied by Murthy [18]. Recently, Suresh Kumar et al. [[21]-[23]] studied the existence of Ψ-bounded solutions for matrix difference system (1.4).

The basic problem under consideration is to determine necessary and sufficient conditions for the existence of a solution with some specified boundedness conditions. Classical results of this type, for linear and nonlinear differential equations were given by Coppel [9] and for difference equations by Agarwal [1]. The problem of Ψ-bounded solutions for the systems of linear ordinary differential equations as well as difference equations have been studied by many authors [[2], [4], [6]-[8], [10]-[13], [17]]. Recently [[13], [19]-[23]] extended the concept of Ψ-bounded solutions for Lyapunov matrix differential and difference equations. The present work unify the results of existence of Ψ-bounded solutions of linear differential equations [13] and linear difference equations [22] and also generalizes to Sylvester matrix dynamical systems on time scales. Here we present a necessary and sufficient condition for the existence of at least one Ψ-bounded solution for linear non-homogenous Sylvester matrix dynamical systems (1.1), using the technique of Kronecker product of matrices on $\mathbb{T}^+$ for every Lebesgue Ψ- deltaintegrable function F, on time scale $\mathbb{T}^+$.

2. Preliminaries

In this section, we present some basic notations, definitions and results of time scales and Kronecker product of matrices. For more details about time scales refer [5] and Kronecker products of matrices refer [3].

Let $\mathbb{T}$ be a time scale, i.e., an arbitrary nonempty closed subset of real numbers. Throughout this paper, the time scale $\mathbb{T}$ is assumed to be unbounded above and below.

In this paper, to facilitate the discussion below, we introduce some notation: $\mathbb{T}^+ = [0,\infty) \cap \mathbb{T}$, $v = \min\{[0,\infty) \cap \mathbb{T}\}$. Denote $\mathbb{R}^d$ the Euclidean d-space. For $x = [x_1, x_2, \ldots, x_d]^* \in \mathbb{R}^d$ (* denotes transpose), let $\|x\| = \max\{|x_1|, |x_2|, \ldots |x_d|\}$ be the norm of x. For a $d \times d$ real matrix $A = \left[a_{ij}\right]$. We define the matrix norm $|A| = \sup_{\|x\| \le 1} \|Ax\|$.

Definition 2.1. *[5] The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$, the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$, and the graininess function $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined by*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \mu(t) = \sigma(t) - t \quad \text{for} \quad t \in \mathbb{T},$$

respectively. If $\sigma(t) = t$, then t is called right-dense (otherwise: right-scattered), and if $\rho(t) = t$, then t is called left-dense (otherwise: left-scattered).

Definition 2.2. *[5] If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$. Otherwise $\mathbb{T}^k = \mathbb{T}$.*

Since $\mathbb{T}$ is assumed to be unbounded above and below, then we have $\mathbb{T}^k = \mathbb{T}$ throughout this paper.

Definition 2.3. *[5] Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}$. Then $f^{\Delta}(t)$ to be the number(provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t (i.e. $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that for all $s \in U$,*

$$\left| [f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s] \right| \leq \epsilon |\sigma(t) - s|.$$

In this case, $f^{\Delta}(t)$ is called the delta (or Hilger) derivative of f at t. Moreover, f is said to be delta (or Hilger) differentiable on $\mathbb{T}$ if $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$.

Definition 2.4. *[5] Let A be an $m \times n$ matrix valued function on time scale $\mathbb{T}$. We say that A is delta differentiable on $\mathbb{T}$ provided each entry of A is delta differentiable on $\mathbb{T}$ and in this case we put*

$$A^{\Delta} = \left[a_{ij}^{\Delta}\right]_{1\leq i\leq m,\ \ 1\leq j\leq n}, \quad \text{where} \quad A = \left[a_{ij}\right]_{1\leq i\leq m,\ \ 1\leq j\leq n}.$$

Theorem 2.5. *[5] Suppose that A and B are delta differentiable $d \times d$ matrix valued functions on $\mathbb{T}$, then*

(1). $A(\sigma(t)) = A(t) + \mu(t)A^{\Delta}(t)$ for $t \in \mathbb{T}$;

(2). $(A + B)^{\Delta} = A^{\Delta} + B^{\Delta}$;

(3). $(\alpha A)^{\Delta} = \alpha A^{\Delta}$, where α is a constant;

(4). $(AB)^{\Delta} = A^{\Delta}B + A^{\sigma}B^{\Delta}$ ($A^{\sigma}(t) = A(\sigma(t))$).

Definition 2.6. *[5] A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}$. Then we define*

$$\int_r^s f(t)\Delta t = F(s) - F(r), \quad \text{for} \quad s, r \in \mathbb{T}.$$

Definition 2.7. *[5] A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exists(finite) at all left-dense points in $\mathbb{T}$.*

The set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}(\mathbb{R}^{d\times d})$ will be denoted by $C_{rd}(\mathbb{T})$; meanwhile, the set of functions $f : \mathbb{T} \to \mathbb{R}(\mathbb{R}^{d\times d})$ that are delta differentiable and whose derivatives are rd-continuous is denoted by $C^1_{rd}(\mathbb{T})$.

Definition 2.8. *[5] A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. A $d \times d$ matrix valued function $A(t)$ on a time scale $\mathbb{T}$ is called regressive provided $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}$.*

The set of such regressive and rd-continuous functions is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}^d) = \mathcal{R}(\mathbb{T}, \mathbb{R}^{d\times d})$.

Definition 2.9. *[3] Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$, then the Kronecker product of A and B is written as $A \otimes B$ and is defined to be the partitioned matrix*

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}$$

which is an $mp \times nq$ matrix and is in $\mathbb{R}^{mp\times nq}$.

From Definitions 2.8 and 2.9, we have the following lemma.

Lemma 2.10. *[5] If $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{m\times n})$ and $B \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{p\times q})$, then $A \otimes B \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{mp\times nq})$.*

Definition 2.11. *[3] Let $A = [a_{ij}] \in \mathbb{R}^{m\times n}$, then the vectorization operator $Vec : \mathbb{R}^{m\times n} \to \mathbb{R}^{mn}$ is defined as*

$$\hat{A} = VecA = \begin{bmatrix} A_{.1} \\ A_{.2} \\ \vdots \\ A_{.n} \end{bmatrix}, \ \ where \ \ A_{.j} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}, \ \ (1 \le \ j \le \ n).$$

From Definitions 2.8 and 2.11, we have that if $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{m\times n})$, then $\hat{A} = VecA \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{mn})$.

Lemma 2.12. *The vectorization operator $Vec : \mathcal{R}(\mathbb{T}, \mathbb{R}^{d\times d}) \to \mathcal{R}(\mathbb{T}, \mathbb{R}^{d^2})$ is a linear and one-to-one operator. In addition, Vec and Vec^{-1} are continuous operators.*

Proof. The fact that the vectorization operator is linear and one-to-one is immediate. Now, for $A = [a_{ij}] \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{d\times d})$, we have

$$\|Vec(A)\| = \max_{1\le i,j\le d} \left\{ \left| a_{ij} \right| \right\} \le \max_{1\le i\le d} \left\{ \sum_{j=1}^{d} \left| a_{ij} \right| \right\} = |A| \, .$$

Thus, the vectorization operator is continuous and $\|Vec\| \le 1$.

In addition, for $A = I_d$ (identity $d \times d$ matrix) we have $\|Vec(I_d)\| = 1 = |I_d|$.

Obviously, the inverse of the vectorization operator, $Vec^{-1} : \mathcal{R}(\mathbb{T}, \mathbb{R}^{d^2}) \to \mathcal{R}(\mathbb{T}, \mathbb{R}^{d\times d})$, is defined by

$$Vec^{-1}(u) = \begin{bmatrix} u_1 & u_{d+1} & \cdots & u_{d^2-d+1} \\ u_2 & u_{d+2} & \cdots & u_{d^2-d+2} \\ \vdots & \vdots & \cdots & \vdots \\ u_d & u_{2d} & \cdots & u_{d^2} \end{bmatrix},$$

where $u = [u_1, u_2, u_3, ..., u_{d^2}]^* \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{d^2})$. We have

$$\left| Vec^{-1}(u) \right| = \max_{1\le i\le d} \left\{ \sum_{j=0}^{d-1} \left| u_{dj+i} \right| \right\} \le d \max_{1\le i\le d} \{|u_i|\} = d \, \|u\|.$$

Thus, Vec^{-1} is a continuous operator. Also, if we take $u = VecA$ in the above inequality, then the following inequality holds

$$|A| \le d\|VecA\|,$$

for every $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{d\times d})$. □

Regarding properties and rules for Kronecker product of matrices we refer [3]. We use some properties and rules, which are stated in the following lemma.

Lemma 2.13. *[3] The following are true provided that the dimension of the matrices are such that the various expressions exist.*

(1). $(A \otimes B)^* = (A^* \otimes B^*)$;

(2). $(A \otimes B)^{-1} = (A^{-1} \otimes B^{-1})$;

(3). $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$;

(4). $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$;

(5). $(A \otimes B)(C \otimes D) = AC \otimes BD$;

(6). $A \otimes (B \otimes C) = (A \otimes B) \otimes C$;

(7). If $A(t), B(t) \in \mathbb{R}^{d\times d}$ *and whose components are continuously differentiable with respect to 't' on* $\mathbb{R}$, *then*

$$(A \otimes B)' = (A' \otimes B) + (A \otimes B'). \qquad \left(' = \frac{d}{dt}\right)$$

(8). If $A, B, X \in \mathbb{R}^{d\times d}$, *then*

(i) $Vec(AXB) = (B^* \otimes A)VecX$;

(ii) $Vec(AX) = (I_d \otimes A)VecX$;

(iii) $Vec(XB) = (B^* \otimes I_d)VecX$.

Remark 2.14. *It is easily seen that the properties and rules in Lemma 2.13 except differentiation rule (7) are satisfies for rd-continuous matrix functions* $A(t), B(t)$ *on* $\mathbb{T}$. *From Theorem 2.5 (4), the differentiation rule (7) of Lemma 2.13 has the following form:*

$$(A \otimes B)^\Delta = (A^\Delta \otimes B) + (A^\sigma \otimes B^\Delta).$$

Now by applying the Vec operator to the linear nonhomogeneous Sylvester matrix dynamical system (1.1) and using Kronecker product properties, we have

$$\hat{X}^\Delta(t) = G(t)\hat{X}(t) + \hat{F}(t), \tag{2.1}$$

where $G(t) = B^*(t) \otimes I_d + I_d \otimes A(t) + \mu(t)(B^*(t) \otimes A(t))$ is a $d^2 \times d^2$ matrix and $\hat{F}(t) = VecF(t)$ is a column matrix of order d^2. The equation (2.1) is called the Kronecker product dynamic equation associated with (1.1). The corresponding homogeneous equation of (2.1) is

$$\hat{X}^\Delta(t) = G(t)\hat{X}(t). \tag{2.2}$$

From the above conversion (matrix dynamical system to vector dynamic equation) and Lemma 2.12, we have the following lemma.

Lemma 2.15. *The matrix valued function X(t) is a solution of* (1.1) *on* $\mathbb{T}^+$ *if and only if the vector valued function* $\hat{X}(t) = Vec(X(t))$ *is a solution of the equation* (2.1) *on* $\mathbb{T}^+$.

Lemma 2.16. *Let* $Y(t)$ *and* $Z(t)$ *be the fundamental matrices for the matrix dynamical systems*

$$X^\Delta(t) = A(t)X(t), \tag{2.3}$$

and

$$X^\Delta(t) = B^*(t)X(t), \quad t \in \mathbb{T}^+ \tag{2.4}$$

respectively. Then the matrix $W(t) = Z(t) \otimes Y(t)$ *is a fundamental matrix of* (2.2). *If in addition,* $Y(v) = I_d$, $Z(v) = I_d$, *then* $W(v) = I_{d^2}$.

Proof. Using Remark 2.14, we have

$$W^\Delta(t) = Z^\Delta(t) \otimes Y(t) + Z(\sigma(t)) \otimes Y^\Delta(t)$$

$$= B^*(t)Z(t) \otimes Y(t) + Z(\sigma(t)) \otimes A(t)Y(t)$$

$$= (B^* \otimes I_d)(Z(t) \otimes Y(t)) + (Z(t) + \mu(t)B^*(t)Z(t)) \otimes A(t)Y(t)$$

$$= (B^* \otimes I_d)(Z(t) \otimes Y(t))$$
$$+[(I_d \otimes A(t) + \mu(t)(B^*(t) \otimes A(t))](Z(t) \otimes Y(t))$$
$$= [(B^* \otimes I_d) + (I_d \otimes A(t)) + \mu(t)(B^*(t) \otimes A(t))](Z(t) \otimes Y(t))$$
$$= G(t)(Z(t) \otimes Y(t)),$$

for all $t \in \mathbb{T}^+$.

On the other hand, the matrix $Z(t) \otimes Y(t)$ is an invertible matrix for all $t \in \mathbb{T}^+$ (because $Z(t)$ and $Y(t)$ are invertible matrices for all $t \in \mathbb{T}^+$). Thus, $Z(t) \otimes Y(t)$ is the fundumental matrix of (2.2). And also $W(v) = Z(v) \otimes Y(v) = I_d \otimes I_d = I_{d^2}$. □

Let $\overline{\mathbf{X}}_1$ denotes the subspace of $\mathbb{R}^{d^2}$ consisting of all vectors which are values of $I_d \otimes \Psi$-bounded solution of (2.2) on $\mathbb{T}^+$ at $t = v$ and $\overline{\mathbf{X}}_2$ a fixed subspace of $\mathbb{R}^{d^2}$, supplementary to $\overline{\mathbf{X}}_1$. Let Q_1, Q_2 denote the corresponding projections of $\mathbb{R}^{d^2 \times d^2}$ onto $\overline{\mathbf{X}}_1$, $\overline{\mathbf{X}}_2$ respectively.

Theorem 2.17. *Let $A(t), B(t) \in \mathcal{R}(\mathbb{T}^+, \mathbb{R}^{d\times d})$ and F(t) be rd-continuous matrix function on $\mathbb{T}^+$. If $Y(t)$ and $Z(t)$ are the fundamental matrices for the matrix dynamical systems* (2.3) *and* (2.4)*, then*

$$\hat{X}(t) = \int_v^t (Z(t) \otimes Y(t))Q_1(Z^{-1}(\sigma(s)) \otimes Y^{-1}(\sigma(s)))\hat{F}(s)\Delta s$$
$$- \int_v^\infty (Z(t) \otimes Y(t))Q_2(Z^{-1}(\sigma(s)) \otimes Y^{-1}(\sigma(s)))\hat{F}(s)\Delta s \tag{2.5}$$

is a solution of (2.1) *on* $\mathbb{T}^+$.

Proof. It is easily seen that $\hat{X}(t)$ is the solution of (2.1) on $\mathbb{T}^+$. □

Let $\Psi_i : \mathbb{T}^+ \to (0, \infty)$, $i = 1, 2, \ldots, d$ be regressive and rd-continuous functions, and define

$$\Psi = \mathrm{diag}[\Psi_1, \Psi_2, \ldots \Psi_d].$$

Then $\Psi(t)$ is regressive, rd-continuous and invertible on $\mathbb{T}^+$.

Definition 2.18. *A function $\phi : \mathbb{T}^+ \to \mathbb{R}^d$ is said to be Ψ- bounded on $\mathbb{T}^+$ if $\Psi(t)\phi(t)$ is bounded on $\mathbb{T}^+$ (i.e., there exists $L_1 > 0$ such that $\|\Psi(t)\phi(t)\| \leq L_1$, for all $t \in \mathbb{T}^+$).*

Extend this definition for matrix functions.

Definition 2.19. *A matrix function $F : \mathbb{T}^+ \to \mathbb{R}^{d\times d}$ is said to be Ψ-bounded on $\mathbb{T}^+$ if the matrix function ΨF is bounded on $\mathbb{T}^+$ (i.e., there exists $L_2 > 0$ such that $|\Psi(t)F(t)| \leq L_2$, for all $t \in \mathbb{T}^+$).*

Definition 2.20. *A function $f : \mathbb{T}^+ \to \mathbb{R}^d$ is said to be Lebesgue Ψ- deltaintegrable on $\mathbb{T}^+$ if f is delta measurable and f is Lebesgue deltaintegrable on $\mathbb{T}^+$*
$\left(i.e., \int_v^\infty \|\Psi(s)f(s)\|\Delta s < \infty\right)$.

Extend this definition for matrix functions.

Definition 2.21. *A matrix function $F : \mathbb{T}^+ \to \mathbb{R}^{d\times d}$ is said to be Lebesgue Ψ-deltaintegrable on $\mathbb{T}^+$ if F is delta measurable and F is Lebesgue deltaintegrable on $\mathbb{T}^+$* $\left(i.e., \int_v^\infty |\Psi(s)F(s)|\Delta s < \infty\right)$.

The following lemmas play a vital role in the proof of main results.

Lemma 2.22. *The matrix function $F : \mathbb{T}^+ \to \mathbb{R}^{d\times d}$ is Ψ-deltaintegrable on time scale $\mathbb{T}^+$ if and only if the vector function $VecF(t)$ is $I_d \otimes \Psi$-deltaintegrable on $\mathbb{T}^+$.*

Proof. From the proof of Lemma 2.12, it follows that

$$\frac{1}{d}\,|A| \le \|VecA\|_{\mathcal{R}(\mathbb{T}^+,\mathbb{R}^{d^2})} \le |A|$$

for every $A \in \mathcal{R}(\mathbb{T}^+, \mathbb{R}^{d\times d})$.

Put $A(t) = \Psi(t)F(t)$ in the above inequality, we have

$$\frac{1}{d}\,|\Psi(t)F(t)| \le \|(I_d \otimes \Psi(t)).VecF(t)\|_{\mathcal{R}(\mathbb{T}^+,\mathbb{R}^{d^2})} \le |\Psi(t)F(t)|\,, \tag{2.6}$$

$t \in \mathbb{T}^+$, for all matrix functions $F(t)$.

Suppose that $F(t)$ is Ψ-deltaintegrable on $\mathbb{T}^+$, From (2.6)

$$\|(I_d \otimes \Psi(t)).VecF(t)\|_{\mathcal{R}(\mathbb{T}^+,\mathbb{R}^{d^2})} \le |\Psi(t)F(t)|\,,$$

Using Definitions 2.21 and 2.20, $\hat{F}(t)$ is Lebesgue $I_d \otimes \Psi$-deltaintegralble on $\mathbb{T}^+$.

Conversely, suppose that $\hat{F}(t)$ is $I_d \otimes \Psi$- deltaintegrable on $\mathbb{T}^+$. Again from (2.6), we have

$$|\Psi(t)F(t)| \le d\,\|(I_d \otimes \Psi(t)).VecF(t)\|_{\mathbb{R}^{d^2}}\,.$$

From Definitions 2.20 and 2.21, $F(t)$ is Ψ- deltaintegrable on $\mathbb{T}^+$. □

Lemma 2.23. *The matrix function $F : \mathbb{T}^+ \to \mathbb{R}^{d\times d}$ is Ψ-bounded on the time scale $\mathbb{T}^+$ if and only if the vector function $VecF(t)$ is $I_d \otimes \Psi$-bounded on $\mathbb{T}^+$.*

Proof. The proof easily follows from the inequality (2.6). □

The following Theorems are easily obtain from [10] and [16].

Theorem 2.24. *Let $A \in \mathcal{R}$ be a square matrix of order $d \times d$, and $y^{\Delta}(t) = A(t)y + f(t)$ has at least one Ψ-bounded solution on time scale $\mathbb{T}^+$ for each Lebesgue Ψ-deltaintegrable function f on $\mathbb{T}^+$ if and only if $\exists$ a constant $L > 0$ such that*

$$|\Psi(\tau)X(\tau)Q_1X^{-1}(\sigma(s))\Psi^{-1}(s)| \le L, \quad \text{if } v \le \sigma(s) \le \tau,$$
$$|\Psi(\tau)X(\tau)Q_2X^{-1}(\sigma(s))\Psi^{-1}(s)| \le L, \quad \text{if } v \le \tau \le s.$$

Theorem 2.25. *Assume that:*

(1) $X(\tau)$ the fundamental matrix of $y^{\Delta}(t) = A(t)y$ obey the following properties:

(a) $\lim_{\tau\to\infty} \Psi(\tau)X(\tau)Q_1 = 0$;

(b) $|\Psi(\tau)X(\tau)Q_1X^{-1}(\sigma(s))\Psi^{-1}(s)| \le L$, for $v \le \sigma(s) \le \tau$,
$|\Psi(\tau)X(\tau)Q_2X^{-1}(\sigma(s))\Psi^{-1}(s)| \le L$, for $v \le \tau \le s$

where L is a +ve constant.

(2) The function $f : \mathbb{T}^+ \to \mathbb{R}^d$ is Lebesgue Ψ-deltaintegrable on $\mathbb{T}^+$.

Then each Ψ-bounded solution $x(\tau)$ of $y^{\Delta}(t) = A(t)y + f(t)$ satisfies

$$\lim_{\tau\to\infty} \|\Psi(\tau)x(\tau)\| = 0.$$

3. Sylvester Matrix Dynamical Systems on Time Scales

In this section, we obtain a necessary and sufficient condition for the the existence of Ψ-bounded solution for Sylvester matrix dynamical system (1.1), via Ψ-deltaintegrable matrix function F on $\mathbb{T}^+$. And also obtain a result relating to asymptotic behavior of Ψ-bounded solution of (1.1).

Theorem 3.1. *If $A(t)$ and $B(t)$ are regressive and rd-continuous $d \times d$ matrices on $\mathbb{T}^+$, and* (1.1) *has at least one Ψ-bounded solution on $\mathbb{T}^+$ for every Lebesgue Ψ-deltaintegrable matrix function $F : \mathbb{T}^+ \to \mathbb{R}^{d\times d}$ on $\mathbb{T}^+$ if and only if there exists a positive constant N such that*

$$\begin{aligned} |(Z(t) \otimes \Psi(t)Y(t))Q_1(Z^{-1}(\sigma(s)) \otimes Y^{-1}(\sigma(s))\Psi^{-1}(s))| \leq N \text{ for } v \leq \sigma(s) \leq t \\ |(Z(t) \otimes \Psi(t)Y(t))Q_2(Z^{-1}(\sigma(s)) \otimes Y^{-1}(\sigma(s))\Psi^{-1}(s))| \leq N \text{ for } v \leq t \leq s \end{aligned} \tag{3.1}$$

Proof. Suppose that the equation (1.1) has atleast one Ψ-bounded solution on $\mathbb{T}^+$ for every Lebesgue Ψ-deltaintegrable matrix function $F : \mathbb{T}^+ \to \mathbb{R}^{d\times d}$.

Let $\hat{F} : \mathbb{R} \to \mathbb{R}^{d^2}$ be a Lebesgue $I_d \otimes \Psi$-deltaintegrable function on $\mathbb{T}^+$. From Lemma 2.22, it follows that the matrix function $F(t) = Vec^{-1}\hat{F}(t)$ is Lebesgue Ψ-deltaintegrable matrix function on $\mathbb{T}^+$. From the hypothesis, the system (1.1) has at least one Ψ-bounded solution $X(t)$ on $\mathbb{T}^+$. From Lemma 2.15, it follows that the vector valued function $\hat{X}(t) = VecX(t)$ is a $I_d \otimes \Psi$-bounded solution of (2.1) on $\mathbb{T}^+$.

Thus, system (2.1) has at least one $I_d\otimes\Psi$-bounded solution on $\mathbb{T}^+$ for every Lebesgue $I_d\otimes\Psi$-deltaintegrable function $\hat{F}$ on $\mathbb{T}^+$.

From Lemma 2.16 and Theorem 2.24 there is a positive constant N such that the fundamental matrix $W(t)$ of the system (2.2) satisfies the condition

$$\begin{aligned} |(I_d \otimes \Psi(t))W(t)Q_1W^{-1}(\sigma(s))(I_d \otimes \Psi^{-1}(s))| \leq N, \quad \text{for} \quad v \leq \sigma(s) \leq t \\ |(I_d \otimes \Psi(t))W(t)Q_2W^{-1}(\sigma(s))(I_d \otimes \Psi^{-1}(s))| \leq N \quad \text{for} \quad v \leq t \leq s \end{aligned}$$

Putting $W(t) = Z(t) \otimes Y(t)$ and using Kronecker product properties, condition (3.1) holds.

Conversely, suppose that the condition (3.1) holds for some $N > 0$.

Let $F : \mathbb{T}^+ \to \mathbb{R}^{d\times d}$ be a lebesgue Ψ-deltaintegrable matrix function on $\mathbb{T}^+$. From Lemma 2.22, it follows that the vector valued function $\hat{F}(t) = VecF(t)$ is a Lebesgue $I_d \otimes \Psi$-deltaintegrable function on $\mathbb{T}^+$.

It follows that from Theorem 2.24 the vector dynamic equation (2.1) has at least one $I_d \otimes \Psi$ - bounded solution on $\mathbb{T}^+$. Let $v(t)$ be this solution.

From Lemma 2.23, it follows that the matrix function $V(t) = Vec^{-1}v(t)$ is a Ψ-bounded solution of the equation (1.1) on $\mathbb{T}^+$ (because $F(t) = Vec^{-1}\hat{F}(t)$).

Thus, the Sylvester matrix dynamical system (1.1) has at least one Ψ-bounded solution on $\mathbb{T}^+$ for every Lebesgue Ψ-deltaintegrable matrix function F on $\mathbb{T}^+$. □

In the following theorem, we obtain sufficient conditions for the asymptotic behaviour of Ψ-bounded solutions of the dynamical system (1.1).

Theorem 3.2. *Suppose that:*

(1) Let $Y(t)$ and $Z(t)$ be fundamental matrices of (2.3), (2.4) *respectively and satisfies the conditions:*

(a) $\lim_{t\to\infty} (Z(t) \otimes \Psi(t)Y(t))\, Q_1 = 0;$

(b) Condition (3.1) *of Theorem 3.1.*

(2) The function $F : \mathbb{T}^+ \to \mathbb{R}^{d\times d}$ is Lebesgue Ψ-deltaintegrable matrix function on $\mathbb{T}^+$.

Then every Ψ-bounded solution $X(t)$ of (1.1) *satisfies*

$$\lim_{t\to\infty} |\Psi(t)X(t)| = 0.$$

Proof. Let $X(t)$ be a Ψ-bounded solution of (1.1). From Lemma 2.23, it follows that the function $\hat{X}(t) = VecX(t)$ is a $I_d \otimes \Psi$-bounded solution on $\mathbb{T}^+$ of the vector dynamic equation (2.1). Also from Lemma 2.22, the function $\hat{F}(t)$ is Lebesgue $I_d \otimes \Psi$-deltaintegrable on $\mathbb{T}^+$. From Theorem 2.25, it follows that

$$\lim_{t\to\infty} \|(I_d \otimes \Psi(t))\hat{X}(t)\| = 0.$$

Now, from the inequality (2.6), we have

$$|\Psi(t)X(t)| \leq d \left\|(I_d \otimes \Psi(t))\, \hat{X}(t)\right\|, \quad t \in \mathbb{T}^+$$

and, then

$$\lim_{t\to\infty} |\Psi(t)X(t)| = 0.$$

□

Remark 3.3. *Theorem 3.2 is no longer true if we require that the function F be Ψ-bounded on $\mathbb{T}^+$, instead of the condition (2) and it doesnot apply even if the function F is such that* $\lim_{t\to\infty} |\Psi(t)F(t)| = 0$.

The following example illustrates Remark 3.3.

Example 3.4. Consider (1.1) with $A(t) = B(t) = O_2$ and

$$F(t) = \begin{bmatrix} \frac{1}{t+1} & 0 \\ 0 & \sqrt{(t+1)^3} \end{bmatrix}.$$

Then, $Y(t) = Z(t) = I_2$ are the fundamental matrices for (2.3) and (2.4) respectively. Consider

$$\Psi(t) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{(1+t)^2} \end{bmatrix},$$

then there exist projections

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

satisfies the condition (3.1) with $N = 1$. In addition, the hypothesis (1a) of Theorem 3.2 is satisfied. Because

$$\Psi(t)F(t) = \begin{bmatrix} \frac{1}{t+1} & 0 \\ 0 & \frac{1}{\sqrt{t+1}} \end{bmatrix},$$

the matrix function F is not Lebesgue Ψ-deltaintegrable on $\mathbb{T}^+$, but it is Ψ-bounded on $\mathbb{T}^+$, with $\lim_{t\to\infty} |\Psi(t)F(t)| = 0$. The solutions of the system (1.1) are

$$X(t) = \begin{cases} \begin{bmatrix} \ln(t+1) + c_1 & c_2 \\ c_3 & \frac{2}{5}\sqrt{(t+1)^5} + c_4 \end{bmatrix} & \text{when } \mathbb{T} = \mathbb{R} \\[2ex] \begin{bmatrix} \sum\limits_{k=1}^{t} \frac{1}{k} + c_1 & c_2 \\ c_3 & \sum\limits_{k=1}^{t} \sqrt{k^3} + c_4 \end{bmatrix} & \text{when } \mathbb{T} = \mathbb{Z}. \end{cases}$$

It is easily seen that $\lim_{t\to\infty} \|\Psi(t)X(t)\| = +\infty$, for all $c_1, c_2, c_3, c_4 \in \mathbb{R}$. It follows that the solutions of the system (1.1) are Ψ-unbounded on $\mathbb{T}^+$.

Note: In Example 3.4, if

$$F(t) = \begin{bmatrix} \frac{1}{(t+1)^2} & 0 \\ 0 & 1 \end{bmatrix},$$

then

$$\int_{\mathbb{T}^+} ||\Psi(t)F(t)||dt = \begin{cases} 1, & \mathbb{T} = \mathbb{R} \\ \frac{\pi^2}{6}, & \mathbb{T} = \mathbb{Z}. \end{cases}$$

On the other hand, the solutions of (1.1) are

$$X(t) = \begin{cases} \begin{bmatrix} -\frac{1}{t+1} + c_1 & c_2 \\ c_3 & t + c_4 \end{bmatrix} & \text{when } \mathbb{T} = \mathbb{R} \\ \begin{bmatrix} \sum\limits_{k=1}^{t} \frac{1}{k^2} + c_1 & c_2 \\ c_3 & t + c_4 \end{bmatrix} & \text{when } \mathbb{T} = \mathbb{Z}. \end{cases}$$

We observe that the asymptotic properties of the components of the solutions are not the same. The second row second column element is unbounded and the remaining elements are bounded on $\mathbb{T}^+$. However, all solutions of (1.1) are Ψ-bounded on $\mathbb{T}^+$ and $\lim_{t\to\infty} |\Psi(t)X(t)| = 0$. This shows that the asymptotic properties of the components of the solutions are the same, via the matrix function Ψ. This is obtained by using a matrix function Ψ rather than a scalar function.

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