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On Another Approach for a Family of Appell Polynomials

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Abstract. The present study is devoted to some new formulas for a family of Appell polynomials. These formulas are expressed in terms of Bernoulli-Euler and Bernoulli-Genocchi polynomials. Moreover, other additive formulas and new combinatorial identities are established. In particular some closed relations with nested sums are exhibited.

1. Introduction

Several families of polynomials play a fundamental role in various fields of mathematics and applied sciences. Especially, after their introduction and study in the seminal work of Appell (see [3]). Appell polynomials are paid much attention and their theory is subject of many studies (see for example [8, 12, 17–22]). Indeed, the active research on this family of polynomials is motivated by its several applications in various fields of mathematics and applied sciences such as combinatorics, analytic number theory and asymptotic approximation theory.

Among the various methods and techniques used in the literature for studying Appell polynomials and their generalizations, are those developed by Srivastava et al.. In the aim to consider the q-Appell polynomials, Srivastava gave in [19] some characterization of Appell polynomials in terms of Stieltjes integrals. Some specializations of the main results of [19], allow to recover several known results. In [22] Verde-Star and Srivastava aim at giving the complete list of the binomial formulas of the generalized Appell polynomials, using a generating-function of the generalized Appell form for a sequence of Newton polynomials. Several related formulas, including the well-known q-analogue of the binomial formula are obtained. Continuing his investigations on Appell polynomials, Srivastava introduced and investigate some of the principal generalizations and unifications of the classical Bernoulli, Euler and Genocchi polynomials, and also their corresponding numbers, by means of suitable generating functions. Moreover, Srivastava presented several interesting properties of these general polynomial systems including some explicit series representations in terms of the Hurwitz (or generalized) zeta function and the familiar Gauss hypergeometric function. In addition, in this interesting study a historical overview of these classes of polynomials and their various extensions are presented. On the other hand, some families of differential equations associated with the

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Hermite-based Appell polynomials and other classes of Hermite-based polynomials, have been investigated recently in [21]. Moreover, the corresponding results for the Hermite-based Genocchi polynomials, and those involving the Hermite-based Euler polynomials are derived.

Several characterizations of the family of Appell polynomials $\{A_n(x)\}_{n\geq 0}$, where $deg(A_n) = n$, were given in various studies (see for example [13, 15, 16]). Mainly, it was shown in [16] that the assertion $\frac{dA_n}{dx}(x) = nA_{n-1}(x)$, for $n = 1, 2, ...$, is equivalent to the existence of a formal power series $f(t) = \sum_{n=0}^{+\infty} \alpha_n \frac{t^n}{n!}$, with $\alpha_0 \neq 0$, satisfying the equation,

$$f(t)e^{xt} = \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!}. \tag{1}$$

Expression (1) makes it possible to recover one of the well known Appell assertion, namely, there is a one-to-one correspondence of the set of numerical sequences $\{\alpha_n\}_n$, with $\alpha_0 \neq 0$, and the set of polynomials sequence $\{A_n(x)\}_n$ given by the following binomial convoluted expression,

$$A_n(x) = \sum_{k=0}^{n} \binom{n}{k} \alpha_{n-k} x^k, \tag{2}$$

or, equivalently $A_n(x) = \alpha_n + \binom{n}{1}\alpha_{n-1}x + \binom{n}{2}\alpha_{n-2}x^2 + \cdots + \alpha_0 x^n$, for $n = 0, 1, 2, \ldots$. It is also known that the family of Appell polynomials contains many important classes of classical polynomials (see [13, 18]). That is, following (1) the polynomials $B_n(x)$ ($n \geq 0$) of Bernoulli, $E_n(x)$ ($n \geq 0$) of Euler and $G_n(x)$ ($n \geq 0$) of Genocchi, are nothing else but the Appell polynomials corresponding, respectively, to the invertible series of the functions $f(t) = \frac{t}{e^t - 1}$, $f(t) = \frac{2}{e^t+1}$ and $f(t) = \frac{2t}{e^t + 1}$. More precisely, we have

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad ; \quad \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \text{ and } \frac{2te^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, \tag{3}$$

where $|t| < 2\pi$ for Bernoulli polynomials and $|t| < \pi$ for Euler and Genocchi polynomials. Several approaches and methods have been considered for studying Appell polynomials, as well as for Bernoulli, Euler and Genocchi polynomials. In particular, formulas of these polynomials and some identities connecting them are developed. Recently, an approach by means of a determinantal form for Appell polynomials has been considered in [7], and its equivalence with the previous characterizations of Appell polynomials has been studied.

The goal of this paper is to explore another determinantal approach to study some properties of a large class of Appell polynomials. This approach involves Bernoulli, Euler and Genocchi polynomials. Moreover, several new combinatorial identities and additional formulas are established, and some related nested sums are exhibited. Our approach is based on standard computations, involving the generating functions (3).

The content of this study is organized as follows. We first introduce the deterministic concept, in connection with new identities, involving Bernoulli, Euler and Genocchi polynomials. Therefore, expressions of Appell polynomials and a determinantal additional formulas for Appell polynomials, are established in terms of Bernoulli, Euler and Genocchi polynomials (Section 2). Secondly, an interesting class of Appell polynomials, extending Euler and Genocchi polynomials is considered, and some new combinatorial identities are established, especially those related to nested sums (Section 3). Finally, concluding remarks and perspectives are presented (Section 4).

2. On Another Determinantal Approach for Appell Polynomials

In this Section we present a determinantal approach for Appell polynomials by means of some expansion formulas. Let x and t be real numbers, with $|t| < \pi$, and set $T(x, t) = \frac{te^{xt}}{e^t - 1} \times \frac{2e^{xt}}{e^t + 1}$. Taking into account the right hand side of (3) a direct computation permits to have

$$T(x+1,t) - T(x,t) = \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \binom{n}{k} (B_{n-k}(x+1)E_k(x+1) - B_{n-k}(x)E_k(x)) \right\} \frac{t^n}{n!}.$$

On the other hand, we have $T(x+1,t) - T(x,t) = 2te^{2xt} = \sum_{n=1}^{+\infty} n2^n x^{n-1} \frac{t^n}{n!}$. Comparing the preceding two expansions of $T(x+1,t) - T(x,t)$, we can formulate the next result.

Proposition 2.1. *Let x be a real number and an integer $n \geq 0$. Then, the two following identities are verified,*

$$x^n = \frac{1}{2^{n+1}(n+1)} \sum_{k=0}^{n+1} \binom{n+1}{k} \begin{vmatrix} B_{n-(k-1)}(x+1) & E_k(x) \\ B_{n-(k-1)}(x) & E_k(x+1) \end{vmatrix}, \tag{4}$$

$$x^n = \frac{1}{2^{n+1}(n+1)(n+2)} \sum_{k=0}^{n+2} \binom{n+2}{k} \begin{vmatrix} B_{n-(k-2)}(x+1) & G_k(x) \\ B_{n-(k-2)}(x) & G_k(x+1) \end{vmatrix}, \tag{5}$$

where $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$ *represents the determinant of the matrix* $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

The proof of the identity (5) is similar to the preceding proof of (4). As a consequence of Proposition 2.1, we show that the Appell polynomials, $\{A_n(x)\}_{n\geq 0}$ given by (2), can be expressed in terms of the Bernoulli, Euler and Genocchi polynomials. That is, by a straightforward calculation, the substitution of x^n given by (4)-(5) in Expression (2) allows us to reach the formulas,

$$A_n(x) = \sum_{k=0}^{n} \alpha_{n-k} \sum_{j=0}^{k+1} \Lambda_{j,k,n} \times \begin{vmatrix} B_{k-(j-1)}(x+1) & E_j(x) \\ B_{k-(j-1)}(x) & E_j(x+1) \end{vmatrix}, \tag{6}$$

where $\Lambda_{j,k,n} = \frac{1}{2^{k+1}(k+1)} \binom{n}{k} \binom{k+1}{j}$, and

$$A_n(x) = \sum_{k=0}^{n} \alpha_{n-k} \sum_{j=0}^{k+2} \Delta_{j,k,n} \times \begin{vmatrix} B_{k-(j-1)}(x+1) & G_j(x) \\ B_{k-(j-1)}(x) & G_j(x+1) \end{vmatrix}, \tag{7}$$

where $\Delta_{j,k,n} = \frac{1}{2^{k+1}(k+1)(k+2)} \binom{n}{k} \binom{k+2}{j}$. Recently, several additional formulas were established in [12], for a large class of Appell polynomials. As a consequence of Proposition 2.1, we derive two new addition formulas for the Appell polynomials, in terms of Bernoulli, Euler and Genocchi polynomials.

Theorem 2.2. *Let $\{A_n(x)\}_{n\geq 0}$ be a sequence of Appell polynomials. Then, each of the following addition formulas holds,*

$$A_n(x+y) = \sum_{k=0}^{n} A_{n-k}(y) \sum_{j=0}^{k+1} \Lambda_{j,k,n} \times \begin{vmatrix} B_{k-(j-1)}(x+1) & E_j(x) \\ B_{k-(j-1)}(x) & E_j(x+1) \end{vmatrix}, \tag{8}$$

$$A_n(x+y) = \sum_{k=0}^{n} A_{n-k}(y) \sum_{j=0}^{k+1} \Delta_{j,k,n} \times \begin{vmatrix} B_{k-(j-1)}(x+1) & G_j(x) \\ B_{k-(j-1)}(x) & G_j(x+1) \end{vmatrix}, \tag{9}$$

where $\Lambda_{j,k,n} = \frac{1}{2^{k+1}(k+1)} \binom{n}{k} \binom{k+1}{j}$ *and* $\Delta_{j,k,n} = \frac{1}{2^{k+1}(k+1)(k+2)} \binom{n}{k} \binom{k+2}{j}$.

Proof. For an Appell polynomial $A_n(x)$, we have

$$A_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} A_{n-k}(y) x^k, \tag{10}$$

(see [12, 13]). Now, by replacing the factor x^k in (10) by its expression given in (4), and by interchanging the order of the resulting double sum, we get the expansion formula (8). Similarly, Expression (9) can be proved by substituting the expansion formula (5) of x^n in Formula (10). □

The determinantal formula (4) of x^n and Expression (8) of $A_n(x+y)$, involve the Bernoulli and Euler polynomials. However, these formulas can be expanded only in terms of the Bernoulli polynomials, using some well known closed relations between Bernoulli and Euler polynomials. Indeed, for every integer $m \geq 2$, it is well known that $E_n(mx) = \dfrac{-2}{n+1} m^n \sum_{k=0}^{m-1} (-1)^k B_{n+1}(x + \frac{k}{m})$, and

$$E_n(x) = \frac{2}{n+1}\left[B_{n+1}(x) - 2^{n+1} B_{n+1}(\frac{x}{2})\right], \tag{11}$$

for $m = 1$ (see [8, 17]). Thus, the determinantal expansion formula (4) of x^n, can be formulated only in terms of the Bernoulli polynomials. By considering (11), we introduce the following useful notation

$$\mathbb{E}_{s,k}(x) := \frac{2}{k+1} \begin{vmatrix} B_s(x+1) & B_k(x) - 2^k B_k(\frac{x}{2}) \\ B_s(x) & B_k(x+1) - 2^k B_k(\frac{x+1}{2}) \end{vmatrix}, \tag{12}$$

where s, k are integers and x is a real number. On the same lines, it is well known that (see [2])

$$G_n(x) = 2B_n(x) - 2^{n+1} B_n(\frac{x}{2}). \tag{13}$$

Hence, similarly the determinantal expansion formula (5) of x^n, can be expressed only in terms of the Bernoulli polynomials. By considering (13), we consider the following notation,

$$\mathbb{G}_{s,k}(x) := 2 \begin{vmatrix} B_s(x+1) & B_k(x) - 2^k B_k(\frac{x}{2}) \\ B_s(x) & B_k(x+1) - 2^k B_k(\frac{x+1}{2}) \end{vmatrix}, \tag{14}$$

for every integer s, k and a real number $x \in \mathbb{R}$.

Proposition 2.3. *Under the preceding data, for every $n \geq 1$ the following expansion formulas hold,*

$$x^n = \frac{1}{2^{n+1}(n+1)} \sum_{k=0}^{n+1} \binom{n+1}{k} \mathbb{E}_{n-k+1,k+1}(x), \tag{15}$$

and

$$x^n = \frac{1}{2^{n+1}(n+1)(n+2)} \sum_{k=0}^{n+2} \binom{n+2}{k} \mathbb{G}_{n-k+2,k}(x). \tag{16}$$

where the $\mathbb{E}_{n,k}(x)$ and $\mathbb{G}_{n,k}(x)$ are as in (12) *and* (14)*, respectively.*

Proof. For $m = 1$ and $s = n-(k-1)$, the substitution of Expression (11) in (4), leads to the following relation,

$$\begin{vmatrix} B_{n-(k-1)}(x+1) & E_k(x) \\ B_{n-(k-1)}(x) & E_k(x+1) \end{vmatrix} = \mathbb{E}_{n-k+1,k+1}(x),$$

for every k $(0 \leq k \leq n+1)$, where $\mathbb{E}_{k,j}(x)$ is as in (12). This completes the proof of (15). The identity (16) is established in a manner similar to that of (15). □

Expression (15) can be used for providing a new additional formula for Appell polynomials. That is, the substitution of Formula (15) in (10), implies the following relation $A_n(x+y) = \sum_{k=0}^{n}\sum_{j=0}^{k+1}\Lambda_{j,k,n}A_{n-k}(y)\times \mathbb{E}_{k-j+1,j+1}(x)$, where $\Lambda_{j,k,n} = \dfrac{1}{2^{k+1}(k+1)}\binom{n}{k}\binom{k+1}{j}$ and the $\mathbb{E}_{k,j}(x)$ are as in (12). We do the same for Expression (16). Therefore, we have the result.

Theorem 2.4. *Let $\{A_n(x)\}_{n\geq 0}$ be a sequence of Appell polynomials. Then, each of the following addition formulas holds,*

$$A_n(x+y) = \sum_{k=0}^{n} A_{n-k}(y)\sum_{j=k}^{n+1}\Lambda_{j,k,n}\times \mathbb{E}_{k-j+1,j+1}(x), \tag{17}$$

where $\Lambda_{j,k,n} = \dfrac{1}{2^{k+1}(k+1)}\dbinom{n}{k}\dbinom{k+1}{j}$, *the* $\mathbb{E}_{k,j}(x)$ *are as in* (12) *and*

$$A_n(x+y) = \sum_{k=0}^{n} A_{n-k}(y)\sum_{j=k}^{n+2}\Delta_{j,k,n}\times \mathbb{G}_{k-j+2,j}(x), \tag{18}$$

where $\Delta_{j,k,n} = \dfrac{1}{2^{k+1}(k+1)(k+2)}\dbinom{n}{k}\dbinom{k+2}{j}$ *and* $\mathbb{G}_{k,j}(x)$ *are as in* (14).

Expressions (17)-(18) can be generalized using the notion of nested sums formula (see [1, 5, 14]). That is, we set $\mathbb{W}_{k,n}(x) = \sum_{j=k}^{n+1}\Lambda_{j,k,n}\times\mathbb{E}_{k-j+1,j+1}(x)$ and $\mathbb{U}_{k,n}(x) = \sum_{j=k}^{n+2}\Delta_{j,k,n}\times\mathbb{G}_{k-j+2,j}(x)$, where the $\mathbb{E}_{k,j}(x)$ and the $\mathbb{G}_{k,j}(x)$ are as in (12) and (14), respectively. Then, Expressions (17)-(18) take the form $A_n(x+y) = \sum_{k=0}^{n}A_{n-k}(y)\mathbb{W}_{k,n}(x)$ and $A_n(x+y) = \sum_{k=0}^{n}A_{n-k}(y)\mathbb{U}_{k,n}(x)$, respectively. Therefore, Formulas (17)-(18) can be extended as follows. Let $n\geq 1$ be an integer and $x_1,\ldots,x_s$, y be real numbers, then we have the formulas,

$$A\Big(\sum_{i=1}^{s}x_i+y\Big) = \sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}\cdots\sum_{k_s=0}^{n-\sum_{i=1}^{s-1}k_i} A_{n-\sum_{i=1}^{s}k_i}(y)\mathbb{Y}_{k_1,\ldots,k_s;n}(x_1,\ldots,x_s) \tag{19}$$

where $\mathbb{Y}_{k_1,\ldots,k_s;n}(x_1,\ldots,x_s) = \mathbb{W}_{k_1,n}(x_1)\mathbb{W}_{k_2,n-k_1}(x_2)\cdots\mathbb{W}_{k_s,n-\sum_{i=1}^{s-1}k_i}(x_s)$ and

$$A\Big(\sum_{i=1}^{s}x_i+y\Big) = \sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}\cdots\sum_{k_s=0}^{n-\sum_{i=1}^{s-1}k_i} A_{n-\sum_{i=1}^{s}k_i}(y)\mathbb{V}_{k_1,\ldots,k_s;n}(x_1,\ldots,x_s) \tag{20}$$

where $\mathbb{V}_{k_1,\ldots,k_s;n}(x_1,\ldots,x_s) = \mathbb{U}_{k_1,n}(x_1)\mathbb{U}_{k_2,n-k_1}(x_2)\cdots\mathbb{U}_{k_s,n-\sum_{i=1}^{s-1}k_i}(x_s)$. In fact, Formulas (19)-(20) represent a nested sums formula. For $s=1$, we show easily that Expressions (17)-(18) represent a particular case of (19)-(20).

Now in a similarly way to Formulas (6)-(7), the result of Proposition 2.3 can be used to establish that the Appell polynomials $\{A_n(x)\}_{n\geq 0}$, can be expressed only in terms of Bernoulli polynomials. Indeed, a straightforward computation using the substitution of the determinental expansion (15)-(16) of x^n in Expression (2), permits us to get the formulas $A_n(x) = \sum_{k=0}^{n}\alpha_{n-k}\sum_{j=0}^{k+1}\Lambda_{j,k,n}\times\mathbb{E}_{k-j+1,j+1}(x)$, where $\Lambda_{j,k,n} = \dfrac{1}{2^{k+1}(k+1)}\binom{n}{k}\binom{k+1}{j}$ and the $\mathbb{E}_{k-j+1,j+1}(x)$ are as in (12), and

$$A_n(x) = \sum_{k=0}^{n}\alpha_{n-k}\sum_{j=k}^{n+2}\Delta_{j,k,n}\times\mathbb{G}_{k-j+2,j}(x),$$

where $\Delta_{j,k,n} = \dfrac{1}{2^{k+1}(k+1)(k+2)}\binom{n}{k}\binom{k+2}{j}$ and the $\mathbb{G}_{k-j+2,j}(x)$ are as in (14).

Remark 2.5. Expression (11) shows that an expansion of the x^n in terms of Bernoulli polynomials, can be also established. On the other hand, the expansion of x^n in terms of Euler and Genocchi polynomials can be also obtained, taking into account the well known formulas,

$$B(x) = \sum_{k=0,k\neq 1}^{n} \binom{n}{k} B_k E_{n-k}(x) = \sum_{k=0,k\neq n-1}^{n} \binom{n}{k} B_{n-k} E_k(x), \tag{21}$$

where the B_m are the Bernoulli numbers (see [9]). Moreover, by using (21) and the well known closed relation between Genocchi and Euler polynomials, namely $E_n(x) = \dfrac{1}{n+1} G_{n+1}(x)$ and $G_n(x) = nE_{n-1}(x)$ (see, for example, [10, p. 5707, Lemma 1]), we derive the following expansion

$$B(x) = \sum_{k=0,k\neq 1}^{n} \beta_{n,k} B_k G_{n-(k-1)}(x) = \sum_{k=0,k\neq n-1}^{n} \eta_{n,k} B_{n-k} G_{k+1}(x),$$

where $\beta_{n,k} = \binom{n}{k} \dfrac{1}{n-(k-1)}$ and $\eta_{n,k} = \binom{n}{k} \dfrac{1}{k+1}$.

3. On a Class of Appell Polynomials of Euler-Genocchi Type

This section is devoted to a class of Appell polynomials, which generalizes the classical Euler $E_n(x)$ and Genocchi $G_n(x)$ polynomials. Some important results (of this class) are established, by considering the determinental approach studied in the preceding section. Let $r \geq 1$ be an integer and $\{A_n^{(r)}(x)\}_{n\geq 0}$ the class of polynomials defined by the generating function,

$$\frac{2t^{r-1}}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} A_n^{(r)}(x) \frac{t^n}{n!}. \tag{22}$$

Differentiation of both sides of (22) with respect to x and equalising the coefficients corresponding to the powers t^n (in both sides), lead to $\dfrac{d}{dx} A_n^{(r)}(x) = nA_{n-1}^{(r)}(x)$. Hence, the polynomials $A_n^{(r)}(x)$ are Appell polynomials. Since for $r \geq 2$ the summation start from $r-1$, then we have $A_j^{(r)}(x) = 0$ for $0 \leq j \leq r-2$. Such kind of generalization of Appell polynomials has been considered for Bernoulli polynomials in [11]. Particularly, we show easily that the Euler and Genocchi polynomials are given by $E_n(x) = A_n^{(1)}(x)$ and $G_n(x) = A_n^{(2)}(x)$, respectively. We present the following determinantal identity.

Proposition 3.1. *For every integer $n \geq r-1$ ($r \geq 1$), the following formula is satisfied,*

$$x^n = \frac{1}{2^{n+1} \prod_{j=1}^{r}(n+j)} \sum_{k=0}^{n+r} \binom{n+r}{k} \begin{vmatrix} B_{n-(k-r)}(x+1) & A_k^{(r)}(x) \\ B_{n-(k-r)}(x) & A_k^{(r)}(x+1) \end{vmatrix}. \tag{23}$$

Proof. The proof is based on the similar process of the proof of Proposition 2.1. That is, set $T(x,t) = \dfrac{2t^{r-1}}{e^t+1} e^{xt} \dfrac{t}{e^t-1} e^{xt}$, then taking into account the series product, a straightforward computation implies that,

$$T(x+1,t) - T(x,t) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \begin{vmatrix} B_{n-k}(x+1) & A_k^{(r)}(x) \\ B_{n-k}(x) & A_k^{(r)}(x+1) \end{vmatrix} \frac{t^n}{n!}.$$

On the other hand, we have

$$T(x+1,t) - T(x,t) = 2t^r e^{2xt} = \sum_{n=0}^{+\infty} 2^{n+1} x^n \frac{t^{n+r}}{n!} = \sum_{n=0}^{+\infty} 2^{n-r+1} x^{n-r} \frac{t^n}{(n-r)!}.$$

By equalising the two preceding power series we easily get the result. □

By combining the Expressions (10) and (23) we can arrive at a new addition formula related to Appell polynomials $A_n^{(r)}(x)$ $(n \geq 0)$.

Theorem 3.2. *Let $\{A_n^{(r)}(x)\}_{n\geq 0}$ be the class of Appell polynomials given by (22), with $r \geq 1$. Then, for every $n \geq r-1$, the following formula holds,*

$$A_n(x+y) = \sum_{k=0}^{n} A_{n-k}(y) \sum_{j=0}^{k+r} \Omega_{r,j,k,n} \times \begin{vmatrix} B_{k-j+1}(x+1) & A_j^{(r)}(x) \\ B_{k-j+1}(x) & A_j^{(r)}(x+1) \end{vmatrix}, \tag{24}$$

where $\Omega_{r,j,k,n} = \dfrac{1}{2^{k+1}\prod_{i=1}^{r}(k+i)}\dbinom{n}{k}\dbinom{k+r}{j}$.

Note that $A_j^{(r)}(y) \equiv 0$, for $0 \leq j \leq r-2$, thus we have $A_{n-k}(y) \equiv 0$ for $0 \leq n-k \leq r-2$. For $r = 1$ and $r = 2$, respectively, Formula (24) is nothing else but Formulas (8) and (9), respectively. Expression (24) can be also generalized using the notion of nested sums formula (see [1, 5, 14]). That is, we set

$$\mathbb{W}_{k,n}^{(r)}(x) = \sum_{j=k}^{n+r} \Omega_{j,k,n}^{(r)}(x) \times \begin{vmatrix} B_{k-j+1}(x+1) & A_j^{(r)}(x) \\ B_{k-j+1}(x) & A_j^{(r)}(x+1) \end{vmatrix},$$

where $\Omega_{j,k,n}^{(r)}(x) = \dfrac{1}{2^{k+1}\prod_{i=1}^{r}(k+i)}\dbinom{n}{k}\dbinom{k+r}{j}$. Therefore, (24) takes the form $A_n^{(r)}(x+y) = \sum_{k=0}^{n} A_{n-k}^{(r)}(y)\mathbb{W}_{k,n}^{(r)}(x)$. On a same way, as for Expression (17) the generalization of Formula (24) is given as follows,

$$A_n^{(r)}(\sum_{i=1}^{s} x_i + y) = \sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1} \cdots \sum_{k_s==0}^{n-\sum_{i=1}^{s-1}k_i} A_{n-\sum_{i=1}^{s}k_i}^{(r)}(y)\Upsilon_{k_1,\ldots,k_s;n}^{(r)}(x_1,\ldots,x_s), \tag{25}$$

where $\Upsilon_{k_1,\ldots,k_s;n}^{(r)}(x_1,\ldots,x_s) = \mathbb{W}_{k_1,n}^{(r)}(x_1)\mathbb{W}_{k_2,n-k_1}^{(r)}(x_2)\cdots\mathbb{W}_{k_s,n-\sum_{i=1}^{s-1}k_i}^{(r)}(x_s)$, for every integer $n \geq 1$ and real numbers $x_1, \ldots, x_s$, y. For $r = 1$ we can show that (25) is nothing else but (19), and when $s = 1$ in (25) we easily recover Formula (24). On the other hand, the substitution of (23) in (2) brings us; through a direct computation, to the following general results.

Theorem 3.3. *Let $\{A_n(x)\}_{n\geq 0}$ be a sequence of Appell polynomials given by* (2), *and $\{A_n^{(r)}(x)\}_{n\geq 0}$ be the class of Appell polynomials defined by (22), with $r \geq 1$. Then, for every $n \geq r-1$, the following formula holds,*

$$A_n^{(r)}(x) = \sum_{k=0}^{n} \alpha_{n-k} \sum_{j=0}^{k+r} \Omega_{j,k,n}^{(r)}(x) \times \begin{vmatrix} B_{k-j+1}(x+1) & A_j^{(r)}(x) \\ B_{k-j+1}(x) & A_j^{(r)}(x+1) \end{vmatrix}.$$

Let $k \geq r-1$ $(r \geq 1)$, $n \geq 0$ be two integers and $\{A_n^{(r)}(x)\}_{n\geq 0}$ be the class of Appell polynomials given by (22). In the sequel of this section $\Theta_{m,k}^{(r)}(x,y)$ denotes the expression

$$\Theta_{m,k}^{(r)}(x,y) = \begin{vmatrix} B_m(x+1) & A_k^{(r)}(y) \\ B_m(x) & A_k^{(r)}(y+1) \end{vmatrix}.$$

The family of determinants $\{\Theta_{m,k}^{(r)}(x,y)\}_{m,k\geq 0}$, appearing in results of the preceding sections, satisfy some interesting identities. These identities are useful for computing them recursively. Indeed, since $\frac{2t^r e^{(x+y)t}}{e^t+1} = \frac{2t^r e^{xt}}{e^t+1}.e^{yt}$ and $e^{yt} = \sum_{k=0}^{+\infty} y^k \frac{t^k}{k!}$, Expression (22) and formula of series product, imply that we have $A_k^{(r)}(y+z) = \sum_{j=0}^{k}\binom{k}{j} y^j A_{k-j}^{(r)}(z)$. A straightforward computation, using the former formula, allows us to formulate the following property.

Proposition 3.4. *Let $r \geq 1$ and $\{A_n^{(r)}(x)\}_{n\geq 0}$ ($r \geq 1$) be the class of Appell polynomials given by (22). Then, we have*

$$\begin{vmatrix} B_m(x+1) & A_k^{(r)}(z+y) \\ B_m(x) & A_k^{(r)}(z+y+1) \end{vmatrix} = \sum_{j=0}^{k} \binom{k}{j} y^j \begin{vmatrix} B_m(x+1) & A_{k-j}^{(r)}(z) \\ B_m(x) & A_{k-j}^{(r)}(z+1) \end{vmatrix},$$

where x, y, z are reals numbers and m, k are integers. In other words, the class of determinants $\Theta_{m,k}^{(r)}(x,y)$ satisfy the following additive relation,

$$\Theta_{m,k}^{(r)}(x, z+y) = \sum_{j=0}^{k} \binom{k}{j} y^j \Theta_{m,k-j}^{(r)}(x,z).$$

In particular, for $y = (s-1)x$ and $z = x$ we come to the corollary,

Corollary 3.5. *Let $r \geq 1$ and $\{A_n^{(r)}(x)\}_{n\geq 0}$ be the class of Appell polynomials defined as in* (22). *Then, for every real number x we have*

$$\begin{vmatrix} B_m(x+1) & A_k^{(r)}(sx) \\ B_m(x) & A_k^{(r)}(sx+1) \end{vmatrix} = \sum_{j=0}^{k} \gamma_{k,j}(s) x^{k-j} \begin{vmatrix} B_m(x+1) & A_j^{(r)}(x) \\ B_m(x) & A_j^{(r)}(x+1) \end{vmatrix}, \tag{26}$$

where m, k, s are integers and $\gamma_{k,j}(s) = \binom{k}{j}(s-1)^{k-j}$. In other words, we have

$$\Theta_{m,k}^{(r)}(x, sx) = \sum_{j=0}^{k} (s-1)^{k-j} \binom{k}{j} x^{k-j} \Theta_{m,j}^{(r)}(x,x).$$

For $y = (s-t)x$ and $z = tx$ we get the corollary,

Corollary 3.6. *Let $r \geq 1$ and $\{A_n^{(r)}(x)\}_{n\geq 0}$ ($r \geq 1$) be the class of Appell polynomials defined as in* (22). *Then, for every real number x, we have*

$$\begin{vmatrix} B_m(x+1) & A_k^{(r)}(sx) \\ B_m(x) & A_k^{(r)}(sx+1) \end{vmatrix} = \sum_{j=0}^{k} \gamma_{k,j,s,t}(x) \begin{vmatrix} B_m(x+1) & A_j^{(r)}(tx) \\ B_m(x) & A_j^{(r)}(tx+1) \end{vmatrix}, \tag{27}$$

where m, k, t, s and $s \geq t$ are integers and $\gamma_{k,j,s,t}(x) = \binom{k}{j}(s-t)^{k-j}x^{k-j}$. In other words, we have

$$\Theta_{m,k}^{(r)}(x, sx) = \sum_{j=0}^{k} (s-t)^{k-j} \binom{k}{j} x^{k-j} \Theta_{m,j}^{(r)}(x,tx).$$

In fact, considering the properties of the nested sums (see [1, 5, 14]), we can see that Proposition 3.4 and its Corollaries 3.5, 3.6 are special cases of the following general theorem.

Theorem 3.7. *Let $r \geq 1$ and $\{A_n^{(r)}(x)\}_{n\geq 0}$ be the class of Appell polynomials given by (22). Then, for every real x, x_1, $\ldots$, x_p, y ($p \geq 1$) and integers m, n, we have*

$$\Theta_{m,n}^{(r)}\Big(x, \sum_{i=1}^{p} x_i + y\Big) = \sum_{j_1=0}^{n} \sum_{j_2=0}^{n-j_1} \cdots \sum_{j_p=0}^{n-\sum_{i=1}^{p-1} j_i} \Gamma_{n,m,j_1,\ldots,j_p}(x_1, \ldots, x_p; x, y), \tag{28}$$

such that $\Gamma_{n,m,j_1,\ldots,j_p}(x_1,\ldots,x_p;x,y) = \binom{n}{j_1,j_2,\ldots,j_p} x_1^{j_1} \cdots x_p^{j_p} \Theta_{m,n-\sum_{i=1}^{p} j_i}^{(r)}(x,y)$, where the summation takes place over all integers $j_1, j_2, \ldots, j_p$ such that

$$\binom{n}{j_1, j_2, \ldots, j_p} = \begin{cases} \dfrac{n!}{j_1! j_2! \cdots j_p!}, & \text{for} \quad j_1 + \cdots + j_p = n; \\ 0, & \text{otherwise}, \end{cases}$$

which are called the multinomial coefficients [6, p.41] and

$$\binom{n}{j_1, j_2, \ldots, j_s} = \binom{n}{j_1}\binom{n-j_1}{j_2}\cdots\binom{n-\sum_{i=1}^{s-1} j_i}{j_s}.$$

For general properties concerning multinomial coefficients see [4]. When $s = 1$ we show easily that (28) is nothing else but formula of Proposition 3.4. Now for $p = s-1$ and $x = x_1 = \cdots = x_p = y$ we can deduce the following combinatorial identity,

$$\Theta_{m,n}^{(r)}(x, sx) = \sum_{j_1=0}^{n}\sum_{j_2=0}^{n-j_1}\cdots\sum_{j_{s-1}=0}^{n-\sum_{i=1}^{s-2} j_i} \Pi_{n,m,j_1,\ldots,j_p}(x_1, \ldots, x_p; x), \tag{29}$$

where $\Pi_{n,m,j_1,\ldots,j_p}(x_1, \ldots, x_p; x) = \binom{n}{j_1,j_2,\ldots,j_{s-1}} x^{\sum_{i=1}^{s-1} j_i}\Theta^{(r)}_{m,n-\sum_{i=1}^{s-1} j_i}(x, x)$, and comparing (26) and (29) we obtain the identity,

$$\sum_{j=0}^{k}\binom{k}{j}(s-1)^{k-j}x^{k-j}\Theta_{m,j}^{(r)}(x, x) = \sum_{j_1+j_2+\cdots+j_{s-1}=n} \Pi_{n,m,j_1,\ldots,j_p}(x_1, \ldots, x_p; x).$$

Using (28) for $p = s-t$, $x = x_1 = \cdots = x_p$ and $y = tx$, we can derive the following combinatorial identity,

$$\Theta_{m,n}^{(r)}(x, sx) = \sum_{j_1=0}^{n}\sum_{j_2=0}^{n-j_1}\cdots\sum_{j_{s-t}=0}^{n-\sum_{i=1}^{s-(t+1)} j_i} \mathcal{H}_{n;j_1,\ldots,j_{s-t}}(x)\Theta^{(r)}_{m,n-\sum_{i=1}^{s-t} j_i}(x, tx), \tag{30}$$

where $\mathcal{H}_{n;j_1,\ldots,j_{s-t}}(x) = \binom{n}{j_1,j_2,\ldots,j_{s-t}} x^{\sum_{i=1}^{s-t} j_i}$, and comparing (27) and (30), we obtain the identity,

$$\sum_{j=0}^{k}\binom{k}{j}(s-t)^{k-j}x^{k-j}\Theta_{m,j}^{(r)}(x, tx) = \sum_{j_1+\cdots+j_{s-t}=n} \mathcal{H}_{n;j_1,\ldots,j_{s-t}}(x)\Theta^{(r)}_{m,n-\sum_{i=1}^{s-t} j_i}(x, tx).$$

4. Concluding Remarks and Perspective

In this paper we have developed a series of identities involving a determinantal form. This determinantal form, elaborated from a simple calculation process, allowed us to express the Appell polynomials with the aid of Bernoulli, Euler and Genocchi polynomials. Moreover, some addition formulas are established. As far as we know, our formulas are not current in the literature.

Several studies and generalisations of Bernoulli, Euler and Genocchi polynomials have been proposed in the literature. Our perspective is to go deeper into our work, taking into account results of these studies. Especially, we are interested in the elaboration of a recursive and combinatorial process for some classes of Appell polynomials. Therefore, we can perform several identities established in Sections 2 and 3.

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