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Pointwise Slant Submanifolds and their Warped Products in Sasakian Manifolds

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Abstract. Recently, B.-Y. Chen and O.J. Garay studied pointwise slant submanifolds of almost Hermitian manifolds. In this paper, first we study pointwise slant and pointwise pseudo-slant submanifolds of almost contact metric manifolds and then using this notion, we show that there exist a non-trivial class of warped product pointwise pseudo-slant submanifolds of Sasakian manifolds by giving some useful results, including a characterization.

1. Introduction

The study of slant submanifolds is an active field of research in differential geometry. The notion of slant submanifolds of almost Hermitian manifolds was introduced by B.-Y. Chen [8, 9]. Many examples of slant submanifolds in $\mathbf{C}^2$ and $\mathbf{C}^4$ were given by B.-Y. Chen and Y. Tazawa in [16]. Later on, A. Lotta [23] has extended this study for almost contact metric manifolds. Later, Cabrerizo et al. investigated slant submanifolds of a Sasakian manifold [6].

As a generalization of slant submanifolds of an almost Hermitian manifold, F. Etayo [19] has introduced the notion of pointwise slant submanifolds of almost Hermitian manifolds under the name of quasi-slant submanifolds. Recently, B.-Y. Chen and O.J. Garay [15] studied pointwise slant submanifolds of almost Hermitan manifolds. They have obtained many interesting results, including a characterization of such submanifolds. They also have given a method that how to construct examples of pointwise slant submanifolds in Euclidean spaces. Later, K.S. Park [25] has extended this study for almost contact metric manifolds. His definition of pointwise slant submanifolds of almost contact metric manifolds is similar to pointwise slant submanifolds of almost Hermitian manifolds which have been discussed in [15].

Motivated by the above studies, we briefly study pointwise slant and pointwise pseudo-slant submanifolds of Sassakian manifolds. Also, we have seen in [29] that there are no warped product submanifolds in a Sasakian manifold, when the spherical manifold of the warped product is slant. In this paper, we study warped product pointwise pseudo-slant sumanifolds of the form $M_\perp \times M_\theta$ of a Sasakian manifold $\tilde{M}$, where $M_\perp$ and M_θ are anti-invariant and pointwise slant submanifolds of $\tilde{M}$, respectively. Warped product

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submanifolds have been studied rapidly and actively, after B.-Y. Chen's papers on CR-warped products of Kaehler manifolds [10, 11]. Warped product submanifolds of Sasakian manifolds have been studied in (see [20], [29, 33]). Moreover, different kinds of warped product submanifolds of almost Hermitian manifolds were studied in (see [26, 27], [28], [1, 30, 31]). For the survey on warped product manifolds and warped product submanifolds we refer to B.-Y. Chen's books [12, 14] and his survey article [13].

The paper is organised as follows: In Section 2, we give basic definitions and preliminaries formulas needed for this paper. In Section 3, we define pointwise slant submanifolds and our definition is quit different from the definition of pointwise slant submanifolds given in [25]. We present an example of such submanifolds for the justification of our definition and we prove a characterization result. In this section, we also define pointwise pseudo-slant submanifolds and give two preparatory lemmas for further study in the next section. Section 4 is devoted to the study of warped product pointwise pseudo-slant submanifolds of Sasakian manifolds. In [29], we have seen that there are no warped products of the form $M_\perp \times_f M_\theta$ of a Sasakian manifold $\tilde{M}$ such that $M_\perp$ is an anti-invariant submanifold and M_θ is proper slant submanifold of $\tilde{M}$, but if we consider M_θ is a pointwise slant submanifold of $\tilde{M}$, then such warped products exist. As a generalization, we give few application of our derived results.

2. Preliminaries

An *almost contact manifold* is a $(2n+1)$ odd-dimensional manifold $\tilde{M}$ which carries a tensor field φ of the tangent space, a vector field ξ, called *characteristic* or *Reeb vector field* and a 1-form η satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{1}$$

where $I : T\tilde{M} \to T\tilde{M}$ is the identity map [3]. From the definition it follows that $\varphi\xi = 0$, $\eta \circ \varphi = 0$ and the $(1,1)$-tensor field φ has constant rank $2n$ (cf. [3]). An almost contact manifold $(\tilde{M}, \varphi, \eta, \xi)$ is said to be *normal* when the tensor field $N_\varphi = [\varphi, \varphi] + 2d\eta \otimes \xi$ vanishes identically, where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ. It is known that any almost contact manifold $(\tilde{M}, \varphi, \eta, \xi)$ admits a Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2}$$

for any $X, Y \in \Gamma(T\tilde{M})$, the Lie algebra of vector fields on $\tilde{M}$. This metric g is called a *compatible metric* and the manifold $\tilde{M}$ together with the structure (φ, ξ, η, g) is called an *almost contact metric manifold*. As an immediate consequence of (2), one has $\eta(X) = g(X, \xi)$ and $g(\varphi X, Y) = -g(X, \varphi Y)$. Hence the fundamental 2-form Φ of $\tilde{M}$ is defined $\Phi(X, Y) = g(X, \varphi Y)$ and the manifold is said to be *contact metric manifold* if $\Phi = d\eta$. If ξ is a Killing vector field with respect to g, then the contact metric structure is called a *$K-$contact structure*. A normal contact metric manifold is said to be a *Sasakian manifold*. In terms of the covariant derivatives of φ, the Sasakian condition can be expressed by

$$(\tilde{\nabla}_X \varphi)Y = g(X, Y)\xi - \eta(Y)X \tag{3}$$

for all $X, Y \in \Gamma(T\tilde{M})$, where $\tilde{\nabla}$ is the Levi-Civita connection of g. From the formula (3), it follows that $\tilde{\nabla}_X \xi = -\varphi X$.

Let M be a Riemannian manifold isometrically immersed in $\tilde{M}$ and denote by the same symbol g the Riemannian metric induced on M. Let $\Gamma(TM)$ be the Lie algebra of vector fields in M and $\Gamma(T^\perp M)$, the set of all vector fields normal to M. Let ∇ be the Levi-Civita connection on M, then the Gauss and Weingarten formulas are respectively given by [34]

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{4}$$

and

$$\tilde{\nabla}_X V = -A_V X + \nabla^\perp_X V \tag{5}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where $\nabla^{\perp}$ is the normal connection in the normal bundle $T^{\perp}M$ and A_V is the shape operator of M with respect to the normal vector V. Moreover, $h : TM \times TM \to T^{\perp}M$ is the second fundamental form of M in $\tilde{M}$. Furthermore, A_V and h are related by

$$g(h(X, Y), V) = g(A_V X, Y) \tag{6}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

For any X tanget to M, we write

$$\varphi X = TX + FX, \tag{7}$$

where TX and FX are the tangential and normal components of φX, respectively. Then T is an endomorphism of tangent bundle TM and F is a normal bundle valued 1-form on TM. Similarly, for any vector field V normal to M, we put

$$\varphi V = tV + fV, \tag{8}$$

where tV and fV are the tangential and normal components of φV, respectively. Moreover, from (2) and (7), we have

$$g(TX, Y) = -g(X, TY), \tag{9}$$

for any $X, Y \in \Gamma(TM)$.

Throughout the paper, we consider the structure field ξ is tangent to M. Let M be a Riemannian manifold, isometrically immersed in an almost contact metric manifold $(\tilde{M}, \varphi, \xi, \eta, g)$. Hence, if we denote $\mathfrak{D}$ the orthogonal distribution to ξ in TM, then $TM = \mathfrak{D} \oplus \langle\xi\rangle$. For each nonzero vector X tangent to M at $p \in M$, such that X is not proportional to ξ_p, we denote the angle $\theta(X)$ the angle between φX and T_pM. In fact, since $\varphi\xi = 0$, $\theta(X)$ agrees with the angle between φX and $\mathfrak{D}_p$. A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be slant [6], if for each non-zero vector X tangent to M such that X is not proportional to $\langle\xi\rangle$, the angle $\theta(X)$ between φX and T_pM is a constant, i.e., it does not depend on the choice of $p \in M$ and $X \in T_pM - \langle\xi_p\rangle$. In this case $\mathfrak{D}$ is a slant distribution with slant angle θ.

A slant submanifold is said to be *proper slant*, if neither $\theta = 0$ nor $\theta = \frac{\pi}{2}$. We note that on a slant submanifold if $\theta = 0$, then it is an invariant submanifold and if $\theta = \frac{\pi}{2}$, then it is an anti-invariant submanifold. A slant submanifold is said to be *p*roper slant if it is neither invariant nor anti-invariant.

3. Pointwise Slant Submanifolds of Almost Contact Metric Manifolds

As a generalization of slant submanifolds F. Etayo [19] introduced pointwise slant submanifolds of an almost Hermitian manifold under the name of quasi-slant submanifolds. After that, B.-Y. Chen and O.J. Garay studied pointwise slant submanifolds of almost Hermitian manifolds and obtained many interesting results [15].

For any nonzero vector $X \in T_pM$, $p \in M$, orthogonal to ξ, the angle $\theta(X)$ between φX and the tangent space T_pM is called the Wirtinger angle of X. The Wirtinger angle gives rise to a real-valued function $\theta : T^*M = T_pM - \{0\} \to \mathbf{R}$, called the Wirtinger function, defined on the set T^*M consisting of all nonzero tangent vectors on M.

In a similar way, we can define these submanifolds of almost contact metric manifolds as follows:

Definition 3.1. A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be *pointwise slant*, if for a nonzero vector X tangent to M at $p \in M$, such that X is orthogonal to ξ_p, the angle $\theta(X)$ between φX and $T^*M = T_pM - \{0\}$ is independent of the choice of nonzero vector $X \in T^*_pM$. In this case, θ can be regarded as a function on M, which is called the *slant function* of the pointwise slant submanifold.

We note that every slant submanifold is a pointwise slant submanifold but converse may not be true. We also note that a pointwise slant submanifold is *invariant* (respectively, *anti-invariant*) if for each point $p \in M$, the slant function $\theta = 0$ (respectively, $\theta = \frac{\pi}{2}$). A pointwise slant submanifold is slant if the slant function θ is constant on M. Moreover, a pointwise slant submanifold is proper if neither $\theta = 0, \frac{\pi}{2}$ nor θ is constant.

Now, we provide a non-trivial example of pointwise slant submanifolds of an almost contact metric manifold.

Example 3.2. Let $(\mathbf{R}^5, \varphi, \xi, \eta, g)$ be an almost contact metric manifold with cartesian coordinates (x_1, y_1, x_2, y_2, t) and the almost contact structure

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial y_i}, \ \varphi\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j}, \ \varphi\left(\frac{\partial}{\partial t}\right) = 0, \ \ 1 \le i, j \le 2$$

such that $\xi = \frac{\partial}{\partial t}$, $\eta = dt$ and g is the standard Euclidean metric on $\mathbf{R}^5$. Then (φ, ξ, η, g) is an almost contact metric structure on $\mathbf{R}^5$. Now, consider a submanifold M of $\mathbf{R}^5$ defined by

$$\chi(u, v, t) = (u\cos v,\ u\sin v,\ -u + v,\ u - v,\ t),$$

where u is a non vanishing real valued function on M. Then the tangent space TM is spanned by the following vector fields

$$\begin{aligned} v_1 &= \cos v\frac{\partial}{\partial x_1} + \sin v\frac{\partial}{\partial y_1} - \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2}, \\ v_2 &= -u\sin v\frac{\partial}{\partial x_1} + u\cos v\frac{\partial}{\partial y_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial y_2}, \\ v_3 &= \frac{\partial}{\partial t}. \end{aligned}$$

From the assumed almost contact structure on $\mathbf{R}^5$, we have

$$\begin{aligned} \varphi v_1 &= -\cos v\frac{\partial}{\partial y_1} + \sin v\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_2} + \frac{\partial}{\partial x_2}, \\ \varphi v_2 &= u\sin v\frac{\partial}{\partial y_1} + u\cos v\frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_2} - \frac{\partial}{\partial x_2}, \\ \varphi v_3 &= \varphi\left(\frac{\partial}{\partial t}\right) = 0. \end{aligned}$$

Thus, M is a pointwise slant submanifold of $\mathbf{R}^5$ with slant function $\theta = \cos^{-1}\left(\frac{u}{3}\right)$.

We note that some examples of pointwise slant submanifolds are given in [25], when the structure vector field ξ is normal to the submanifold.

Now, we prove the following characterization theorem.

Theorem 3.3. *Let M be a submanifold of an almost contact metric manifold $\tilde{M}$ such that $\xi \in \Gamma(TM)$. Then, M is pointwise slant if and only if*

$$T^2 = \cos^2\theta\,(-I + \eta \otimes \xi)\,, \tag{10}$$

for some real valued function θ defined on the tangent bundle TM of M.

Proof. If M is a pointwise slant submanifold with slant function $\theta : M \to \mathbf{R}$, then at any point $p \in M$, from the definition we have

$$\cos\theta\,(p) = \frac{\|TX\|}{\|\varphi X\|}$$

for any $X \in T_pM$, which gives

$$g(TX, TX) = \cos^2\theta\,(p)\; g(\varphi X, \varphi X)$$
$$= \cos^2\theta\,(p)\; \{g(X, X) - \eta^2(X)\}.$$

Using polarization identity, we obtain

$$g(TX, TY) = \cos^2\theta\,(p)\; \{g(X, Y) - \eta(X)\eta(Y)\}. \tag{11}$$

Then, from (9) and (11), we derive

$$T^2 = -\cos^2\theta\;(I - \eta \otimes \xi)\,.$$

Conversely, if M is a submanifold of $\tilde{M}$ such that $T^2 = \cos^2\theta\;(-I + \eta \otimes \xi)$ holds, for some function θ on M, then

$$g(TX, TX) = -g(T^2X, X) = \cos^2\theta\,\{g(X, X) - \eta^2(X)\},$$

which means that the Wirtinger angle is independent of the choice of $X \in T^*_pM$ at each given point $p \in M$. Hence the submanifold is pointwise slant. This ends the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 3.3.

Corollary 3.4. *Let M be a pointwise slant submanifold of an almost contact metric manifold $\tilde{M}$. Then, we have*

$$g(TX, TY) = \cos^2\theta\,[g(X, Y) - \eta(X)\eta(Y)] \tag{12}$$

$$g(FX, FY) = \sin^2\theta\,[g(X, Y) - \eta(X)\eta(Y)] \tag{13}$$

for any $X, Y \in \Gamma(TM)$.

Proof. The proof follows from Theorem 3.3 and the relations (2) and (7). □

Another useful relation for a pointwise slant submanifold of an almost contact metric manifold is obtained by using (7), (8) and (10) as follows:

$$tFX = \sin^2\theta\,(-X + \eta(X)\xi)\,, \quad fFX = -FTX \tag{14}$$

for any $X \in \Gamma(TM)$.

Now, we define and study pointwise pseudo-slant submanifolds, the generalised version of pseudo-slant submanifolds defined and studied by Cabrerizo [5] and Carriazo [7] under the name of anti-slant submanifolds.

Definition 3.5. A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be a pointwise pseudo-slant submanifold if there exist a pair of orthogonal distributions $\mathfrak{D}^\perp$ and $\mathfrak{D}^\theta$ on M such that

(i) The tangent bundle TM admits the orthogonal direct decomposition $TM = \mathfrak{D}^\perp \oplus \mathfrak{D}^\theta \oplus \langle\xi\rangle$.
(ii) The distribution $\mathfrak{D}^\perp$ is anti-invariant, i.e., $\varphi\mathfrak{D}^\perp \subset T^\perp M$.
(iii) The distribution $\mathfrak{D}^\theta$ is pointwise slant with slant function θ.

Notice that the anti-invariant distribution $\mathfrak{D}^\perp$ of a pointwise pseudo-slant submanifold is a pointwise slant distribution with slant function $\theta = \frac{\pi}{2}$. Moreover, if we denote the dimensions of $\mathfrak{D}^\perp$ and $\mathfrak{D}^\theta$ by m_1 and m_2, respectively, then we have the following possible cases:

(i) If $m_1 = 0$, then M is a pointwise slant submanifold.
(ii) If $m_2 = 0$, then M is an anti-invariant submanifold.

(iii) If $m_1 = 0$ and $\theta = 0$, then M is an invariant submanifold.
(iv) If $\theta = 0$, then M is a contact CR-submanifold.
(v) If θ is constant on M, then M is a pseudo-slant submanifold with slant angle θ.

We note that a pointwise pseudo-slant submanifold is *proper* if $m_1 \neq 0$ and $\theta \neq 0, \frac{\pi}{2}$, which should not be a constant.

The normal bundle $T^\perp M$ of a pointwise pseudo-slant submanifold M is decomposed by

$$T^\perp M = \varphi\mathfrak{D}^\perp \oplus F\mathfrak{D}^\theta \oplus \nu, \ \ \varphi\mathfrak{D}^\perp \perp F\mathfrak{D}^\theta,$$

where ν is a $\varphi-$invariant normal subbundle of $T^\perp M$.

For the integrability of the involved distributions in the Definition 3.5, we give the following useful lemmas for Sasakian manifolds.

Lemma 3.6. *Let M be a pointwise pseudo-slant submanifold of a Sasakian manifold $\tilde{M}$. Then the anti-invariant distribution $\mathfrak{D}^\perp$ is always integrable.*

The proof of Lemma 3.6 is similar to a result of [22].

Lemma 3.7. *Let M be a pointwise pseudo-slant submanifold of a Sasakian manifold $\tilde{M}$. Then*

(i) For any $Z, W \in \Gamma(\mathfrak{D}^\perp \oplus \langle\xi\rangle)$ and $X \in \Gamma(\mathfrak{D}^\theta)$, we have

$$g(\nabla_Z W, TX) = g(h(X,Z), \varphi W) - g(h(Z,W), FX).$$

(ii) For any $X, Y \in \Gamma(\mathfrak{D}^\theta)$ and $Z \in \Gamma(\mathfrak{D}^\perp \oplus \langle\xi\rangle)$, we have

$$\cos^2\theta\, g(\nabla_X Y, Z) = g(h(X,TY), \varphi Z) - g(h(X,Z), FTY) - \eta(Z)\, g(X,TY).$$

Proof. From (4) and (7), we have

$$g(\nabla_Z W, TX) = g(\tilde{\nabla}_Z W, \varphi X) - g(h(Z,W), FX)$$

for any $Z, W \in \Gamma(\mathfrak{D}^\perp \oplus \langle\xi\rangle)$ and $X \in \Gamma(\mathfrak{D}^\theta)$. Using (2), (3) and the orthogonality of the distributions, we obtain

$$g(\nabla_Z W, TX) = -g(\tilde{\nabla}_Z \varphi W, X) - g(h(Z,W), FX).$$

Thus (i) follows from above equation by using (5) and (6). Now, we have

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X Y, Z) = g(\varphi\tilde{\nabla}_X Y, \varphi Z) + \eta(Z)\, g(\tilde{\nabla}_X Y, \xi),$$

for any $X, Y \in \Gamma(\mathfrak{D}^\theta)$ and $Z \in \Gamma(\mathfrak{D}^\perp \oplus \langle\xi\rangle)$. Using the covariant derivative property of the Riemannian connection, we derive

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X \varphi Y, \varphi Z) - g((\tilde{\nabla}_X \varphi)Y, \varphi Z) - \eta(Z)\, g(\tilde{\nabla}_X \xi, Y).$$

Then from (3), (4), (7) and the fact that ξ is orthogonal to $\mathfrak{D}^\theta$, we get

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\tilde{\nabla}_X TY, \varphi Z) + g(\tilde{\nabla}_X FY, \varphi Z) + \eta(Z)\, g(\varphi X, Y) \\ &= g(h(X,TY), \varphi Z) - g(\tilde{\nabla}_X \varphi FY, Z) + g((\tilde{\nabla}_X \varphi)FY, Z) \\ &\quad + \eta(Z)\, g(TX, Y). \end{aligned}$$

Again, using (3), (8) and (9), we obtain

$$\begin{aligned} g(\nabla_X Y, Z) &= g(h(X,TY), \varphi Z) - g(\tilde{\nabla}_X tFY, Z) - g(\tilde{\nabla}_X fFY, Z) \\ &\quad - \eta(Z)\, g(X,TY). \end{aligned}$$

From (14), we find that

$$\begin{aligned} g(\nabla_X Y, Z) &= g(h(X,TY), \varphi Z) + \left(\sin^2\theta\right) g(\tilde{\nabla}_X Y, Z) + (\sin 2\theta)\ X(\theta)\, g(Y,Z) \\ &\quad + g(\tilde{\nabla}_X FTY, Z) - \eta(Z)\, g(X,TY). \end{aligned}$$

Hence, the second part of the lemma follows from the above equation by using the orthogonality of vector fields and the relations (4)-(6), which proves the lemma completely. □

4. Warped Products $M_\perp \times_f M_\theta$ with Pointwise Slant Factor

In this section, we study warped product submanifolds of Sasakian manifolds, by considering that one of the factor is a pointwise slant submanifold. First, we give a brief introduction of warped product manifolds.

In [2], Bishop and O'Neill introduced the notion of warped product manifolds as follows: Let M_1 and M_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively, and a positive differentiable function f on M_1. Consider the product manifold $M_1 \times M_2$ with its projections $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$. Then their warped product manifold $M = M_1 \times_f M_2$ is the Riemannian manifold $M_1 \times M_2 = (M_1 \times M_2, g)$ equipped with the Riemannian structure such that

$$g(X,Y) = g_1(\pi_{1\star}X, \pi_{1\star}Y) + (f \circ \pi_1)^2 g_2(\pi_{2\star}X, \pi_{2\star}Y)$$

for any vector field X, Y tangent to M, where $\star$ is the symbol for the tangent maps. A warped product manifold $M = M_1 \times_f M_2$ is said to be *trivial* or simply a *Riemannian product manifold* if the warping function f is constant. Let X be an unit vector field tangent to M_1 and Z be an another unit vector field on M_2, then from Lemma 7.3 of [2], we have

$$\nabla_X Z = \nabla_Z X = (X \ln f)Z \tag{15}$$

where ∇ is the Levi-Civita connection on M. If $M = M_1 \times_f M_2$ be a warped product manifold then the base manifold M_1 is totally geodesic in M and the fiber M_2 is totally umbilical in M [2, 10].

Analogous to CR-warped products introduced in [10], we define the warped product pointwise pseudo-slant submanifolds as follows.

Definition 4.1. A warped product of anti-invariant and pointwise slant submanifolds, say $M_\perp$ and M_θ, of a Sasakian manifold $\tilde{M}$ is called a *w*arped product pointwise pseudo-slant submanifold.

A warped product pointwise pseudo-slant submanifold is called *p*roper if M_θ is a proper pointwise slant submanifold and $M_\perp$ is an anti-invariant submanifold of $\tilde{M}$.

There are two kinds of warped product pointwise pseudo-slant submanifolds $M_\perp \times_f M_\theta$ and $M_\theta \times_f M_\perp$ in a Sasakian manifold $\tilde{M}$ such that $M_\perp$ is an anti-invariant submanifold and M_θ is a pointwise slant submanifold of $\tilde{M}$.

Note that a warped product pointwise pseudo-slant submanifold $M = M_\perp \times_f M_\theta$ or $M = M_\theta \times_f M_\perp$ is a warped product contact CR-submanifold if the slant function $\theta = 0$. Similarly, the warped product pointwise pseudo-slant submanifold of the form $M = M_\perp \times_f M_\theta$ or $M = M_\theta \times_f M_\perp$ is a warped product pseudo-slant submanifold if θ is constant, i.e., M_θ is proper slant.

In this section, we study the warped product pointwise pseudo-slant submanifold of the form $M = M_\perp \times_f M_\theta$ of a Sasakian manifold $\tilde{M}$. If we consider the structure vector field ξ is tangent to M, then either $\xi \in \Gamma(TM_\perp)$ or $\xi \in \Gamma(TM_\theta)$. When ξ is tangent to M_θ, then it is easy to check that warped product is trivial (see [29]), therefore we consider $\xi \in \Gamma(TM_\perp)$, and in this case we prove the following useful results for such warped products.

Lemma 4.2. *Let $M = M_\perp \times_f M_\theta$ be a warped product pointwise pseudo-slant submanifold of a Sasakian manifold $\tilde{M}$ such that $\xi \in \Gamma(TM_\perp)$, where $M_\perp$ is an anti-invariant submanifold and M_θ is a proper pointwise slant submanifold of $\tilde{M}$. Then, we have*

$$g(h(Y,Z), FTX) - g(h(TX,Z), FY) = (\sin 2\theta)\, Z(\theta)\, g(X,Y) \tag{16}$$

for any $X, Y \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$.

Proof. For any $X, Y \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$, we have

$$g(\tilde{\nabla}_Z X, Y) = Z(\ln f)\, g(X,Y). \tag{17}$$

On the other hand, we also have

$$g(\tilde{\nabla}_Z X, Y) = g(\varphi\tilde{\nabla}_Z X, \varphi Y) + \eta(Y)\, g(\tilde{\nabla}_Z X, \xi).$$

The second term in the right hand side of the above equation vanishes identically by using the fact that $\xi \in \Gamma(TM_\perp)$, thus we derive

$$g(\tilde{\nabla}_Z X, Y) = g(\tilde{\nabla}_Z \varphi X, \varphi Y) - g((\tilde{\nabla}_Z \varphi)X, \varphi Y).$$

Using (3), (4), (7), (15) and the orthogonality of vector fields, we find

$$\begin{aligned} g(\tilde{\nabla}_Z X, Y) &= g(\tilde{\nabla}_Z TX, TY) + g(\tilde{\nabla}_Z TX, FY) + g(\tilde{\nabla}_Z FX, \varphi Y) \\ &= Z(\ln f)\, g(TX, TY) + g(h(TX, Z), FY) - g(\varphi\tilde{\nabla}_Z FX, Y) \\ &= \left(\cos^2\theta\right) Z(\ln f)\, g(X, Y) + g(h(TX, Z), FY) - g(\tilde{\nabla}_Z \varphi FX, Y) \\ &\quad + g((\tilde{\nabla}_Z \varphi)FX, Y). \end{aligned}$$

The last relation in the above equation is zero by using (3) and the orthogonality of vector fields. Then, from (8) and (14), the above equation takes the form

$$\begin{aligned} g(\tilde{\nabla}_Z X, Y) &= \left(\cos^2\theta\right) Z(\ln f)\, g(X, Y) + g(h(TX, Z), FY) - g(\tilde{\nabla}_Z(-\sin^2\theta)X, Y) \\ &\quad + g(\tilde{\nabla}_Z FTX, Y) \\ &= \left(\cos^2\theta\right) Z(\ln f)\, g(X, Y) + g(h(TX, Z), FY) + \left(\sin^2\theta\right) g(\tilde{\nabla}_Z X, Y) \\ &\quad + (\sin 2\theta) Z(\theta)\, g(X, Y) - g(A_{FTX} Z, Y). \end{aligned}$$

Again, using (4) and (15), we derive

$$\begin{aligned} g(\tilde{\nabla}_Z X, Y) &= Z(\ln f)\, g(X, Y) + g(h(TX, Z), FY) + (\sin 2\theta) Z(\theta)\, g(X, Y) \\ &\quad - g(h(Y, Z), FTX). \end{aligned} \tag{18}$$

Then, from (17) and (18), we get (16), which completes the proof. □

Lemma 4.3. *Let $M = M_\perp \times_f M_\theta$ be a warped product pointwise pseudo-slant submanifold of a Sasakian manifold $\tilde{M}$ such that $\xi \in \Gamma(TM_\perp)$, where $M_\perp$ and M_θ are anti-invariant and pointwise slant submanifolds of $\tilde{M}$, respectively. Then*

$$g(h(X, Y), \varphi Z) - g(h(X, Z), FY) = Z(\ln f)\, g(X, TY) + \eta(Z)\, g(X, Y) \tag{19}$$

for any $X, Y \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$.

Proof. From (4), we have

$$g(h(X, Y), \varphi Z) = g(\tilde{\nabla}_X Y, \varphi Z) = -g(\varphi\tilde{\nabla}_X Y, Z)$$

for any $X, Y \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$. Then, by using the covariant derivative formula of the Riemannain connection, we derive

$$g(h(X, Y), \varphi Z) = g((\tilde{\nabla}_X \varphi)Y, Z) - g(\tilde{\nabla}_X \varphi Y, Z).$$

Thus, from (3) and (7), we obtain

$$g(h(X, Y), \varphi Z) = \eta(Z)\, g(X, Y) - g(\tilde{\nabla}_X TY, Z) - g(\tilde{\nabla}_X FY, Z).$$

The result follows from the above relation by using (4)-(6) and (15). Hence, the proof is complete. □

Lemma 4.4. *Let $M = M_\perp \times_f M_\theta$ be a warped product pointwise pseudo-slant submanifold of a Sasakian manifold $\tilde{M}$ such that $\xi \in \Gamma(TM_\perp)$, where $M_\perp$ and M_θ are anti-invariant and pointwise slant submanifolds of $\tilde{M}$, respectively. Then*

$$g(h(Y,Z),FTX) - g(h(TX,Z),FY) = \left(2\cos^2\theta\right) Z(\ln f)\, g(X,Y) \tag{20}$$

for any $X, Y \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$.

Proof. Interchanging X by Y in (19), we get

$$g(h(X,Y),\varphi Z) - g(h(Y,Z),FX) = -Z(\ln f)\, g(X,TY) + \eta(Z)\, g(X,Y). \tag{21}$$

Subtracting (21) from (19), we obtain

$$2Z(\ln f)\, g(X,TY) = g(h(Y,Z),FX) - g(h(X,Z),FY).$$

Interchanging X by TX and using (12), we get the required result, which completes the proof. □

A warped product submanifold $M = M_1 \times_f M_2$ of a Sasakian manifold $\tilde{M}$ is said to be *mixed totally geodesic* if $h(X,Z) = 0$, for any $X \in \Gamma(TM_1)$ and $Z \in \Gamma(TM_2)$.

The following corollary is an immediate consequence of Lemma 4.4.

Corollary 4.5. *There does not exist any proper warped product mixed totally geodesic submanifold of the form $M = M_\perp \times_f M_\theta$ of a Sasakian manifold $\tilde{M}$ such that $M_\perp$ is an anti-invariant submanifold and M_θ is a proper pointwise slant submanifold of $\tilde{M}$.*

Proof. From (20) and the mixed totally geodesic condition, we have

$$\left(\cos^2\theta\right) Z(\ln f)\, g(X,Y) = 0.$$

Since g is Riemannian metric and M is proper, then $\cos^2\theta \neq 0$. Thus the proof follows from the above relation. □

Now, from Lemma 4.2 and Lemma 4.4, we have following result.

Theorem 4.6. *Let $M = M_\perp \times_f M_\theta$ be a warped product pointwise pseudo-slant submanifold of a Sasakian manifold $\tilde{M}$ such that $\xi \in \Gamma(TM_\perp)$, where $M_\perp$ is an anti-invariant submanifold and M_θ is a proper pointwise slant submanifold of $\tilde{M}$. Then, one of the following statements holds:*

(i) Either M is warped product of anti-invariant submanifods, i.e., $\theta = \frac{\pi}{2}$,
(ii) or if $\theta \neq \frac{\pi}{2}$, then $Z(\ln f) = (\tan\theta)\, Z(\theta)$, for any $Z \in \Gamma(TM_\perp)$.

Proof. From (16) and (20), we have

$$\cos^2\theta\{Z(\ln f) - (\tan\theta)\, Z(\theta)\}\, g(X,Y) = 0. \tag{22}$$

Since g is Riemannian metric, therefore from (22), we conclude that either $\cos^2\theta = 0$ or $Z(\ln f) - (\tan\theta)\, Z(\theta) = 0$. Consequently, either $\theta = \frac{\pi}{2}$ or $Z(\ln f) = (\tan\theta)\, Z(\theta)$, which proves the theorem completely. □

Now, we have the following applications of the above theorem.

1. If we consider the slant function $\theta = 0$, which is of course differ from $\frac{\pi}{2}$ in Theorem 4.7, then, we get $Z(\ln f) = 0$, i.e., f is constant. Thus, the Theorem 2.1 of [20] is a special case of Theorem 4.7.

2. Also, if we assume that the slant function θ is constant, i.e., M_θ is a proper slant submanifold in Theorem 4.7, then again $Z(\ln f) = 0$. In this case the warped is of the form $M_\perp \times_f M_\theta$ such that M_θ is proper slant and in this case also, the warped products do not exist. Hence, Theorem 4.7 is the generalization of Theorem 4.1 of [29] as well.

In the sequel, we give the following characterization by using a well-known result of Hiepko [21].

Theorem 4.7. *Let M be a pointwise pseudo-slant submanifold of a Sasakian manifold $\tilde{M}$. Then M is locally a warped product submanifold of the form $M_\perp \times_f M_\theta$ if and only if*

$$A_{\varphi Z}TX - A_{FTX}Z = \eta(Z)TX - \left(\cos^2\theta\right)Z(\mu)X, \quad \forall\, Z \in \Gamma(\mathfrak{D}^\perp),\ X \in \Gamma(\mathfrak{D}^\theta), \tag{23}$$

for some smooth function μ on M satisfying $Y(\mu) = 0$, for any $Y \in \Gamma(\mathfrak{D}^\theta)$.

Proof. Let $M = M_\perp \times_f M_\theta$ be a warped product pointwise pseudo-slant submanifold of a Sasakian manifold $\tilde{M}$. Then for any $X \in \Gamma(TM_\theta)$ and $Z, W \in \Gamma(TM_\perp)$, we have

$$g(A_{\varphi Z}X, W) = g(h(X, W), \varphi Z) = g(\tilde{\nabla}_W X, \varphi Z).$$

Using (4), (15) and the orthogonality of vector fields, we get

$$g(A_{\varphi Z}X, W) = W(\ln f)\, g(X, \varphi Z) = 0,$$

which means that $A_{\varphi Z}X$ has no component in $TM_\perp$. Similarly, we can find that $g(A_{FX}Z, W) = 0$, for any $X \in \Gamma(TM_\theta)$ and $Z, W \in \Gamma(TM_\perp)$, i.e., $A_{FX}Z$ also has no component in $TM_\perp$. Therefore, we conclude that $A_{\varphi Z}X - A_{FX}Z$ lies in TM_θ. Then, from Lemma 4.3 with this fact, we get (23).

Conversely, if M is a pointwise pseudo-slant submanifold such that (23) holds, then from Lemma 3.7 (i), we have

$$g(\nabla_Z W, TX) = g(A_{\varphi W}X - A_{FX}W, Z)$$

for any $X \in \Gamma(\mathfrak{D}^\theta)$ and $Z, W \in \Gamma(\mathfrak{D}^\perp \oplus \langle\xi\rangle)$. Interchanging X by TX and using (10), we obtain

$$g(\nabla_Z W, X) = -\left(\sec^2\theta\right)\, g(A_{\varphi W}TX - A_{FTX}W, Z)$$

Then from the given condition (23), we get

$$g(\nabla_Z W, X) = (Z\mu)\, g(X, Z) - \left(\sec^2\theta\right)\eta(Z)\, g(TX, Z) = 0,$$

which means that the leaves of the distribution $\mathfrak{D}^\perp \oplus \langle\xi\rangle$ are totally geodesic in M. On the other hand, from Lemma 3.7 (ii), we also have,

$$\left(\cos^2\theta\right)\, g(\nabla_X Y, Z) = g(A_{\varphi Z}TY - A_{FTY}Z, X) - \eta(Z)\, g(X, TY), \tag{24}$$

By polarization identity, we find

$$\left(\cos^2\theta\right)\, g(\nabla_Y X, Z) = g(A_{\varphi Z}TX - A_{FTX}Z, Y) + \eta(Z)\, g(X, TY), \tag{25}$$

Subtracting (25) from (24) and using (23), we get

$$\left(\cos^2\theta\right)\, g([X, Y], Z) = 0. \tag{26}$$

Since $\mathfrak{D}^\theta$ is a proper pointwise slant distribution, i.e., $\cos^2\theta \neq 0$, then (26) implies that $\mathfrak{D}^\theta$ is integrable. If we consider M_θ be a leaf of $\mathfrak{D}^\theta$ and h^θ is the second fundamental form of M_θ in M, then we have

$$g(h^\theta(X, Y), Z) = g(\nabla_X Y, Z).$$

Using (24), we derive

$$g(h^\theta(X, Y), Z) = \left(\sec^2\theta\right)\{g(A_{\varphi Z}TY - A_{FTY}Z, X) - \eta(Z)\, g(X, TY)\}.$$

Then from (23), we obtain

$$g(h^{\theta}(X,Y),Z) = -Z(\mu)\,g(X,Y),$$

or equivalently, we find

$$h^{\theta}(X,Y) = -\vec{\nabla}\mu\,g(X,Y).$$

Hence, M_θ is a totally umbilical submanifold of M with the mean curvature vector $H^\theta = -\vec{\nabla}\mu$, where $\vec{\nabla}\mu$ is the gradient vector of the function μ. Since $Y(\mu) = 0$, for any $Y \in \Gamma(\mathfrak{D}^\theta)$, then we show that $H^\theta = -\vec{\nabla}\mu$ is parallel with respect to the normal connection, say D^n of M_θ in M (see [1]). Thus, M_θ is a totally umbilical submanifold of M with a non vanishing parallel mean curvature vector $H^\theta = -\vec{\nabla}\mu$, i.e., M_θ is an extrinsic sphere in M. Then from a result of Heipko [21], we conclude that M is a warped product manifold of the form $M_\perp \times_\mu M_\theta$. Hence, the proof is complete. $\square$

We note that the inequality for the squared norm of the second fundamental form $\|h\|^2$ of these kinds of warped products is not sharp because to evaluate the squared norm of the second fundamental form, we have to assume that the warped product is mixed totally geodesic but this is a case of non-existence of such warped products (see; Corollary 4.5). Secondly, if we do not assume that the warped product is mixed totally geodesic, then to discuss the equality case in the inequality, from the leaving terms in inequality, we will get again that M is a mixed totally geodesic submanifold and consequently it is again a case of the non-existence of such warped products. Hence, the inequality is not sharp.

In the end of discussion, we also note that we haven't considered the study of warped product pointwise pseudo-slant submanifolds of the form $M_\theta \times_f M_\perp$ in this paper. The reason is that: These kinds of warped products are the special case of pseudo-slant warped products $M_\theta \times_f M_\perp$ studied in [33], where M_θ is a proper slant submanifold.

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