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From Hom$(A, X) \cong$ Hom(B, X) to $A \cong B$

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Abstract. Let A and B be two R-modules. We examine conditions under which Hom$(A, X) \cong$ Hom(B, X), implies that $A \cong B$, where X belongs to an appropriate class of R-modules. Different perspectives of the question are studied. In the case of abelian groups ($\mathbb{Z}$-modules), this investigation gives a partial answer to an old problem of L. Fuchs.

1. Introduction

In group theory, if G_1, G_2 are finite groups and $|\mathrm{Hom}(G_1, H)| = |\mathrm{Hom}(G_2, H)|$ for every finite group H, then G_1 is isomorphic to G_2 (this result is an outcome of L. Lovász's works in [10],[11] and [12]). On the other hand, L. Fuchs posed in [5, Page 208, Problem 34] the following problem: does there exist a set $\mathcal{X}$ of (abelian) groups X such that Hom$(A, X) \cong$ Hom(B, X) for every $X \in \mathcal{X}$ implies that $A \cong B$? This problem has been extensively studied in [1],[2] and [3] and some classes of abelian groups were obtained which give some answers to Fuchs's problem 34.

In this article, every ring R is associative with identity and any module is a unitary module. Posing the Fuchs 34 question in R-Mod, the category of unitary modules over a ring R, one has to distinguish three possibilities one is confronted with. In the sequel by Hom$(A, X) \overset{T}{\cong}$ Hom(B, X), we mean that these two structures are isomorphic as T-modules. Moreover, suppose that $\mathcal{X}$ is a "suitable" subclass of R-Mod. The first and perhaps most common version of this question is as follows:

Question 1. Let R be a commutative ring and A and B be two R-modules and Hom$(A, X) \overset{R}{\cong}$ Hom(B, X) for every R-module $X \in \mathcal{X}$. Is it true that $A \overset{R}{\cong} B$? Though, as we already asked, this question can be posed for every commutative ring, in this paper, we mainly focus on the case $R = \mathbb{Z}$, i.e., on the category of abelian groups. In Section 2, we determine several classes of abelian groups in which this question has a positive answer. The reader is reminded that in this section, we follow a more elementary approach than [1],[2] and [3].

The second version which is a stronger form than the above one is the following. Remember that when R is commutative, there is a ring homomorphism from R to End(X), for any R-module X:

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Question 2. Let R be an arbitrary ring, A and B two R-modules and

$$\mathrm{Hom}(A, X) \overset{S}{\cong} \mathrm{Hom}(B, X)$$

for every R-module $X \in \mathcal{X}$, where $S = \mathrm{End}_R(X)$. Then is it true that $A \overset{R}{\cong} B$? Section 3 is devoted to this question.

The third version is the strongest one (with respect to its hypothesis):

Question 3. Let A and B be two modules over an arbitrary ring R such that the two functors $\mathrm{Hom}(A, -)$ and $\mathrm{Hom}(B, -)$ are (naturally) isomorphic. Then is it true that $A \overset{R}{\cong} B$? The answer of this question is affirmative and is actually an immediate consequence of Yoneda's Lemma. The reader may find a proof, for example, in [13, 44.6]. A partial case of of this question, when R is an integral domain has been solved in [1, Theorem 3.1]. Note that, in the proof of [1, Theorem 3.1], R is not needed to be an integral domain and also the proof works for any locally small category in the place of R-Mod. Regarding to Question 3, the reader may be curious on behavior of derived functors of Hom functor. Let A and B be two non-isomorphic projective R-modules. Then $\mathrm{Ext}(A, -)$ and $\mathrm{Ext}(B, -)$ are naturally isomorphic due to the fact that for a projective module P, $\mathrm{Ext}(P, X) = 0$ for every R-module X.

Along this line, we may pose one further question: let R be an arbitrary ring and A and B two R-modules with $\mathrm{Hom}(A, X) \overset{\mathbb{Z}}{\cong} \mathrm{Hom}(B, X)$ for every R-module $X \in \mathcal{X}$. Is it true that $A \overset{R}{\cong} B$? However, the next example gives a negative answer to this question immediately, even when $\mathcal{X} = R$-Mod.

Example 1.1. Let $R = M_{2\times 2}(\mathbb{R})$ (two by two matrices over the real field $\mathbb{R}$), and T be a simple R-module. It is well-know that $\mathrm{End}_R(T) = \mathbb{R}$. Suppose that $A = T$ and $B = T \oplus T$. Then for every R-module K we have

$$\mathrm{Hom}_R(A, K) \overset{\mathbb{Z}}{\cong} \mathrm{Hom}_R(B, K)$$

because K is nothing but $\bigoplus_I T$, hence

$$\mathrm{Hom}_R(T, \oplus_I T) \cong \bigoplus_I \mathrm{Hom}(T, T) \cong \oplus_I \mathbb{R}$$

and $\mathrm{Hom}_R(B, K) \cong (\bigoplus_I \mathbb{R}) \oplus (\bigoplus_I \mathbb{R})$. Since $\mathbb{R} \overset{\mathbb{Z}}{\cong} \mathbb{R} \oplus \mathbb{R}$, we have $\mathrm{Hom}_R(A, K) \overset{\mathbb{Z}}{\cong} \mathrm{Hom}_R(B, K)$ for every R-module K, but $A \ncong B$.

As far as the first question is concerned, the following example shows that, sometimes, one has to restrict oneself to finitely generated modules, even if R is a field. In the next example we use a result (it is also named as Erdös-Kaplansky Theorem) which says: If F is a field, I is an infinite set and $V = \prod V_i$, where V_i's are non-zero vector spaces over F, then $\dim V = |V| = \prod_I |V_i|$ (see [8, Chapter 9, Section 5]).

Example 1.2. Let F be a field such that $|F| \geq 2^c$, where by c we mean the continuum (i.e., $2^{\aleph_0}$). Now consider two sets I and J with $|I| = c$ and $|J| = \aleph_0$. Put $A = F^{(I)}$ and $B = F^{(J)}$. In this case, $\mathrm{Hom}_F(A, W) \cong \mathrm{Hom}_F(B, W)$ for every F-module W. Because $\mathrm{Hom}(A, W) \cong \prod_I \mathrm{Hom}(F, W) = W^I$ and on the other hand $\mathrm{Hom}(B, W) = \prod_J \mathrm{Hom}(F, W) = W^J$. Since by Erdös-Kaplansky Theorem $\dim W^I = |W|^{|I|}$ and $\dim W^J = |W|^{|J|}$ and $|W| \geq 2^c$, we have $|W|^{|I|} = |W|^{|J|}$ and hence $\mathrm{Hom}_F(A, W) \cong \mathrm{Hom}_F(B, W)$, but $A \ncong B$.

2. Abelian Groups

As we mentioned in the introduction, a special but very important case of the first question is the case $R = \mathbb{Z}$. L. Fuchs in [5, Page 208, Problem 34] posed the following problem: does there exist a set $\mathcal{X}$ of (abelian) groups X such that $\mathrm{Hom}(A, X) \cong \mathrm{Hom}(B, X)$ for every $X \in \mathcal{X}$ implies that $A \cong B$? The next results answer this question for some classes of abelian groups. In this section, by $A \cong B$ we mean $A \overset{\mathbb{Z}}{\cong} B$, unless stated otherwise. Following [2], a class $\mathcal{X}$ of abelian groups is called *a Fuchs 34 class*, when A and B in $\mathcal{X}$ are isomorphic if and only if $\mathrm{Hom}(A, X) \cong \mathrm{Hom}(B, X)$ for every $X \in \mathcal{X}$.

We begin with finitely generated abelian groups which are easier to deal with because of the fundamental theorem of finitely generated abelian groups .

Proposition 2.1. *Let A and B be two finitely generated abelian groups and $\mathrm{Hom}(A, X) \cong \mathrm{Hom}(B, X)$ for every cyclic group X, then $A \cong B$. In particular, the class of finitely generated abelian groups is a Fuchs 34 class.*

Proof. By the fundamental theorem of finitely generated abelian groups, we have that $A \cong \mathbb{Z}^n \oplus H_1$ and $B \cong \mathbb{Z}^m \oplus H_2$, where H_1, H_2 are two finite abelian groups. First we show that $n = m$ and after that we prove that $H_1 \cong H_2$. We know that

$$\mathbb{Z}^n \cong \mathrm{Hom}(A, \mathbb{Z}) \overset{\mathbb{Z}}{\cong} \mathrm{Hom}(B, \mathbb{Z}) \cong \mathbb{Z}^m$$

This implies that $n = m$. Choose $d \in \mathbb{N}$ such that both the order of H_1 and the order of H_2 divide d. Then it is obvious that

$$\mathrm{Hom}(H_1, \mathbb{Z}_d) = H_1, \quad \mathrm{Hom}(H_2, \mathbb{Z}_d) = H_2.$$

Hence $\mathbb{Z}_d^n \oplus H_1 \cong \mathrm{Hom}(A, \mathbb{Z}_d) \cong \mathrm{Hom}(B, \mathbb{Z}_d) \cong \mathbb{Z}_d^n \oplus H_2$, consequently, $H_1 \cong H_2$. By the above steps we conclude that $A \cong B$. □

Proposition 2.2. *Let R be a P.I.D and A,B be two finitely generated R-modules. If $\mathrm{Hom}_R(A, X) \overset{R}{\cong} \mathrm{Hom}_R(B, X)$ for all cyclic modules X, then $A \overset{R}{\cong} B$.*

Proof. The proof is similar to the proof of Proposition 2.1. □

Convention 1. In the sequel, we suppose the weak Generalized Continuum Hypothesis (the weak GCH), that is, "If α and β are two infinite cardinals and $2^\alpha = 2^\beta$, then $\alpha = \beta$". This property follows from GCH (the generalized continuum hypothesis). Although independent of the axioms of ZFC (the Zermelo-Fraenkel set theory with the Axiom of Choice), the statement is weaker than the GCH in the frame of ZFC (see for example [7]).

We also need the following lemma before establishing our result on divisible groups.

Lemma 2.3. *Let p be a prime number, J_p be the group of p-adic integers and $\mathbb{Z}_{p^\infty}$ be the Prüfer p-group. Then $\mathrm{Hom}(\mathbb{Z}_{p^\infty}, \mathbb{Z}_{p^\infty}) \cong J_p$ and $\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}_{p^\infty}) \cong \mathbb{Q}^{(c)} \cong \mathbb{R}$.*

Proof. By [5, Page 181, Example 3], $\mathrm{Hom}(\mathbb{Z}_{p^\infty}, \mathbb{Z}_{p^\infty}) \cong J_p$. Now, let $\mathbb{Z}[1/p] = \{\frac{m}{p^n} \mid m, n \in \mathbb{Z}\}$. Consider the following exact sequence:

$$0 \longrightarrow \mathbb{Z}[1/p] \longrightarrow \mathbb{Q} \longrightarrow \frac{\mathbb{Q}}{\mathbb{Z}[1/p]} \longrightarrow 0$$

Applying $\mathrm{Hom}(-, \mathbb{Z}_{p^\infty})$ (recall that $\mathbb{Z}_{p^\infty}$ is an injective $\mathbb{Z}$-module and hence $\mathrm{Hom}(-, \mathbb{Z}_{p^\infty})$ is exact) and observing that $\mathrm{Hom}(\frac{\mathbb{Q}}{\mathbb{Z}[1/p]}, \mathbb{Z}_{p^\infty}) = 0$ we obtain

$$\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}_{p^\infty}) \cong \mathrm{Hom}(\mathbb{Z}[1/p], \mathbb{Z}_{p^\infty}).$$

By [5, Page 181, Example 4], we know that $\mathrm{Hom}(\mathbb{Z}[1/p], \mathbb{Z}_{p^\infty}) \cong \mathbb{R}$ and therefore $\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}_{p^\infty}) \cong \mathbb{R}$. □

It is well-known that every abelian group G can be written as $G = G_d \oplus G_r$, where G_d is the unique maximal divisible subgroup of G and G_r is the reduced part of G.

Remark 2.4. Let A and B be two abelian groups. If $\mathrm{Hom}(A, B_r) \cong \mathrm{Hom}(B, B_r)$ and A is divisible, then B is divisible too. If B is not divisible, then $B_r \neq 0$ and hence $\mathrm{Hom}(B, B_r) \neq 0$, but $\mathrm{Hom}(A, B_r) = 0$ because A is divisible.

Theorem 2.5. *Let A, B be two divisible abelian groups. If $\mathrm{Hom}(A, X) \overset{\mathbb{Z}}{\cong} \mathrm{Hom}(B, X)$ where $X \in \{\mathbb{Q}, \mathbb{Z}_{p^\infty}$: p is prime $\}$, then $A \cong B$. In particular the class of divisible abelian groups is a Fuchs 34 class.*

Proof. It is well-known that $A \cong \mathbb{Q}^{(I)} \oplus (\bigoplus_{p\in P} \mathbb{Z}_{p^\infty}^{(I_p)})$ and $B \cong \mathbb{Q}^{(L)} \oplus (\bigoplus_{p\in P} \mathbb{Z}_{p^\infty}^{(L_p)})$. Since $\mathrm{Hom}(A, \mathbb{Q}) \cong \mathrm{Hom}(B, \mathbb{Q})$ we have $\mathbb{Q}^I \cong \mathbb{Q}^L$. Now if I or L is finite, then we have $|I| = |L|$. If on the other hand, I, L are infinite sets, then we have $\aleph_0^{|I|} = \aleph_0^{|L|}$, which implies that $2^{|I|} = 2^{|L|}$ and now by the weak GCH, $|I| = |L|$.
Now consider $\mathrm{Hom}(A, \mathbb{Z}_{p^\infty}) \cong \mathrm{Hom}(B, \mathbb{Z}_{p^\infty})$, which implies that

$$\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}_{p^\infty})^I \oplus \mathrm{Hom}(\mathbb{Z}_{p^\infty}, \mathbb{Z}_{p^\infty})^{I_p} \cong \mathrm{Hom}(\mathbb{Q}, \mathbb{Z}_{p^\infty})^L \oplus \mathrm{Hom}(\mathbb{Z}_{p^\infty}, \mathbb{Z}_{p^\infty})^{L_p}$$

By Lemma 2.3, $\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}_{p^\infty}) \cong \mathbb{R}$ and $\mathrm{Hom}(\mathbb{Z}_{p^\infty}, \mathbb{Z}_{p^\infty}) \cong J_p$. Thus we can write

$$\mathbb{R}^I \oplus J_p^{I_p} \cong \mathbb{R}^L \oplus J_p^{L_p}.$$

Tensoring the above formula by $\mathbb{Z}_p$, we have

$$J_p^{I_p} \otimes \mathbb{Z}_p \cong J_p^{L_p} \otimes \mathbb{Z}_p,$$

Inasmuch as $\mathbb{Z}_p$ is finitely presented, the above relation can be written as

$$(J_p \otimes \mathbb{Z}_p)^{I_p} \cong (J_p \otimes \mathbb{Z}_p)^{L_p},$$

Since $J_p \otimes \mathbb{Z}_p \cong \mathbb{Z}_p$, we conclude that

$$(\mathbb{Z}_p)^{I_p} \cong (\mathbb{Z}_p)^{L_p}.$$

If I_p or L_p is finite, we have $|I_p| = |L_p|$. If I_p and L_p are infinite sets, we have

$$p^{|I_p|} = p^{|L_p|},$$

Now using the weak GCH, $|I_p| = |L_p|$. And this implies that $A \cong B$. □

Before we state our main results on bounded torsion groups (Theorem 2.11 and Corollary 2.12), we need some auxiliary lemmas.

Lemma 2.6. *If A, B are two abelian groups, $\mathrm{Hom}(A, \mathbb{Q}) \cong \mathrm{Hom}(B, \mathbb{Q})$ and A is torsion, then B is also torsion.*

The proof is a consequence of the injectivity of $\mathbb{Q}$.

Lemma 2.7. *Let A and B be two torsion abelian groups and $\mathrm{Hom}(A, X) \cong \mathrm{Hom}(B, X)$, for torsion divisible groups X. If A is bounded, then so is B.*

Proof. If A is bounded, then it is easy to observe that $\mathrm{Hom}(A, X)$ is bounded for every X. Now suppose that B is not bounded. We will show that $\mathrm{Hom}(B, \frac{\mathbb{Q}}{\mathbb{Z}})$ is not bounded and get a contradiction. Choose an arbitrary $n \in \mathbb{N}$. Then there exists $b \in B$ whose order is $> n$. Since $\frac{\mathbb{Q}}{\mathbb{Z}}$ is divisible (injective), there exists $f : B \longrightarrow \frac{\mathbb{Q}}{\mathbb{Z}}$ such that $nf \neq 0$. □

Let G be an abelian group and I be a set. In the following, by G^I and $G^{(I)}$ we mean the direct product and the direct sum of I copies of G respectively.

Lemma 2.8. *Let p be a prime number, $n \in \mathbb{N}$ and I an infinite set. Then $\mathbb{Z}_{p^n}^I \cong \mathbb{Z}_{p^n}^{(J)}$ (as $\mathbb{Z}$-modules or $\mathbb{Z}_{p^n}$-modules), where $|J| = 2^{|I|}$.*

Proof. Consider $\mathbb{Z}_{p^n}^I$ as a $\mathbb{Z}_{p^n}$-module. Inasmuch as $\mathbb{Z}_{p^n}$ is a noetherian self-injective local ring, by Matlis Theorem (see [9, Theorem 3.48, Theorem 3.62]), $\mathbb{Z}_{p^n}^I \cong \mathbb{Z}_{p^n}^{(J)}$, for some set J. Now we infer that $2^{|I|} = |J|$. □

In the sequel, we use two fundamental results in abelian groups.

Theorem 2.9 (Prüfer-Baer). *A bounded group is a direct sum of cyclic groups.*

Proof. See [5, 17.2]. □

Theorem 2.10. *Any two decompositions of an abelian group into direct sums of cyclic groups of prime power orders are isomorphic.*

The proof is immediate by [5, 17.4].

Theorem 2.11. *Let A and B be two p-groups. If $\mathrm{Hom}(A, \mathbb{Z}_{p^\infty}) \cong \mathrm{Hom}(B, \mathbb{Z}_{p^\infty})$ and A is bounded, then $A \cong B$.*

Proof. First of all, we may infer by Lemma 2.7 that B is also bounded. Theorem 2.9 implies that $A \cong \mathbb{Z}_p^{(I_1)} \oplus \mathbb{Z}_{p^2}^{(I_2)} \oplus \cdots \oplus \mathbb{Z}_{p^n}^{(I_n)}$ and $B \cong \mathbb{Z}_p^{(J_1)} \oplus \mathbb{Z}_{p^2}^{(J_2)} \oplus \cdots \oplus \mathbb{Z}_{p^n}^{(J_n)}$ for suitable sets $I_1, \cdots, I_n$ and $J_1, \cdots, J_n$. Now from the fact that $\mathrm{Hom}(A, \mathbb{Z}_{p^\infty}) \cong \mathrm{Hom}(B, \mathbb{Z}_{p^\infty})$ and $\mathrm{Hom}(\mathbb{Z}_{p^i}, \mathbb{Z}_{p^\infty}) \cong \mathbb{Z}_{p^i}$, for $i \in \mathbb{N}$, we get that

$$\mathbb{Z}_p^{I_1} \oplus \mathbb{Z}_{p^2}^{I_2} \oplus \cdots \oplus \mathbb{Z}_{p^n}^{I_n} \cong \mathbb{Z}_p^{J_1} \oplus \mathbb{Z}_{p^2}^{J_2} \oplus \cdots \oplus \mathbb{Z}_{p^n}^{J_n}.$$

Using Lemma 2.8, we have that $\mathbb{Z}_{p^i}^{I_i} \cong \mathbb{Z}_{p^i}^{(K_i)}$ and $\mathbb{Z}_{p^i}^{J_i} \cong \mathbb{Z}_{p^i}^{(L_i)}$, where $K_i = I_i$ if I_i is finite, and $|K_i| = 2^{|I_i|}$ if I_i is infinite. The same holds for J_i and L_i. Therefore we have that

$$\mathbb{Z}_p^{(K_1)} \oplus \mathbb{Z}_{p^2}^{(K_2)} \oplus \cdots \oplus \mathbb{Z}_{p^n}^{(K_n)} \cong \mathbb{Z}_p^{(L_1)} \oplus \mathbb{Z}_{p^2}^{(L_2)} \oplus \cdots \oplus \mathbb{Z}_{p^n}^{(L_n)}.$$

Now by Theorem 2.10, $|K_i| = |L_i|$ for $i = 1, 2, \ldots, n$. From the weak GCH, we conclude that $|I_i| = |J_i|$ for $i = 1, 2, \ldots, n$, and hence $A \cong B$. □

Let A be an abelian group and p be a prime number. By $A(p)$ we indicate the subgroup $\{x \in A \mid p^n x = 0$ for some $n \in \mathbb{N}\}$, called the p-component of A.

Corollary 2.12. *Let A and B be two torsion abelian groups. If $\mathrm{Hom}(A, \frac{\mathbb{Q}}{\mathbb{Z}}) \cong \mathrm{Hom}(B, \frac{\mathbb{Q}}{\mathbb{Z}})$ and the p-components of A are bounded for any prime number p, then $A \cong B$.*

Proof. It is well-known that every torsion abelian group is the direct sum of its p-components. Therefore $A = \bigoplus A(p)$ and $B = \bigoplus B(p)$. Also recall that $\frac{\mathbb{Q}}{\mathbb{Z}} \cong \oplus \mathbb{Z}_{p^\infty}$ and $\mathrm{Hom}(A(p), \oplus_{q \neq p} \mathbb{Z}_{q^\infty}) = (0)$. This implies that

$$\mathrm{Hom}(A, \frac{\mathbb{Q}}{\mathbb{Z}}) \cong \prod \mathrm{Hom}(A(p), \frac{\mathbb{Q}}{\mathbb{Z}}) \cong \prod \mathrm{Hom}(A(p), \mathbb{Z}_{p^\infty}).$$

Inasmuch as, for every prime p, $A(p)$ is bounded, $\mathrm{Hom}(A(p), \mathbb{Z}_{p^\infty})$ is a torsion p-group and hence $\mathrm{Hom}(A, \frac{\mathbb{Q}}{\mathbb{Z}})(p) \cong \mathrm{Hom}(A(p), \mathbb{Z}_{p^\infty})$. Similarly, for B, we have that $\mathrm{Hom}(B, \frac{\mathbb{Q}}{\mathbb{Z}})(p) \cong \mathrm{Hom}(B(p), \mathbb{Z}_{p^\infty})$. Since $\mathrm{Hom}(A, \frac{\mathbb{Q}}{\mathbb{Z}}) \cong \mathrm{Hom}(B, \frac{\mathbb{Q}}{\mathbb{Z}})$, we may infer that

$$\mathrm{Hom}(A(p), \mathbb{Z}_{p^\infty}) \cong \mathrm{Hom}(B(p), \mathbb{Z}_{p^\infty}).$$

By Theorem 2.11, we conclude that $A(p) \cong B(p)$, and hence $A \cong B$. □

Corollary 2.13. *Let A and B be two abelian groups and suppose A finitely generated torsion. If $\mathrm{Hom}(A, \frac{\mathbb{Q}}{\mathbb{Z}}) \cong \mathrm{Hom}(B, \frac{\mathbb{Q}}{\mathbb{Z}})$, then $A \cong B$. In particular, B is finitely generated.*

Proof. Since A is a finitely generated torsion group, every p-component of A is bounded. Now we may apply Corollary 2.12. □

A subset $\{a_\alpha\}$ of an abelian group A is linearly independent (over $\mathbb{Z}$) if the only linear combination of these elements that is equal to zero is trivial: if

$$\sum_\alpha n_\alpha a_\alpha = 0, \quad n_\alpha \in \mathbb{Z},$$

where all but finitely many coefficients n_α are zero (so that the sum is, in effect, finite), then all coefficients are 0. Any two maximal linearly independent sets in A have the same cardinality, which is called the rank of A. The factor-group $\frac{A}{T(A)}$ is the unique maximal torsion-free quotient of A where by $T(A)$ we mean the torsion subgroup of A. The rank of $\frac{A}{T(A)}$ coincides with the rank of A, because $\operatorname{rank} A = \dim A \otimes \mathbb{Q} = \dim \frac{A}{T(A)} \otimes \mathbb{Q} = \operatorname{rank} \frac{A}{T(A)}$.

Proposition 2.14. *Let A and B be two abelian groups. If* $\operatorname{Hom}(A, \mathbb{Q}) \cong \operatorname{Hom}(B, \mathbb{Q})$*, then* $\operatorname{rank} A = \operatorname{rank} B$.

Proof. Suppose $\operatorname{Hom}(A, \mathbb{Q}) \cong \operatorname{Hom}(B, \mathbb{Q})$. Equivalently,

$$\operatorname{Hom}(A, \operatorname{Hom}(\mathbb{Q}, \mathbb{Q})) \cong \operatorname{Hom}(B, \operatorname{Hom}(\mathbb{Q}, \mathbb{Q})).$$

Then

$$\operatorname{Hom}(A \otimes \mathbb{Q}, \mathbb{Q}) \cong \operatorname{Hom}(B \otimes \mathbb{Q}, \mathbb{Q}),$$

so that

$$\operatorname{Hom}_{\mathbb{Q}}(A \otimes \mathbb{Q}, \mathbb{Q}) \overset{\mathbb{Q}}{\cong} \operatorname{Hom}_{\mathbb{Q}}(B \otimes \mathbb{Q}, \mathbb{Q}).$$

Since $A \otimes \mathbb{Q} \cong \mathbb{Q}^{(I)}$ and $B \otimes \mathbb{Q} \cong \mathbb{Q}^{(J)}$ for suitable sets I and J, taking dual, we can deduce that $\mathbb{Q}^I \cong \mathbb{Q}^J$. If either I or J is finite, then $|I| = |J|$. If I and J are infinite, by the weak GCH, we have that $|I| = |J|$. So in both cases we conclude that $A \otimes \mathbb{Q} \cong B \otimes \mathbb{Q}$, and hence $\operatorname{rank} A = \operatorname{rank} B$. □

Corollary 2.15. *Suppose that F_1 and F_2 are two free abelian groups, and* $\operatorname{Hom}(F_1, \mathbb{Q}) \cong \operatorname{Hom}(F_2, \mathbb{Q})$*, then* $F_1 \cong F_2$.

Remark 2.16. As far as Proposition 2.14 is concerned, we can add some comments on integral domains. Since the dual space of a finite dimensional vector space is isomorphic to the space itself, we have that if R is an integral domain with field of fractions Q, A and B are two finitely generated torsion-free R-modules and $\operatorname{Hom}(A, Q) \overset{R}{\cong} \operatorname{Hom}(B, Q)$, then $E(A) \cong E(B)$, where $E(A)$ indicates the injective hull of A. In particular, A and B have the same Goldie dimension. In order to see this, we have from the hypothesis that

$$\operatorname{Hom}_R(A, \operatorname{Hom}_R(Q, Q)) \overset{R}{\cong} \operatorname{Hom}_R(B, \operatorname{Hom}_R(Q, Q)).$$

Hence, by the Hom-tensor relation, we can write

$$\operatorname{Hom}_R(A \otimes_R Q, Q) \overset{R}{\cong} \operatorname{Hom}_R(B \otimes_R Q, Q).$$

Since $\operatorname{Hom}_R(M, N) = \operatorname{Hom}_Q(M, N)$ for every $M, N \in Q$-Mod, we get that

$$\operatorname{Hom}_Q(A \otimes_R Q, Q) \overset{Q}{\cong} \operatorname{Hom}_Q(B \otimes_R Q, Q).$$

Hence $A \otimes_R Q \overset{Q}{\cong} B \otimes Q$. This implies that $A \otimes_R Q \overset{R}{\cong} B \otimes Q$. But it is well-known that $E(M) \cong M \otimes_R Q$ for every finitely generated torsion-free R-module M.

We are ready to express our main result on torsion-free groups of rank 1. These kind of groups are (up to isomorphism) the subgroups of $\mathbb{Q}$. For undefined terms and concepts, the reader is referred to [6, Chapter 13]. Before we state our result, we need two basic results from [6].

Theorem 2.17 (Baer). *Two torsion-free groups of rank 1 are isomorphic if and only if they are of the same type*

Proof. See [6, Theorem 85.1]. □

In the sequel by $t(-)$ we mean the type of a torsion-free abelian group of rank 1, as defined in [6, Section 85].

Proposition 2.18. *If A and B are torsion-free groups of rank 1, then $\mathrm{Hom}(A, B)$ is 0 if $t(A) \not\leq t(B)$, and is a torsion-free group of rank 1 and of type $t(B) : t(A)$ if $t(A) \leq t(B)$.*

Proof. See [6, Proposition 85.4]. □

Theorem 2.19. *Let A and B be two torsion free abelian groups. Suppose $\mathrm{Hom}(A, X) \cong \mathrm{Hom}(B, X)$ for $X \in \{A, B, \mathbb{Q}\}$ and $\mathrm{rank}\, A = 1$. Then $A \cong B$. In particular, the class of torsion free abelian groups of rank 1 is a Fuchs 34 class.*

Proof. By Proposition 2.14, we know that $\mathrm{rank}\, B = 1$. Since

$$\mathrm{Hom}(A, A) \cong \mathrm{Hom}(B, A),$$

we have from Proposition 2.18 that $t(B) \leq t(A)$. Similarly, $t(A) \leq t(B)$. So $t(A) = t(B)$. By Theorem 2.17, $A \cong B$. □

Now we are ready to summarize what we have done in this section. This gives a partial answer to [5, Page 208, Problem 34]. The answer is provided under ZFC together with the weak GCH.

Corollary 2.20. *When A and B belong to each of the following classes of abelian groups, the relation $\mathrm{Hom}(A, X) \cong \mathrm{Hom}(B, X)$ for $X \in \mathcal{X}$ implies that $A \cong B$.*

1. *Finitely generated abelian groups; $\mathcal{X}$ the class of cyclic groups.*
2. *Divisible groups; $\mathcal{X} = \{\mathbb{Q}, \mathbb{Z}_{p^\infty}: p$ is prime $\}$.*
3. *Torsion abelian groups with bounded p-components; $\mathcal{X} = \{\frac{\mathbb{Q}}{\mathbb{Z}}\}$.*
4. *Torsion-free abelian groups of rank 1; $\mathcal{X}$ the class of torsion-free abelian groups of rank 1.*

3. Partial Answers to the Second Question

This section is devoted to the second question. We begin with a useful observation.

Proposition 3.1. *Let A and B be two finitely generated semisimple R-modules. Suppose $\mathrm{Hom}(A, X) \overset{S}{\cong} \mathrm{Hom}(B, X)$ for every simple module $X \in R$-Mod, where $S = \mathrm{End}(X)$. Then $B \overset{R}{\cong} A$.*

Proof. Let T be a simple R-module with $D = \mathrm{End}(T)$.

Since $D^n \cong \mathrm{Hom}(A, T) \overset{D}{\cong} \mathrm{Hom}(B, T) \cong D^m$ for $n, m \geq 0$, we conclude that $n = m$. Hence $\mathrm{Tr}(T, A) \cong \mathrm{Tr}(T, B)$ for every simple R-module T. This implies that $A \cong B$. □

Lemma 3.2. *Let Q be a quasi-injective R-module with $S = \mathrm{End}(Q)$. Then the S-module $\mathrm{Hom}(T, Q)$ is either simple or 0 for every simple R-module T.*

Proof. See [4, Page 191]. □

The next proposition can be compared with Corollary 2.13. Remember that for an R-module M, by $E(M)$ we mean the injective hull of M.

Proposition 3.3. *Let A and B be two finitely generated R-modules. Suppose $\mathrm{Hom}(A, I) \overset{S}{\cong} \mathrm{Hom}(B, I)$ for every injective module $I \in R$-Mod, where $S = \mathrm{End}(I)$. If A is simple, then $A \cong B$.*

Proof. Let B be non-semisimple. Then there exists a proper essential submodule K in B which is maximal. Since $\mathrm{Hom}(B, E(B/K)) \neq 0$, we have $\mathrm{Hom}(A, E(B/K)) \neq 0$. This implies that $A \cong B/K$ due to A and B/K being simple. On the other hand, $\mathrm{Hom}(B, E(B)) \neq 0$ which implies that $\mathrm{Hom}(A, E(B)) \neq 0$. Hence there exists a map $f : B \longrightarrow E(B)$ with $\ker(f) = K$. By injectivity of $E(B)$, we have an R-homomorphism extension (of f) $g : E(B) \longrightarrow E(B)$ with $K \subseteq \ker(g)$ and hence $\ker(g) \leq_e E(B)$ due to K being essential in B. Consider the following diagram, where $\phi : \mathrm{Hom}(A, E(B)) \longrightarrow \mathrm{Hom}(B, E(B))$, is an S-module isomorphism with $S = \mathrm{End}(E(B))$:

$$\begin{array}{ccc} \mathrm{Hom}(A, E(B)) & \stackrel{\phi}{\longrightarrow} & \mathrm{Hom}(B, E(B)) \\ {\scriptstyle \mathrm{Hom}(A,g)} \downarrow & & \downarrow {\scriptstyle \mathrm{Hom}(B,g)} \\ \mathrm{Hom}(A, E(B)) & \stackrel{\phi}{\longrightarrow} & \mathrm{Hom}(B, E(B)) \end{array}$$

This diagram is commutative. To see this, let $h \in \mathrm{Hom}(A, E(B))$. Since $g \in S$, $\phi(g \circ h) = g \circ \phi(h)$. Therefore $\phi \circ \mathrm{Hom}(A, g) = \mathrm{Hom}(B, g) \circ \phi$. Now consider, the inclusion map $\iota : B \longrightarrow E(B)$, so there exists $\alpha \in \mathrm{Hom}(A, E(B))$ such that $\phi(\alpha) = \iota$. From the one hand, $g \circ \phi(\alpha) = g \circ \iota = g|_B = f \neq 0$. On the other hand, $\phi(g \circ \alpha) = 0$ because $g \circ \alpha = 0$, which is a contradiction with the commutativity of the above diagram. So B is semisimple. Let T be a simple submodule of B. Since $\mathrm{Hom}(B, E(T)) \neq 0$, hence $\mathrm{Hom}(A, E(T)) \neq 0$, and therefore $T \cong A$. This implies that $B \cong A^n$, for some $n \geq 1$. Since by Lemma 3.2, $\mathrm{Hom}(A, E(A))$ is a simple S-module, where $S = \mathrm{End}(E(A))$, so $n = 1$ and hence $A \cong B$. □

In the following by a *coretractable* R-module M we mean a module M such that $\mathrm{Hom}_R(\frac{M}{K}, M) \neq 0$ for every proper submodule K of M. In the sequel, by a homogenous seimisimple module we mean a semisimple module which is the direct sum of isomorphic simple modules.

Proposition 3.4. *Let A and B be two finitely generated R-modules and $\mathrm{Hom}(A, X) \overset{S}{\cong} \mathrm{Hom}(B, X)$, where $X = B$ or X is a simple R-module and $S = \mathrm{End}(X)$. If A is semisimple, then under each of the following conditions, $A \overset{R}{\cong} B$:*

a. A is homogenous;

b. B is coretractable.

Proof. By Proposition 3.1, it is enough to show that B is also semisimple. Suppose that B is not semisimple, hence there exists a maximal submodule K of B which is essential. Now, consider the map $\pi : B \longrightarrow \frac{B}{K}$. If B is coretractable, there exists a non-zero map $\beta : \frac{B}{K} \longrightarrow B$ and hence $0 \neq \beta \circ \pi : B \longrightarrow B$ with $K = \ker(\beta \circ \pi)$. In case A is homogenous, since $\mathrm{Hom}(B, \frac{B}{K}) \neq 0$, we have $\mathrm{Hom}(A, \frac{B}{K}) \neq 0$. It is not difficult to observe that, in this case too, there exists a non-zero map $f : B \longrightarrow B$ with $K = \ker f$. So, in either case, we have such a map $g : B \longrightarrow B$ with $\ker g = K$. Now, consider the following diagram which is commutative due to $\mathrm{Hom}(A, B) \overset{S}{\cong} \mathrm{Hom}(B, B)$, where $S = \mathrm{End}(B)$ and $g \in S$:

$$\begin{array}{ccc} \mathrm{Hom}(A, B) & \stackrel{\phi}{\longrightarrow} & \mathrm{Hom}(B, B) \\ {\scriptstyle \mathrm{Hom}(A,g)} \downarrow & & \downarrow {\scriptstyle \mathrm{Hom}(B,g)} \\ \mathrm{Hom}(A, B) & \stackrel{\phi}{\longrightarrow} & \mathrm{Hom}(B, B) \end{array}$$

which, similar to the proof of Proposition 3.3, leads us to a contradiction. Therefore, B is semisimple. □

Recall that an R-module M, is called reflexive if the canonical map $M \longrightarrow M^{**} = \mathrm{Hom}(M^*, R)$, is an isomorphism. Knowing that for a ring R, $\mathrm{End}(R) \cong R$, we have the following result.

Proposition 3.5. *Let A and B be two reflexive modules over a ring R. If $\mathrm{Hom}(A, R) \overset{R}{\cong} \mathrm{Hom}(B, R)$, then $A \overset{R}{\cong} B$.*

Proof. The verification is immediate. □

A ring is said to be quasi-Frobenius if the class of its projective modules coincides with the class of its injective modules.

Corollary 3.6. *Let A and B be two modules over a ring R. Under each of the following cases, from* $\mathrm{Hom}(A,R) \overset{R}{\cong} \mathrm{Hom}(B,R)$ *we conclude that* $A \cong B$.

1. *A and B are finitely generated projective modules.*
2. *R is quasi-Frobenius and A,B are finitely generated modules.*

Proof. Recall that in these cases A and B are reflexive (see [9, Theorem 15.11]). □

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