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On k-circulant Matrices with the Lucas Numbers

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Abstract. Let k be a nonzero complex number. In this paper, we determine the eigenvalues of a k-circulant matrix whose first row is $(L_1, L_2, \ldots, L_n)$, where L_n is the n^{th} Lucas number, and improve the result which can be obtained from the result of Theorem 7. [28]. The Euclidean norm of such matrix is obtained. Bounds for the spectral norm of a k-circulant matrix whose first row is $(L_1^{-1}, L_2^{-1}, \ldots, L_n^{-1})$ are also investigated. The obtained results are illustrated by examples.

1. Introduction

In this paper, for an arbitrary positive integer n and a nonzero complex number k, symbols ψ and ω denote any n^{th} root of k and any primitive n^{th} root of unity, respectively. For $A \in \mathbb{C}^{n\times n}$, where $\mathbb{C}^{n\times n}$ is the set of all complex matrices of order n, the symbols $\lambda_j(A)$, $j = \overline{0, n-1}$, $|A|$, $||A||_E$, $||A||_2$ and $A^{\circ -1}$ are used to designate the eigenvalues, the determinant, the Euclidean norm, the spectral norm and the Hadamard inverse of A, respectively. Let us recall that, for $A = \left[a_{i,j}\right] \in \mathbb{C}^{n\times n}$,

$$||A||_E = \sqrt{\sum_{i,j=1}^{n} |a_{i,j}|^2}, \quad ||A||_2 = \sqrt{\max_{1\le i\le n} \lambda_i(A^*A)}, \tag{1}$$

where A^* is the conjugate transpose of A, and $A^{\circ -1} = \left[a_{i,j}^{-1}\right]$.

The Lucas numbers $\{L_n\}$ satisfy the following recursive relation:

$$L_n = L_{n-2} + L_{n-1}, \quad n \ge 2, \tag{2}$$

with initial conditions $L_0 = 2$ and $L_1 = 1$.

Let α and β be the roots of the equation $x^2 - x - 1 = 0$ i.e.

$$\alpha = \frac{1+\sqrt{5}}{2}, \ \beta = \frac{1-\sqrt{5}}{2}, \ \alpha\beta = -1, \ \alpha + \beta = 1 \quad \text{and} \quad \alpha - \beta = \sqrt{5}. \tag{3}$$

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Binet's formula for the Lucas numbers is:

$$L_n = \alpha^n + \beta^n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n. \tag{4}$$

The following identities hold for the Lucas numbers:

$$\sum_{i=0}^{n} L_i = L_{n+2} - 1 \quad \text{and} \quad \sum_{i=0}^{n} L_i^2 = L_n L_{n+1} + 2\,[= L_{2n+1} + (-1)^n + 2]\,. \tag{5}$$

For more information about these numbers we recommend: [3], [4], [13], [16], [17] and [26].

A matrix C of order n with the first row $(c_0, c_1, c_2, \ldots, c_{n-1})$ is called a *k-circulant matrix* if C has the following form:

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ kc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ kc_{n-2} & kc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ kc_2 & kc_3 & kc_4 & \cdots & c_0 & c_1 \\ kc_1 & kc_2 & kc_3 & \cdots & kc_{n-1} & c_0 \end{bmatrix} \tag{6}$$

i.e. C satisfies the following conditions:

$$c_{i,j} = \begin{cases} c_{j-i}, & i \le j \\ k\,c_{n+j-i}, & \text{otherwise} \end{cases}, \; i = \overline{2,n},\; j = \overline{1,n}\,. \tag{7}$$

If C is a k-circulant matrix with the first row $(c_0, c_1, c_2, \ldots, c_{n-1})$, then we shall write $C = circ_n\{_k(c_0, c_1, c_2, \ldots, c_{n-1})\}$. The designation for the order of a matrix can be omitted if the order of a matrix is known. Circulant (skew circulant) matrices are k-circulant matrices for $k = 1$ ($k = -1$). R. E. Cline, R. J. Plemmons and G. Worm presented, in the paper [2], necessary and sufficient conditions for a complex square matrix to be a k-circulant matrix (Lemma 2.[2] and Lemma 3.[2]).

It is important to point out that k-circulant matrices play important role in many areas (coding theory, probability, statistics, numerical analysis, signal and image processing, engineering model etc.). There are many papers devoted to k-circulant matrices (especially to circulant and skew circulant matrices). Let us mention some of them: [5], [6], [10]−[12], [14], [15], [18]−[22], [27]−[32]. For example, the paper [11] is devoted to skew circulant matrices involving the sum of Fibonacci and Lucas numbers. The four kinds of norms (*the Euclidean norm, the spectral norm, the maximum column sum matrix norm, the maximum row sum matrix norm*), and bounds for the spread of these matrices were given in that paper. Norms of circulant and semicirculant matrices with Horadam's numbers were considered in the paper [14]. In the paper [15], the authors considered the spectral norms of k-circulant matrices whose entries are the biperiodic Fibonacci and biperiodic Lucas numbers and obtained bounds for the spectral norms of such matrice. They also obtained bounds for the spectral norms of Kronecker and Hadamard products of such matrices. The paper [27] is devoted to circulant matrices involving the generalized r-Horadam numbers $\{H_{r,n}\}$ which are defined as follows:

$$H_{r,n+2} = f(r)H_{r,n+1} + g(r)H_{r,n}, \quad n \ge 0,$$

where $r \in \mathbb{R}^+$, $H_{r,0} = a$, $H_{r,1} = b$, $a, b \in \mathbb{R}$ and $f^2(r) + 4g(r) > 0$. Except for the eigenvalues and the determinants of such matrices, the authors also investigated their spectral norms. In the paper [28], the authors investigated k- circulant matrices with the generalized r-Horadam numbers and determined the upper and lower bounds for the spectral norms of such matrices. The formulae for the eigenvalues and determinant of a k-circulant matrix with the generalized r-Horadam numbers were also derived in that paper.

Namely, the results obtained in [28] are:

Theorem 1.1. *(Theorem 5. [28]) Let $H = circ\{_k(H_{r,0}, H_{r,1}, \dots, H_{r,n-1})\}$.*
a) If $|k| \geq 1$, then

$$\sqrt{\sum_{i=0}^{n-1} H_{r,i}^2} \leq ||H||_2 \leq \sqrt{\left(a^2(1-|k|^2) + |k|^2 \sum_{i=0}^{n-1} H_{r,i}^2\right)\left(1 - a^2 + \sum_{i=0}^{n-1} H_{r,i}^2\right)} \tag{8}$$

b) If $|k| < 1$, then

$$|k|\sqrt{\sum_{i=0}^{n-1} H_{r,i}^2} \leq ||H||_2 \leq \sqrt{n\sum_{i=0}^{n-1} H_{r,i}^2}\ . \tag{9}$$

Theorem 1.2. *(Theorem 7. [28]) Let $H = circ\{_k(H_{r,0}, H_{r,1}, \dots, H_{r,n-1})\}$. The eigenvalues of H are:*

$$\lambda_j(H) = \frac{kH_{r,n} - H_{r,0} + (g(r)kH_{r,n-1} - b + af(r))\psi\omega^{-j}}{g(r)(\psi\omega^{-j})^2 + f(r)\psi\omega^{-j} - 1},\ j = \overline{0, n-1}\,. \tag{10}$$

Theorem 1.3. *(Theorem 8. [28]) Let $H = circ\{_k(H_{r,0}, H_{r,1}, \dots, H_{r,n-1})\}$. The determinant of H is:*

$$|H| = \frac{(H_{r,0} - kH_{r,n})^n - (g(r)kH_{r,n-1} - b + af(r))^n k}{(1 - k\alpha^n)(1 - k\beta^n)}, \tag{11}$$

where α and β are the roots of the equation $x^2 - f(r)x - g(r) = 0$.

From the previous theorems, the results for the matrix

$$L = circ\{_k(L_1, L_2, \dots, L_n)\} \tag{12}$$

can be obtained. In this paper, we shall improve the result in relation to the eigenvalues of (12) which can be obtained from (10) because the authors did not consider the case when the denominator is equal to zero. Also, the Euclidean norm of (12) and bounds for the spectral norm of a k-circulant matrix with the first row $(L_1^{-1}, L_2^{-1}, \dots, L_n^{-1})$ will be determined. Before we present our results, let us mention that, in [1], the authors presented bounds for the Euclidean norm and the spectral norm of a circulant matrix with the first row $(L_1, L_2, \dots, L_n)$. Also, bounds for the Euclidean norm and the spectral norm of the Hadamard inverse of such matrix were obtained in that paper. In [23], the author considered circulant matrices whose first rows are $(L_0, L_1, \dots, L_{n-1})$ and $(F_0, F_1, \dots, F_{n-1})$, where F_n is the n^{th} Fibonacci number, and obtained some bounds for the spectral norms of such matrices (see also [24] and [25]). The results presented in the paper [23] were improved, in [9], by A. Ipek. Namely, A. Ipek computed the spectral norms of circulant matrices with the first rows $(L_0, L_1, \dots, L_{n-1})$ and $(F_0, F_1, \dots, F_{n-1})$. The paper [7] is devoted to k-circulant matrices with the Fibonacci and Lucas numbers. In that paper, an upper bound estimation of the spectral norms for such matrices was given.

Our main results will be presented in the next section.

2. Main Results

Throughout this section, α and β are the roots of the equation $x^2 - x - 1 = 0$. First, we shall determine the eigenvalues of (12). Before that, let us recall that ψ is any n^{th} root of a nonzero complex number k and ω is any primitive n^{th} root of unity. We use the following lemma.

Lemma 2.1. *(Lemma 4. [2]) The eigenvalues of $C = circ\{_k(c_0, c_1, c_2, \dots, c_{n-1})\}$ are:*

$$\lambda_j(C) = \sum_{i=0}^{n-1} c_i(\psi\omega^{-j})^i,\ j = \overline{0,\ n-1}\,. \tag{13}$$

Moreover, in this case:

$$c_i = \frac{1}{n}\sum_{j=0}^{n-1} \lambda_j(C)(\psi\omega^{-j})^{-i},\ i = \overline{0,\ n-1}\,. \tag{14}$$

Theorem 2.2. *Let L be the matrix as in (12). The eigenvalues of L are given by the following formulae:*

1) If $\psi\omega^{-j} = \frac{1}{\alpha}$, *then*

$$\lambda_j(L) = n\alpha - \frac{1-(-1)^n\beta^{2n}}{\sqrt{5}}, \tag{15}$$

2) If $\psi\omega^{-j} = \frac{1}{\beta}$, *then*

$$\lambda_j(L) = \frac{1-(-1)^n\alpha^{2n}}{\sqrt{5}} + n\beta, \tag{16}$$

3) If $\psi\omega^{-j} \neq \frac{1}{\alpha}$ *and* $\psi\omega^{-j} \neq \frac{1}{\beta}$, *then*

$$\lambda_j(L) = \frac{kL_{n+1} - 1 - (2-kL_n)\,\psi\omega^{-j}}{(\psi\omega^{-j})^2 + \psi\omega^{-j} - 1}\,. \tag{17}$$

Proof. Based on Lemma 2.1. and (4), it follows:

1) Suppose that $\psi\omega^{-j} = \frac{1}{\alpha}$. Then,

$$\begin{aligned}\lambda_j(L) &= \sum_{i=0}^{n-1} L_{i+1}(\psi\omega^{-j})^i = \sum_{i=0}^{n-1}\left(\alpha^{i+1}+\beta^{i+1}\right)(\frac{1}{\alpha})^i = \alpha\sum_{i=0}^{n-1}1 + \beta\sum_{i=0}^{n-1}(\frac{\beta}{\alpha})^i \\ &= n\alpha + \beta\,\frac{1-(\frac{\beta}{\alpha})^n}{1-\frac{\beta}{\alpha}} = n\alpha - \frac{1-(-1)^n\beta^{2n}}{\sqrt{5}}\,,\end{aligned}$$

2) Suppose that $\psi\omega^{-j} = \frac{1}{\beta}$. Then,

$$\begin{aligned}\lambda_j(L) &= \sum_{i=0}^{n-1} L_{i+1}(\psi\omega^{-j})^i = \sum_{i=0}^{n-1}\left(\alpha^{i+1}+\beta^{i+1}\right)(\frac{1}{\beta})^i = \alpha\sum_{i=0}^{n-1}(\frac{\alpha}{\beta})^i + \beta\sum_{i=0}^{n-1}1 \\ &= \alpha\,\frac{1-(\frac{\alpha}{\beta})^n}{1-\frac{\alpha}{\beta}} + n\beta = \frac{1-(-1)^n\alpha^{2n}}{\sqrt{5}} + n\beta\,,\end{aligned}$$

3) Suppose that $\psi\omega^{-j} \neq \frac{1}{\alpha}$ and $\psi\omega^{-j} \neq \frac{1}{\beta}$. Then, $\lambda_j(L)$ follows from (10). □

The following example illustrates the results of the previous theorem.

Example 2.3. *Let*

$$L = circ\{_{9+4\sqrt{5}}(1,3,4,7,11,18)\}$$

i.e.

$$L = \begin{bmatrix} 1 & 3 & 4 & 7 & 11 & 18 \\ 18(9+4\sqrt{5}) & 1 & 3 & 4 & 7 & 11 \\ 11(9+4\sqrt{5}) & 18(9+4\sqrt{5}) & 1 & 3 & 4 & 7 \\ 7(9+4\sqrt{5}) & 11(9+4\sqrt{5}) & 18(9+4\sqrt{5}) & 1 & 3 & 4 \\ 4(9+4\sqrt{5}) & 7(9+4\sqrt{5}) & 11(9+4\sqrt{5}) & 18(9+4\sqrt{5}) & 1 & 3 \\ 3(9+4\sqrt{5}) & 4(9+4\sqrt{5}) & 7(9+4\sqrt{5}) & 11(9+4\sqrt{5}) & 18(9+4\sqrt{5}) & 1 \end{bmatrix}.$$

Since $n = 6$ *and* $k = 9 + 4\sqrt{5}$ *i.e.* $\psi = -\alpha$ *and* $\omega = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, *based on Theorem 2.2., it follows that*

★: *$\psi\omega^0 = \frac{1}{\beta}$, so $\lambda_0(L)$ is obtained based on 2) of Theorem 2.2.: $\lambda_0(L) = -69 - 35\sqrt{5}$;*

★: *$\psi\omega^{-j} \neq \frac{1}{\alpha}$ and $\psi\omega^{-j} \neq \frac{1}{\beta}$, for $j = \overline{1,5}$, so $\lambda_j(L)$, for $j = \overline{1,5}$, are obtained based on 3) of Theorem 2.2.: $\lambda_{1,5}(L) = -\frac{1}{2}\left[115 + 51\sqrt{5} \pm i\sqrt{3}(65 + 29\sqrt{5})\right]$, $\lambda_{2,4}(L) = 15 + 7\sqrt{5} \mp i\sqrt{3}(65 + 29\sqrt{5})$, $\lambda_3(L) = 160 + 72\sqrt{5}$.*

Since $|\, L \,| = \prod_{j=0}^{n-1} \lambda_j(L)$, it follows that

$$|\, L \,| = -31\,367\,216\,640\,000 \; - \; 14\,027\,845\,734\,400\,\sqrt{5}\,.$$

Let us remark, in relation to the previous example, that the determinant of $L = circ_{\{9+4\sqrt{5}}(1, 3, 4, 7, 11, 18)\}$ is not possible to obtain using the result of Theorem 1.3.

Now, we determine the Euclidean norm of (12). The formula

$$\sum_{i=1}^{n-1} ix^i = \frac{x - nx^n + (n-1)x^{n+1}}{(1-x)^2} \ , \tag{18}$$

which holds for all x, will be used.

Theorem 2.4. *Let L be the matrix as in (12). The Euclidean norm of L is:*

$$\|L\|_E = \sqrt{n\,(L_{2n+1} + (-1)^n - 2) + (|k|^2 - 1)\left(-L_{2n} + (n-1)L_{2n+1} + \frac{7}{2} + \frac{2n-1}{2}(-1)^n\right)}. \tag{19}$$

Proof. From the definition of the Euclidean norm of a matrix, using (4), (5) and (18), we obtain:

$$\begin{aligned}
(\|L\|_E)^2 &= \sum_{i,j=1}^{n} |l_{i,j}|^2 \\
&= nL_1^2 + \left((n-1) + |k|^2\right)L_2^2 + \left((n-2) + 2|k|^2\right)L_3^2 + \cdots + \left(1 + (n-1)|k|^2\right)L_n^2 \\
&= \sum_{i=0}^{n-1}(n-i)L_{i+1}^2 + |k|^2 \sum_{i=1}^{n-1} iL_{i+1}^2 \\
&= n\sum_{i=0}^{n-1} L_{i+1}^2 + (|k|^2 - 1)\sum_{i=1}^{n-1} iL_{i+1}^2 \\
&= n\,(L_{2n+1} + (-1)^n - 2) + (|k|^2 - 1)\sum_{i=1}^{n-1} i\left(\alpha^{2i+2} + 2(\alpha\beta)^{i+1} + \beta^{2i+2}\right) \\
&= n\,(L_{2n+1} + (-1)^n - 2) + (|k|^2 - 1)\left(\alpha^2\,\frac{\alpha^2 - n\alpha^{2n} + (n-1)\alpha^{2n+2}}{\alpha^2}\right. \\
&\quad \left. -2\,\frac{-1 - n(-1)^n + (n-1)(-1)^{n+1}}{4} + \beta^2\,\frac{\beta^2 - n\beta^{2n} + (n-1)\beta^{2n+2}}{\beta^2}\right) \\
&= n\,(L_{2n+1} + (-1)^n - 2) + (|k|^2 - 1)\left(-n\alpha^{2n} + (n-1)\alpha^{2n+2} + \frac{7}{2} + \frac{n}{2}(-1)^n\right. \\
&\quad \left. + \frac{n-1}{2}(-1)^n - n\beta^{2n} + (n-1)\beta^{2n+2}\right) \\
&= n\,(L_{2n+1} + (-1)^n - 2) + (|k|^2 - 1)\left(-nL_{2n} + (n-1)L_{2n+2} + \frac{7}{2} + \frac{2n-1}{2}(-1)^n\right) \\
&= n\,(L_{2n+1} + (-1)^n - 2) + (|k|^2 - 1)\left(-L_{2n} + (n-1)L_{2n+1} + \frac{7}{2} + \frac{2n-1}{2}(-1)^n\right).
\end{aligned}$$

Therefore,

$$||L||_E = \sqrt{n\,(L_{2n+1} + (-1)^n - 2) + (|k|^2 - 1)\left(-L_{2n} + (n-1)L_{2n+1} + \frac{7}{2} + \frac{2n-1}{2}(-1)^n\right)}.$$

□

At the end of this paper, we obtain the upper and lower bounds for the spectral norm of $L_k^{\circ -1} = circ\{_k(L_1^{-1}, L_2^{-1}, L_3^{-1}, \ldots, L_n^{-1})\}$. The well-known inequalities (see Theorem 1. [33] and Table 1. [33])

$$\frac{||A||_E}{\sqrt{n}} \leq ||A||_2 \leq ||A||_E, \tag{20}$$

which hold for any complex matrix A of order n, will be used, and the following lemma.

Lemma 2.5. *([8]) Let $A = \left[a_{i,j}\right]$ and $B = \left[b_{i,j}\right]$ be matrices of order $m \times n$. Then,*

$$||A \circ B||_2 \leq r_1(A) \cdot c_1(B), \tag{21}$$

where $A \circ B = [a_{i,j} b_{i,j}]$ is the Hadamard product (or the Schur product) of matrices A and B,

$$r_1(A) = \max_{1 \leq i \leq m} \sqrt{\sum_{j=1}^{n} |\, a_{i,j} \,|^2} \quad and \quad c_1(B) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^{m} |\, b_{i,j} \,|^2}.$$

Theorem 2.6. *Let $L_k^{\circ -1} = circ\{_k(L_1^{-1}, L_2^{-1}, L_3^{-1}, \ldots, L_n^{-1})\}$.*
1) If $|k| \geq 1$, then

$$\sqrt{\frac{n}{L_{2n+1} + (-1)^n}} \leq ||L_k^{\circ -1}||_2 \leq \sqrt{n(1 + (n-1)|k|^2)}\ , \tag{22}$$

2) If $|k| < 1$, then

$$|k| \sqrt{\frac{n}{L_{2n+1} + (-1)^n}} \leq ||L_k^{\circ -1}||_2 \leq n\ . \tag{23}$$

Proof. From the definition of the Euclidean norm of a matrix, it follows that

$$||L_k^{\circ -1}||_E^2 = \sum_{i=0}^{n-1} (n-i) \frac{1}{L_{i+1}^2} + |k|^2 \sum_{i=1}^{n-1} i \frac{1}{L_{i+1}^2}\ . \tag{24}$$

1) If $|k| \geq 1$, then

$$||L_k^{\circ -1}||_E^2 \geq \sum_{i=0}^{n-1} (n-i) \frac{1}{L_{i+1}^2} + \sum_{i=1}^{n-1} i \frac{1}{L_{i+1}^2} = n \sum_{i=0}^{n-1} \frac{1}{L_{i+1}^2} = n \sum_{i=1}^{n} \frac{1}{L_i^2}$$

$$\geq n \sum_{i=1}^{n} \frac{1}{L_n^2} = (\frac{n}{L_n})^2 > \frac{n^2}{L_n L_{n+1}} = \frac{n^2}{L_{2n+1} + (-1)^n}\ .$$

Therefore,

$$\frac{||L_k^{\circ -1}||_E}{\sqrt{n}} \geq \sqrt{\frac{n}{L_{2n+1} + (-1)^n}}\ .$$

We conclude from (20) that

$$\|L_k^{\circ -1}\|_2 \geq \sqrt{\frac{n}{L_{2n+1}+(-1)^n}}\,.$$

Now, we shall obtain the upper bound for the spectral norm of $L_k^{\circ -1}$. Let R and S be the following matrices:

$$R=\begin{bmatrix} \frac{1}{L_1} & \frac{1}{L_2} & \frac{1}{L_3} & \cdots & \frac{1}{L_n} \\ k & \frac{1}{L_1} & \frac{1}{L_2} & \cdots & \frac{1}{L_{n-1}} \\ k & k & \frac{1}{L_1} & \cdots & \frac{1}{L_{n-2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k & k & k & \cdots & \frac{1}{L_1} \end{bmatrix} \quad \text{and} \quad S=\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \frac{1}{L_n} & 1 & 1 & \cdots & 1 \\ \frac{1}{L_{n-1}} & \frac{1}{L_n} & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{L_2} & \frac{1}{L_3} & \frac{1}{L_4} & \cdots & 1 \end{bmatrix}.$$

Then,

$$r_1(R) = \max_{1\leq i\leq n} \sqrt{\sum_{j=1}^{n} \mid r_{i,j} \mid^2} = \sqrt{1+(n-1)|k|^2}$$

and

$$c_1(S) = \max_{1\leq j\leq n} \sqrt{\sum_{i=1}^{n} \mid s_{i,j} \mid^2} = \sqrt{n}\ .$$

Since $L_k^{\circ -1} = R \circ S$, based on Lemma 2.5., we can write

$$\|L_k^{\circ -1}\|_2 \leq r_1(R)\cdot c_1(S) = \sqrt{n(1+(n-1)|k|^2)}\ .$$

2) If $|k| < 1$, then

$$\|L_k^{\circ -1}\|_E^2 \geq \sum_{i=0}^{n-1}(n-i)|k|^2\frac{1}{L_{i+1}^2} + \sum_{i=1}^{n-1} i|k|^2\frac{1}{L_{i+1}^2} = n|k|^2\sum_{i=0}^{n-1}\frac{1}{L_{i+1}^2} = n|k|^2\sum_{i=1}^{n}\frac{1}{L_i^2}$$

$$\geq n|k|^2\sum_{i=1}^{n}\frac{1}{L_n^2} = |k|^2(\frac{n}{L_n})^2 > |k|^2\frac{n^2}{L_nL_{n+1}} = |k|^2\frac{n^2}{L_{2n+1}+(-1)^n}\,.$$

Therefore,

$$\frac{\|L_k^{\circ -1}\|_E}{\sqrt{n}} \geq |k|\sqrt{\frac{n}{L_{2n+1}+(-1)^n}}\,.$$

We conclude from (20) that

$$\|L_k^{\circ -1}\|_2 \geq |k|\sqrt{\frac{n}{L_{2n+1}+(-1)^n}}\,.$$

Now, we shall obtain the upper bound for the spectral norm of $L_k^{\circ -1}$. Let Q and W be the following matrices:

$$Q=\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \frac{k}{L_n} & 1 & 1 & \cdots & 1 \\ \frac{k}{L_{n-1}} & \frac{k}{L_n} & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{k}{L_2} & \frac{k}{L_3} & \frac{k}{L_4} & \cdots & 1 \end{bmatrix} \quad \text{and} \quad W=\begin{bmatrix} \frac{1}{L_1} & \frac{1}{L_2} & \frac{1}{L_3} & \cdots & \frac{1}{L_n} \\ 1 & \frac{1}{L_1} & \frac{1}{L_2} & \cdots & \frac{1}{L_{n-1}} \\ 1 & 1 & \frac{1}{L_1} & \cdots & \frac{1}{L_{n-2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & \frac{1}{L_1} \end{bmatrix}.$$

Then,

$$r_1(Q) = \max_{1\le i\le n} \sqrt{\sum_{j=1}^{n} | q_{i,j} |^2} = \sqrt{n}$$

and

$$c_1(W) = \max_{1\le j\le n} \sqrt{\sum_{i=1}^{n} | w_{i,j} |^2} = \sqrt{n}\ .$$

Since $L_k^{\circ-1} = Q \circ W$, based on Lemma 2.5., we can write

$$||L_k^{\circ-1}||_2 \le r_1(Q)\cdot c_1(W) = n\,.$$

□

Example 2.7. *Let* $L_k^{\circ-1} = circ\{_k(L_1^{-1}, L_2^{-1}, L_3^{-1}, \dots, L_n^{-1})\}$.

The lower bounds for the spectral norm of $L_k^{\circ-1}$

a) for $|k| \ge 1$,

n	
2	$\sqrt{\frac{1}{6}} \approx 0.40825$
3	$\sqrt{\frac{3}{28}} \approx 0.32733$
4	$\sqrt{\frac{4}{77}} \approx 0.22792$
5	$\sqrt{\frac{5}{198}} \approx 0.15891$
6	$\sqrt{\frac{1}{87}} \approx 0.10721$
7	$\sqrt{\frac{7}{1363}} \approx 0.07166$

b) for $k = -\frac{1}{3}, \frac{1}{3}$,

n	
2	$\frac{1}{3}\sqrt{\frac{1}{6}} \approx 0.13608$
3	$\frac{1}{3}\sqrt{\frac{3}{28}} \approx 0.10911$
4	$\frac{1}{3}\sqrt{\frac{4}{77}} \approx 0.07597$
5	$\frac{1}{3}\sqrt{\frac{5}{198}} \approx 0.05297$
6	$\frac{1}{3}\sqrt{\frac{1}{87}} \approx 0.03574$
7	$\frac{1}{3}\sqrt{\frac{7}{1363}} \approx 0.02389$

c) for $k = -\frac{1}{2}, \frac{1}{2}$.

n	
2	$\frac{1}{2}\sqrt{\frac{1}{6}} \approx 0.20412$
3	$\frac{1}{2}\sqrt{\frac{3}{28}} \approx 0.16366$
4	$\frac{1}{2}\sqrt{\frac{4}{77}} \approx 0.11396$
5	$\frac{1}{2}\sqrt{\frac{5}{198}} \approx 0.07946$
6	$\frac{1}{2}\sqrt{\frac{1}{87}} \approx 0.05361$
7	$\frac{1}{2}\sqrt{\frac{7}{1363}} \approx 0.03583$

The upper bounds for the spectral norm of $L_k^{\circ-1}$

a) for $k = -3, 3$,

n	
2	$\sqrt{20} \approx 4.47214$
3	$\sqrt{57} \approx 7.54983$
4	$\sqrt{112} \approx 10.58301$
5	$\sqrt{185} \approx 13.60147$
6	$\sqrt{276} \approx 16.61325$
7	$\sqrt{385} \approx 19.62142$

b) for $k = -2, 2$,

n	
2	$\sqrt{10} \approx 3.16228$
3	$\sqrt{27} \approx 5.19615$
4	$\sqrt{52} \approx 7.21110$
5	$\sqrt{85} \approx 9.21954$
6	$\sqrt{126} \approx 11.22497$
7	$\sqrt{175} \approx 13.22876$

c) for $|k| < 1$.

n	
2	2
3	3
4	4
5	5
6	6
7	7

3. Conclusion

In this paper, we considered the matrix

$$L = circ\{_k(L_1, L_2, \dots, L_n)\},$$

where L_n is the n^{th} Lucas number and k is a nonzero complex number, and investigated the eigenvalues and the Euclidean norm of such matrix. Also, the upper and lower bounds for the spectral norm of a k-circulant matrix whose first row is $(L_1^{-1}, L_2^{-1}, \ldots, L_n^{-1})$ were determined.

We did not consider the matrix L provided that $k = 0$ i.e. if L is a semicirculant matrix. But, since such matrix belongs to the class of upper-triangular matrices, we conclude that the eigenvalues of such matrix are: $\lambda_j(L) = 1$, $(j = \overline{0,\, n-1})$. The Euclidean norm of such matrix can be obtained from (19) i.e. in (19), k can be equal to 0. The upper and lower bounds for the spectral norm of a semicirculant matrix whose first row is $(L_1^{-1}, L_2^{-1}, \ldots, L_n^{-1})$ can be obtained from (23).

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