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Categorical Properties of L-Fuzzifying Convergence Spaces

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Abstract. In this paper, categorical properties of L-fuzzifying convergence spaces are investigated. It is shown that (1) the category L-**FYC** of L-fuzzifying convergence spaces is a strong topological universe; (2) the category L-**FYKC** of L-fuzzifying Kent convergence spaces, as a bireflective and bicoreflective subcategory of L-**FYC**, is also a strong topological universe; (3) the category L-**FYLC** of L-fuzzifying limit spaces, as a bireflective subcategory of L-**FYKC**, is a topological universe.

1. Introduction

In the realm of the theory of topological spaces, natural function spaces cannot be discussed in a satisfactory way, i.e., continuous convergence cannot be described via a topology. This leads to the result that the category of topological spaces and continuous mappings is not Cartesian closed. Kowalsky and Fischer independently enlarged the setting of topological spaces to limit spaces, where the continuous convergence could be always induced by a limit structure. Moreover, the resulting category of limit spaces and continuous mappings is not only Cartesian closed, but also a strong topological universe, which means that it is Cartesian closed, extensional and has the property that quotient mappings are productive.

With the development of fuzzy mathematics, many researchers generalized convergence theory to the fuzzy setting [1, 6, 7, 10, 20–22, 24, 38, 40]. In 2001, Jäger [10] introduced a definition of stratified L-generalized convergence spaces by means of stratified L-filters and showed that the resulting category SL-**FCS** is Cartesian closed, which can embed the category of stratified L-topological spaces as a reflective full subcategory. Afterwards, so many researchers investigated stratified L-generalized convergence spaces from different aspects [1, 2, 11–19, 28, 40]. In [41], Yao made use of L-filters of ordinary subsets to introduce the concept of L-fuzzifying convergence spaces and also showed that the resulting category is Cartesian closed, which can embed the category of L-fuzzifying topological spaces as a reflective full subcategory. In a more general sense, Pang and Fang [22, 23] proposed the notion of L-fuzzy Q-convergence structures and established its relationship with L-fuzzy topologies. In this direction, Pang further introduced (L, M)-fuzzy convergence structures [24, 29], enriched (L, M)-fuzzy convergence structures [25] and stratified L-prefilter convergence structures [33], and investigated their categorical properties and topological properties.

In the above-mentioned works, researchers mainly concerned on the categorical relationship between fuzzy convergence spaces and fuzzy topological spaces, and the Cartesian closedness of fuzzy convergence

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spaces. Actually, categorical properties are important spatial properties in many different mathematical research areas. By this motivation, researchers explored the categorical properties of fuzzy filter spaces, fuzzy (semi, pre, quasi) uniform convergence spaces and fuzzy convex spaces, such as Yang and Li [39], Pang et al. [26, 27, 30–32, 34], Fang [3–5], Xiu et al. [36, 37]. Besides the Cartesian closedness, extensionality and the productivity of quotient mappings also are important categorical properties. In this paper, we will make a systematic investigation on categorical properties of L-fuzzifying convergence spaces. That is, we will explore the Cartesian-closedness, extensionality and the productivity of quotient mappings in the category of L-fuzzifying convergence spaces as well as its subcategories.

This paper is organized as follows. In Section 2, we recall some necessary concepts and notations. In Section 3, we show the category L-**FYC** of L-fuzzifying convergence spaces is a strong topological universe. In Section 4, we propose the concept of L-fuzzifying Kent convergence spaces and prove that the resulting category L-**FYKC**, as a bireflective and bicoreflective subcategory of L-**FYC**, is also a strong topological universe. In Section 5, we show the category L-**FYLC** of L-fuzzifying limit spaces, as a a bireflective full subcategory of L-**FYKC**, is a topological universe.

2. Preliminaries

Throughout this paper, let $(L, \bigvee, \bigwedge, \rightarrow)$ be a complete Heyting algebra unless otherwise statement. The smallest element and the greatest element of L are denoted by $\perp$ and $\top$, respectively. For a given set X, let 2^X denote the powerset of X and let L^X denote the set of all L-subsets on X.

Definition 2.1. ([9]) A mapping $\mathcal{F} : 2^X \longrightarrow L$ is called an L-filter of ordinary subsets on X if it satisfies:
(F1) $\mathcal{F}(\emptyset) = \perp, \mathcal{F}(X) = \top$,
(F2) $A \subseteq B$ implies $\mathcal{F}(A) \leqslant \mathcal{F}(B)$,
(F3) $\mathcal{F}(A \cap B) \geqslant \mathcal{F}(A) \wedge \mathcal{F}(B)$.

The family of all L-filters of ordinary subsets on X will be denoted by $\mathcal{F}_L(X)$. An order on $\mathcal{F}_L(X)$ is defined as follows: $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L(X), \mathcal{F} \leqslant \mathcal{G} \Longleftrightarrow \forall\, A \in 2^X, \mathcal{F}(A) \leqslant \mathcal{G}(A)$.

For every $x \in X$, $[x] \in \mathcal{F}_L(X)$ defined by

$$\forall A \in 2^X,\ [x](A) = \begin{cases} \top, & x \in A, \\ \perp, & otherwise, \end{cases}$$

is an L-filter of ordinary subsets on X.

The following conclusions with respect to L-filters of ordinary subsets parallel to those stratified L-filters possess, we will omit the proofs. For more details we refer to [8, 10].

Proposition 2.2. *For a nonempty family $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ of L-filters of ordinary subsets on X, there exists an L-filter of ordinary subsets $\mathcal{F}$ such that $\mathcal{F}_\lambda \leqslant \mathcal{F}$ $(\forall \lambda \in \Lambda)$, if and only if*

$$\mathcal{F}_{\lambda_1}(A_1) \wedge \cdots \wedge \mathcal{F}_{\lambda_n}(A_n) = \perp \text{ whenever } A_1 \cap \cdots \cap A_n = \emptyset,$$

for $n \in \mathbb{N}, A_1, \cdots, A_n \in 2^X, \{\lambda_1, \cdots, \lambda_n\} \subseteq \Lambda$. In the case of existence, the supremum $\bigvee_{\lambda\in\Lambda} \mathcal{F}_\lambda$ of a nonempty family $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ of L-filters of ordinary subsets is given by

$$\left(\bigvee_{\lambda\in\Lambda} \mathcal{F}_\lambda\right)(A) = \bigvee_{n\in\mathbb{N}} \bigvee \{\mathcal{F}_{\lambda_1}(A_1) \wedge \cdots \wedge \mathcal{F}_{\lambda_n}(A_n) \mid A_1 \cap \cdots \cap A_n \subseteq A\}$$

for all $A \in 2^X$.

Let $\varphi : X \longrightarrow Y$ be a mapping and let $\mathcal{F}$ be an L-filter of ordinary subsets on X. Let $\varphi^{\rightarrow}(A) = \{\varphi(x) \mid x \in A\}$ for all $A \in 2^X$ and $\varphi^{\leftarrow}(B) = \{x \in X \mid \varphi(x) \in B\}$ for all $B \in 2^Y$. Then the mapping $\varphi^{\Rightarrow}(\mathcal{F}) : 2^Y \longrightarrow L$ defined by $\varphi^{\Rightarrow}(\mathcal{F})(A) = \mathcal{F}(\varphi^{\leftarrow}(A))$ for each $A \in 2^Y$, is an L-filter of ordinary subsets on Y, which is called the image of $\mathcal{F}$ under φ. Given a mapping $\varphi : X \longrightarrow Y$ and an L-filter of ordinary subsets $\mathcal{F}$ on Y, the mapping $\varphi^{\Leftarrow}(\mathcal{F}) : 2^X \longrightarrow L$ defined by

$$\forall A \in 2^X,\ \ \varphi^{\Leftarrow}(\mathcal{F})(A) = \bigvee_{\varphi^{\leftarrow}(B)\subseteq A} \mathcal{F}(B)$$

is an L-filter of ordinary subsets if and only if $\mathcal{F}(B) = \bot$ whenever $\varphi^{\leftarrow}(B) = \emptyset$ for all $B \in 2^Y$. In case $\varphi^{\Leftarrow}(\mathcal{F}) \in \mathcal{F}_L(X)$, it is called the inverse image of $\mathcal{F}$ under φ.

Proposition 2.3. *Let $\mathcal{F} \in \mathcal{F}_L(X)$ and let $\varphi : X \longrightarrow Y$ be a mapping. Then the following statements are equivalent:*
(1) *$\varphi^{\Leftarrow}(\mathcal{F})$ is an L-filter of ordinary subsets.*
(2) *$\mathcal{F}(B) = \bot$ whenever $\varphi^{\leftarrow}(B) = \emptyset$ for all $B \in 2^Y$.*
(3) *$\mathcal{F}(Y - \varphi^{\rightarrow}(X)) = \bot$.*

Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of nonempty sets and $\mathcal{F}_\lambda \in \mathcal{F}_L(X_\lambda)$ for each $\lambda \in \Lambda$. Define $\prod_{\lambda\in\Lambda} \mathcal{F}_\lambda$ as follows:

$$\prod_{\lambda\in\Lambda} \mathcal{F}_\lambda := \bigvee_{\lambda\in\Lambda} p_\lambda^{\Leftarrow}(\mathcal{F}_\lambda) \in \mathcal{F}_L\left(\prod_{\lambda\in\Lambda} X_\lambda\right),$$

where for each $\lambda \in \Lambda$, $p_\lambda : \prod_{\mu\in\Lambda} X_\mu \longrightarrow X_\lambda$ is the projection mapping. We call $\prod_{\lambda\in\Lambda} \mathcal{F}_\lambda$ the product of $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$. For two L-filters of ordinary subsets $\mathcal{F}$ and $\mathcal{G}$, their product is usually denoted by $\mathcal{F} \times \mathcal{G}$.

Proposition 2.4. *Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of nonempty sets, $p_\lambda : \prod_{\mu\in\Lambda} X_\mu \longrightarrow X_\lambda$ be the projection mapping, $\mathcal{F}_\lambda \in \mathcal{F}_L(X_\lambda)$ $(\forall \lambda \in \Lambda)$ and $\mathcal{F} \in \mathcal{F}_L(\prod_{\lambda\in\Lambda} X_\lambda)$. Then the following statements hold:*
(1) $\prod_{\mu\in\Lambda} p_\mu^{\Rightarrow}(\mathcal{F}) \leqslant \mathcal{F}$.
(2) $p_\lambda^{\Rightarrow}\left(\prod_{\mu\in\Lambda} \mathcal{F}_\mu\right) \geqslant \mathcal{F}_\lambda$, $\forall \lambda \in \Lambda$.
(3) $p_\lambda^{\Rightarrow}\left(\prod_{\mu\in\Lambda} p_\mu^{\Rightarrow}(\mathcal{F})\right) = \mathcal{F}_\lambda$, $\forall \lambda \in \Lambda$.

Definition 2.5. ([41]) An L-fuzzifying convergence structure on X is a mapping $\lim : \mathcal{F}_L(X) \longrightarrow L^X$ satisfying
(LFY1) $\forall x \in X,\ \lim[x](x) = \top$,
(LFY2) $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L(X),\ \mathcal{F} \leqslant \mathcal{G}$ implies $\lim \mathcal{F} \leqslant \lim \mathcal{G}$.
The pair $(X, \lim)$ is called an L-fuzzifying convergence space.

A continuous mapping between two L-fuzzifying convergence spaces $(X, \lim_X)$ and $(Y, \lim_Y)$ is a mapping $\varphi : X \longrightarrow Y$ such that for each $\mathcal{F} \in \mathcal{F}_L(X)$ and $x \in X$, $\lim_X \mathcal{F}(x) \leqslant \lim_Y \varphi^{\Rightarrow}(\mathcal{F})(\varphi(x))$. Let L-**FYC** denote the category with L-fuzzifying convergence spaces as objects and with continuous mappings as morphisms.

From a categorical aspect, initial structures, final structures and power structures in L-**FYC** can be easily described as follows:

Initial structures: Let $(\varphi_\lambda : X \longrightarrow (X_\lambda, \lim_\lambda))_{\lambda\in\Lambda}$ be a source. Then

$$\lim \mathcal{F}(x) = \bigwedge_{\lambda\in\Lambda} \lim_\lambda(\varphi_\lambda^{\Rightarrow}(\mathcal{F}))(\varphi_\lambda(x))$$

is the initial L-fuzzifying convergence structure on X [41].

Final structures: Let $(\varphi_\lambda : (X_\lambda, \lim_\lambda) \longrightarrow X)_{\lambda\in\Lambda}$ be a sink and define for each $\mathcal{F} \in \mathcal{F}_L(X)$ and $x \in X$,

$$\lim \mathcal{F}(x) = \begin{cases} \top, & \text{if } \mathcal{F} \geqslant [x], \\ \bigvee_{\lambda\in\Lambda} \bigvee_{\varphi_\lambda(x_\lambda)=x} \bigvee_{\varphi^{\Rightarrow}(\mathcal{F}_\lambda)\leqslant\mathcal{F}} \lim_\lambda \mathcal{F}_\lambda(x_\lambda), & \text{if } \mathcal{F} \ngeqslant [x]. \end{cases}$$

Then lim is final with respect to the given sink. In particular, for a surjective mapping $\varphi : (X, \lim_X) \longrightarrow Y$ as a sink, the final structure $\lim_Y$ with respect to the sink is called the quotient structure, and the surjective mapping $\varphi : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ is called the quotient mapping.

The product space, subspace and quotient space are formed in the natural way.

Power structures: Let $(X, \lim_X)$ and $(Y, \lim_Y)$ be L-fuzzifying convergence spaces. We define the L-fuzzifying convergence structure of continuous convergence on the set of morphisms from X to Y ($C(X,Y)$) as follows:

$$c\text{-}\lim \mathcal{F}(\varphi) = \bigwedge_{\mathcal{H}\in\mathcal{F}_L(X)} \bigwedge_{x\in X} \Big(\lim_X \mathcal{H}(x) \to \lim_Y ev^{\Rightarrow}(\mathcal{F}\times\mathcal{H})(\varphi(x))\Big),$$

where $\mathcal{F} \in \mathcal{F}_L(C(X,Y))$ and $ev : C(X,Y)\times X \longrightarrow Y$, $(\varphi, x) \longmapsto \varphi(x)$ is the evaluation mapping.

3. *L*-FYC is a Strong Topological Universe

Recall in a topological category **C**, a partial morphism from X to Y is a **C**-morphism $\varphi : Z \longrightarrow Y$ whose domain is a subobject of X. A topological category **C** is called extensional provided that every **C**-object Y has a one-point extension Y^*, in the sense that every **C**-object Y can be embedded via the addition of a single point ∞_Y into a **C**-object Y^* such that for every partial morphism $\varphi : Z \longrightarrow Y$ from X to Y, the mapping φ^*: $X \longrightarrow Y^*$ defined by

$$\varphi^*(x) = \begin{cases} \varphi(x), & \text{if } x \in Z; \\ \infty_Y, & \text{if } x \notin Z. \end{cases}$$

is a **C**-morphism.

Several categorical properties for a topological category are proposed by Preuss in the book [35], namely:
(CP1) It is Cartesian-closed,
(CP2) It is extensional,
(CP3) It is closed under formations of products of quotient mappings.
According to the terminology of [35], a topological category **C** is called:
(1) strongly Cartesian-closed provided that **C** fulfills (CP1) and (CP3),
(2) a topological universe provided that **C** fulfills (CP1) and (CP2),
(3) a strong topological universe provided that **C** fulfills (CP1)–(CP3).

Theorem 3.1. ([41]) *The category L-***FYC** *is a Cartesian closed topological category.*

For convenience, in the sequel, we suppose that X is a nonempty set and $\infty_X \notin X$. Put $X^* = X \cup \{\infty_X\}$ and $i : X \longrightarrow X^*$ be the inclusion mapping. By Proposition 2.3, we know for each $\mathcal{F} \in \mathcal{F}_L(X^*)$, $i^{\Leftarrow}(\mathcal{F})$ exists if and only if $\mathcal{F}(\{\infty_X\}) = \bot$. Then we have the following proposition.

Proposition 3.2. *Let* $(X, \lim)$ *be an L-fuzzifying convergence space and define* $\lim^* : \mathcal{F}_L(X^*) \longrightarrow L^{X^*}$ *as follows:*

$$\lim^*\mathcal{F}(x) = \begin{cases} \top, & x = \infty_X \text{ or } \mathcal{F}(\{\infty_X\}) \neq \bot; \\ \lim i^{\Leftarrow}(\mathcal{F})(x), & x \neq \infty_X \text{ and } \mathcal{F}(\{\infty_X\}) = \bot. \end{cases}$$

Then $\lim^*$ *is an L-fuzzifying convergence structure on* X^*.

Proof. It suffices to verify that $\lim^*$ satisfies (LFY1) and (LFY2). Indeed,
(LFY1) For each $x \in X$, it follows from $[x](\{\infty_X\}) = \bot$ that

$$\lim^*[x](x) = \lim i^{\Leftarrow}([x])(x) = \lim[x](x) = \top.$$

Further, if $x = \infty_X$, then $\lim^*[x](\infty_X) = \top$.
(LFY2) Take $\mathcal{F}$, $\mathcal{G} \in \mathcal{F}_L(X)$ with $\mathcal{F} \leqslant \mathcal{G}$. If $x = \infty_X$, then $\lim^*\mathcal{F}(x) = \top = \lim^*\mathcal{G}(x)$. Now suppose that $x \neq \infty_X$. If $\mathcal{F}(\{\infty_X\}) \neq \bot$, then $\mathcal{G}(\{\infty_X\}) \neq \bot$. Thus we have $\lim^*\mathcal{F}(x) = \top = \lim^*\mathcal{G}(x)$. If $\mathcal{F}(\{\infty_X\}) = \bot$, then there are two cases.
Case 1: $\mathcal{G}(\{\infty_X\}) = \bot$, we have

$$\lim^*\mathcal{F}(x) = \lim i^{\Leftarrow}(\mathcal{F})(x) \leqslant \lim i^{\Leftarrow}(\mathcal{G})(x) = \lim^*\mathcal{G}(x).$$

Case 2: $\mathcal{G}(\{\infty_X\}) \neq \bot$, the conclusion $\lim^*\mathcal{F}(x) \leqslant \lim^*\mathcal{G}(x)$ follows from the fact that $\lim^*\mathcal{G}(x) = \top$.
As a result, $\lim^*$ is an L-fuzzifying convergence structure on X^*. □

Theorem 3.3. *The category L-**FYC** is extensional.*

Proof. Suppose that $(X, \lim_X)$ is an L-fuzzifying convergence space. By Proposition 3.2, we obtain an L-fuzzifying convergence space $(X^*, \lim^*_X)$. It suffices to show that $(X^*, \lim^*_X)$ is the one-point extension of $(X, \lim_X)$.

Firstly, we show that $(X, \lim_X)$ is a subspace of $(X^*, \lim^*_X)$. Let $\overline{\lim}_X$ be the initial structure on X with respect to the inclusion mapping $i : X \longrightarrow X^*$, i.e., $\overline{\lim}_X\mathcal{F}(x) = \lim^*_X i^{\Rightarrow}(\mathcal{F})(x)$ for all $\mathcal{F} \in \mathcal{F}_L(X)$ and $x \in X$. Since $i^{\Rightarrow}(\mathcal{F})(\{\infty_X\}) = \mathcal{F}(i^{\leftarrow}(\{\infty_X\})) = \mathcal{F}(\emptyset) = \bot$, we have

$$\lim{}^*_X i^{\Rightarrow}(\mathcal{F})(x) = \lim{}_X i^{\Leftarrow} \circ i^{\Rightarrow}(\mathcal{F})(x) = \lim{}_X\mathcal{F}(x).$$

Thus, $\overline{\lim}_X = \lim_X$. That is to say, $(X, \lim_X)$ is a subspace of $(X^*, \lim^*_X)$.

Next, we show that $(X^*, \lim^*_X)$ is the one-point extension of $(X, \lim_X)$. Let $(Y, \lim_Y)$ be an L-fuzzifying convergence space, $(Z, \lim_Z)$ be a subspace of $(Y, \lim_Y)$ and $\varphi : (Z, \lim_Z) \longrightarrow (X, \lim_X)$ be a continuous mapping. Let $k : Z \longrightarrow Y$ denote the inclusion mapping and $\varphi^* : Y \longrightarrow X^*$ denote the extensional mapping of φ, i.e., $\varphi^*(y) = \varphi(y)$ for all $y \in Z$, and $\varphi^*(y) = \infty_X$ otherwise. In order to prove that $\varphi^* : (Y, \lim_Y) \longrightarrow (X^*, \lim^*_X)$ is continuous, it suffices to verify that $\lim_Y \mathcal{G} \leqslant \lim^*_X(\varphi^*)^{\Rightarrow}(\mathcal{G})$ for all $\mathcal{G} \in \mathcal{F}_L(Y)$, which will be shown in the following cases.

Case 1: $k^{\Leftarrow}(\mathcal{G})$ does not exist, i.e., $\mathcal{G}(Y - k^{\rightarrow}(Z)) = \mathcal{G}(Y - Z) \neq \bot$. In this case, it follows from $Y - Z = (\varphi^*)^{\leftarrow}(\{\infty_X\})$ that

$$(\varphi^*)^{\Rightarrow}(\mathcal{G})(\{\infty_X\}) = \mathcal{G}((\varphi^*)^{\leftarrow}(\{\infty_X\})) = \mathcal{G}(Y - Z) \neq \bot.$$

Hence, $\lim^*_X(\varphi^*)^{\Rightarrow}(\mathcal{G})(\varphi^*(y)) = \top \geqslant \lim_Y \mathcal{G}(y)$.

Case 2: $k^{\Leftarrow}(\mathcal{G})$ exists. First of all, we prove the following conclusions:

(1) $i^{\leftarrow}((\varphi^*)^{\rightarrow}(B)) = \varphi^{\rightarrow}(k^{\leftarrow}(B))$ for all $B \in 2^Y$,

(2) $\varphi^{\Rightarrow}(k^{\Leftarrow}(\mathcal{G})) \leqslant i^{\Leftarrow}((\varphi^*)^{\Rightarrow}(\mathcal{G}))$ for all $\mathcal{G} \in \mathcal{F}_L(Y)$.

For (1), take any $x \in X$. Then

$$\begin{aligned}
x \in i^{\leftarrow}((\varphi^*)^{\rightarrow}(B)) &\iff x = i(x) \in (\varphi^*)^{\rightarrow}(B)\\
&\iff \exists y \in B, \text{ s.t. } x = \varphi^*(y) \quad (x \in X)\\
&\iff \exists y \in B \cap Z, \text{ s.t. } x = \varphi^*(y) = \varphi(y)\\
&\iff \exists y \in k^{\leftarrow}(B), \text{ s.t. } x = \varphi(y)\\
&\iff x \in \varphi^{\rightarrow}(k^{\leftarrow}(B)).
\end{aligned}$$

For (2), take any $\mathcal{G} \in \mathcal{F}_L(Y)$ and $A \in 2^X$. Then

$$\begin{aligned}
\varphi^{\Rightarrow}(k^{\Leftarrow}(\mathcal{G}))(A) &= k^{\Leftarrow}(\mathcal{G})(\varphi^{\leftarrow}(A)) = \bigvee_{k^{\leftarrow}(B)\subseteq\varphi^{\leftarrow}(A)} \mathcal{G}(B)\\
&\leqslant \bigvee_{\varphi^{\rightarrow}(k^{\leftarrow}(B))\subseteq A} \mathcal{G}((\varphi^*)^{\leftarrow} \circ (\varphi^*)^{\rightarrow}(B))\\
&= \bigvee_{i^{\leftarrow}((\varphi^*)^{\rightarrow}(B))\subseteq A} \mathcal{G}((\varphi^*)^{\leftarrow}((\varphi^*)^{\rightarrow}(B))) \quad \text{(by (1))}\\
&\leqslant \bigvee_{i^{\leftarrow}(B^*)\subseteq A} \mathcal{G}((\varphi^*)^{\leftarrow}(B^*))\\
&= \bigvee_{i^{\leftarrow}(B^*)\subseteq A} (\varphi^*)^{\Rightarrow}(\mathcal{G})(B^*)\\
&= i^{\Leftarrow}((\varphi^*)^{\Rightarrow}(\mathcal{G}))(A).
\end{aligned}$$

Now we show that $\varphi^* : (Y, \lim_Y) \longrightarrow (X^*, \lim_X^*)$ is continuous. Take any $y \in Y$. If $y \in Z$, then

$$\begin{array}{rcl} \lim_Y \mathcal{G}(y) & \leqslant & \lim_Y k^{\Rightarrow}(k^{\Leftarrow}(\mathcal{G}))(y) \\ & = & \lim_Z k^{\Leftarrow}(\mathcal{G})(y) \\ & \leqslant & \lim_X \varphi^{\Rightarrow}(k^{\Leftarrow}(\mathcal{G}))(\varphi(y)) \\ & \leqslant & \lim_X i^{\Leftarrow}((\varphi^*)^{\Rightarrow}(\mathcal{G}))(\varphi^*(y)) \\ & = & \lim_X^*(\varphi^*)^{\Rightarrow}(\mathcal{G})(\varphi^*(y)). \end{array}$$

If $y \in Y - Z$, then $\varphi^*(y) = \infty_X$. It follows from $\lim_X^*(\varphi^*)^{\Rightarrow}(\mathcal{G})(\infty_X) = \top$ that $\lim_Y \mathcal{G}(y) \leqslant \lim_X^*(\varphi^*)^{\Rightarrow}(\mathcal{G})(\varphi^*(y))$. This implies the continuity of $\varphi^* : (Y, \lim_Y) \longrightarrow (X^*, \lim_X^*)$.

As a consequence, the category L-**FYC** is extensional. □

By Theorems 3.1 and 3.3, we obtain

Theorem 3.4. *The category L-**FYC** is a topological universe.*

In the sequel, we will show that the category L-**FYC** satisfies (CP3). To this end, the following lemma is necessary.

Lemma 3.5. *Let $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ be a family of L-filters of ordinary subsets with $\mathcal{F}_\lambda \in \mathcal{F}_L(X_\lambda)$ $(\lambda \in \Lambda)$. Then for each $A \in 2^{\prod_{\lambda\in\Lambda} X_\lambda}$,*

$$\left(\prod_{\lambda\in\Lambda} \mathcal{F}_\lambda\right)(A) = \bigvee_{n\in\mathbb{N}} \bigvee \left\{ \wedge_{i=1}^n \mathcal{F}_{\lambda_i}(B_{\lambda_i}) \mid B_\lambda = X_\lambda \text{ when } \lambda \notin \{\lambda_i\}_{i=1}^{i=n}, \prod_{\lambda\in\Lambda} B_\lambda \subseteq A \right\}$$

Proof. For each $A \in 2^{\prod_{\lambda\in\Lambda} X_\lambda}$, we have

$$\begin{array}{rl} & \left(\prod_{\lambda\in\Lambda} \mathcal{F}_\lambda\right)(A) \\ = & \left(\bigvee_{\lambda\in\Lambda} p_\lambda^{\Leftarrow}(\mathcal{F}_\lambda)\right)(A) \\ = & \bigvee_{n\in\mathbb{N}} \bigvee \{\wedge_{i=1}^n p_{\lambda_i}^{\Leftarrow}(\mathcal{F}_{\lambda_i})(A_{\lambda_i}) \mid \cap_{i=1}^n A_{\lambda_i} \subseteq A\} \quad \text{(by Proposition 2.2)} \\ = & \bigvee_{n\in\mathbb{N}} \bigvee_{\wedge_{i=1}^n A_{\lambda_i} \subseteq A} \wedge_{i=1}^n \bigvee_{p_{\lambda_i}^{\leftarrow}(B_{\lambda_i}) \subseteq A_{\lambda_i}} \mathcal{F}_{\lambda_i}(B_{\lambda_i}) \\ = & \bigvee_{n\in\mathbb{N}} \bigvee_{\wedge_{i=1}^n A_{\lambda_i} \subseteq A} \bigvee_{p_{\lambda_1}^{\leftarrow}(B_{\lambda_1}) \subseteq A_{\lambda_1}} \cdots \bigvee_{p_{\lambda_n}^{\leftarrow}(B_{\lambda_n}) \subseteq A_{\lambda_n}} \wedge_{i=1}^n \mathcal{F}_{\lambda_i}(B_{\lambda_i}) \\ = & \bigvee_{n\in\mathbb{N}} \bigvee_{\wedge_{i=1}^n p_{\lambda_i}^{\leftarrow}(B_{\lambda_i}) \subseteq A} \wedge_{i=1}^n \mathcal{F}_{\lambda_i}(B_{\lambda_i}) \\ = & \bigvee_{n\in\mathbb{N}} \bigvee \left\{ \wedge_{i=1}^n \mathcal{F}_{\lambda_i}(B_{\lambda_i}) \mid B_\lambda = X_\lambda \text{ when } \lambda \notin \{\lambda_i\}_{i=1}^{i=n}, \prod_{\lambda\in\Lambda} B_\lambda \subseteq A \right\}. \end{array}$$

This proves the conclusion. □

Proposition 3.6. *Let $\{\varphi_\lambda : X_\lambda \longrightarrow Y_\lambda\}$ be a family of surjective mappings and let $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ be a family of L-filters of ordinary subsets with $\mathcal{F}_\lambda \in \mathcal{F}_L(X_\lambda)$. Then*

$$\left(\prod_{\lambda\in\Lambda} \varphi_\lambda\right)^{\Rightarrow} \left(\prod_{\lambda\in\Lambda} \mathcal{F}_\lambda\right) = \prod_{\lambda\in\Lambda} \varphi_\lambda^{\Rightarrow}(\mathcal{F}_\lambda).$$

Proof. Let

$$\begin{array}{ccc}\prod_{\lambda\in\Lambda} X_\lambda & \xrightarrow{\prod_{\lambda\in\Lambda}\varphi_\lambda} & \prod_{\lambda\in\Lambda} Y_\lambda \\ {\scriptstyle p_\lambda}\downarrow & & {\scriptstyle q_\lambda}\downarrow \\ X_\lambda & \xrightarrow{\varphi_\lambda} & Y_\lambda\end{array}$$

be the product commutation diagram. Firstly, the inequality

$$\left(\prod_{\lambda\in\Lambda}\varphi_\lambda\right)^{\Rightarrow}\left(\prod_{\lambda\in\Lambda}\mathcal{F}_\lambda\right) \geqslant \prod_{\lambda\in\Lambda}\varphi_\lambda^{\Rightarrow}(\mathcal{F}_\lambda)$$

can be verified as follows:

$$\begin{aligned}
&\left(\prod_{\lambda\in\Lambda}\varphi_\lambda\right)^{\Rightarrow}\left(\prod_{\lambda\in\Lambda}\mathcal{F}_\lambda\right)\\
\geqslant\ & \prod_{\lambda\in\Lambda} q_\lambda^{\Rightarrow}\left(\left(\prod_{\lambda\in\Lambda}\varphi_\lambda\right)^{\Rightarrow}\left(\prod_{\lambda\in\Lambda}\mathcal{F}_\lambda\right)\right) \quad \text{(by Proposition 2.4 (1))}\\
=\ & \prod_{\lambda\in\Lambda}\left(q_\lambda\circ\prod_{\lambda\in\Lambda}\varphi_\lambda\right)^{\Rightarrow}\left(\prod_{\lambda\in\Lambda}\mathcal{F}_\lambda\right)\\
=\ & \prod_{\lambda\in\Lambda}(\varphi_\lambda\circ p_\lambda)^{\Rightarrow}\left(\prod_{\lambda\in\Lambda}\mathcal{F}_\lambda\right)\\
=\ & \prod_{\lambda\in\Lambda}\varphi_\lambda^{\Rightarrow}\left(p_\lambda^{\Rightarrow}\left(\prod_{\lambda\in\Lambda}\mathcal{F}_\lambda\right)\right) \quad \text{(by Proposition 2.4 (2))}\\
\geqslant\ & \prod_{\lambda\in\Lambda}\varphi_\lambda^{\Rightarrow}(\mathcal{F}_\lambda).
\end{aligned}$$

Conversely, for all $A \in 2^{\prod_{\lambda\in\Lambda} Y_\lambda}$, by Lemma 3.5, we have

$$\begin{aligned}
&\left(\prod_{\lambda\in\Lambda}\varphi_\lambda\right)^{\Rightarrow}\left(\prod_{\lambda\in\Lambda}\mathcal{F}_\lambda\right)(A)\\
=\ & \left(\prod_{\lambda\in\Lambda}\mathcal{F}_\lambda\right)\left(\left(\prod_{\lambda\in\Lambda}\varphi_\lambda\right)^{\leftarrow}(A)\right)\\
=\ & \bigvee_{n\in\mathbb{N}}\bigvee\left\{\wedge_{i=1}^{n}\mathcal{F}_{\lambda_i}(B_{\lambda_i}) \mid B_\lambda = X_\lambda \text{ when } \lambda\notin\{\lambda_i\}_{i=1}^{i=n}, \prod_{\lambda\in\Lambda}B_\lambda \subseteq \left(\prod_{\lambda\in\Lambda}\varphi_\lambda\right)^{\leftarrow}(A)\right\}
\end{aligned}$$

and

$$\begin{aligned}
&\left(\prod_{\lambda\in\Lambda}\varphi_\lambda^{\Rightarrow}(\mathcal{F}_\lambda)\right)(A)\\
=\ & \bigvee_{n\in\mathbb{N}}\bigvee\left\{\wedge_{i=1}^{n}\mathcal{F}_{\lambda_i}(\varphi_{\lambda_i}^{\leftarrow}(E_{\lambda_i})) \mid E_\lambda = Y_\lambda \text{ when } \lambda\notin\{\lambda_i\}_{i=1}^{i=n}, \prod_{\lambda\in\Lambda}E_\lambda\subseteq A\right\}.
\end{aligned}$$

Take each $n\in\mathbb{N}$ and $B_\lambda \in 2^{X_\lambda}$ $(\lambda\in\Lambda)$ such that $B_\lambda = X_\lambda$ when $\lambda\notin\{\lambda_i\}_{i=1}^{i=n}$ and $\prod_{\lambda\in\Lambda}B_\lambda \subseteq \left(\prod_{\lambda\in\Lambda}\varphi_\lambda\right)^{\leftarrow}(A)$. Let $E_\lambda = \varphi_\lambda^{\rightarrow}(B_\lambda)$ for all $\lambda\in\Lambda$. Then

$$\wedge_{i=1}^{n}\mathcal{F}_{\lambda_i}(\varphi_{\lambda_i}^{\leftarrow}(E_{\lambda_i})) = \wedge_{i=1}^{n}\mathcal{F}_{\lambda_i}(\varphi_{\lambda_i}^{\leftarrow}(\varphi_{\lambda_i}^{\rightarrow}(B_{\lambda_i}))) \geqslant \wedge_{i=1}^{n}\mathcal{F}_{\lambda_i}(B_{\lambda_i}).$$

Since φ_λ is surjective, we have $E_\lambda = Y_\lambda$ when $\lambda \notin \{\lambda_i\}_{i=1}^{i=n}$. Further, we have

$$\prod_{\lambda\in\Lambda} E_\lambda = \prod_{\lambda\in\Lambda} \varphi_\lambda^{\rightarrow}(B_\lambda) = \left(\prod_{\lambda\in\Lambda}\varphi_\lambda\right)^{\rightarrow}\left(\prod_{\lambda\in\Lambda} B_\lambda\right) \subseteq \left(\prod_{\lambda\in\Lambda}\varphi_\lambda\right)^{\rightarrow}\left(\left(\prod_{\lambda\in\Lambda}\varphi_\lambda\right)^{\leftarrow}(A)\right) \subseteq A,$$

Thus, we obtain $\left(\prod_{\lambda\in\Lambda}\varphi_\lambda\right)^{\Rightarrow}\left(\prod_{\lambda\in\Lambda}\mathcal{F}_\lambda\right)(A) \leqslant \left(\prod_{\lambda\in\Lambda}\varphi_\lambda^{\Rightarrow}(\mathcal{F}_\lambda)\right)(A)$ for all $A \in L^{\prod_{\lambda\in\Lambda} Y_\lambda}$, as desired. □

Theorem 3.7. *Suppose that L is a completely distributive lattice. If $\{\varphi_\lambda : (X_\lambda, \lim_{X_\lambda}) \longrightarrow (Y_\lambda, \lim_{Y_\lambda})\}$ is a family of quotient mappings in L-**FYC**, then the product mapping*

$$\prod_{\lambda\in\Lambda}\varphi_\lambda : \left(\prod_{\lambda\in\Lambda} X_\lambda, \prod_{\lambda\in\Lambda}\lim{}_{X_\lambda}\right) \longrightarrow \left(\prod_{\lambda\in\Lambda} Y_\lambda, \prod_{\lambda\in\Lambda}\lim{}_{Y_\lambda}\right)$$

*is a quotient mapping in L-**FYC**.*

Proof. Suppose that $\varphi := \prod_{\lambda\in\Lambda}\varphi_\lambda$, $(X, \lim_X) := (\prod_{\lambda\in\Lambda} X_\lambda, \prod_{\lambda\in\Lambda}\lim_{X_\lambda})$ and $(Y, \lim_Y) := (\prod_{\lambda\in\Lambda} Y_\lambda, \prod_{\lambda\in\Lambda}\lim_{Y_\lambda})$. Let

$$\begin{array}{ccc}
(X, \lim_X) & \xrightarrow{\varphi} & (Y, \lim_Y) \\
{\scriptstyle p_\lambda}\downarrow & & {\scriptstyle q_\lambda}\downarrow \\
(X_\lambda, \lim_{X_\lambda}) & \xrightarrow{\varphi_\lambda} & (Y_\lambda, \lim_{Y_\lambda})
\end{array}$$

be the product commutation diagram. Since $\varphi_\lambda : (X_\lambda, \lim_{X_\lambda}) \longrightarrow (Y_\lambda, \lim_{Y_\lambda})$ is a quotient mapping, i.e., $\lim_{Y_\lambda}$ is the final structure with respect to the sink $\varphi_\lambda : (X_\lambda, \lim_{X_\lambda}) \longrightarrow Y_\lambda$, we have

$$\forall \mathcal{G}_\lambda \in \mathcal{F}_L(Y_\lambda),\ y_\lambda \in Y_\lambda,\ \lim{}_{Y_\lambda}\mathcal{G}_\lambda(y_\lambda) = \bigvee_{\varphi_\lambda^{\Rightarrow}(\mathcal{F}_\lambda)\leqslant\mathcal{G}_\lambda}\ \bigvee_{\varphi_\lambda(x_\lambda)=y_\lambda} \lim{}_{X_\lambda}\mathcal{F}_\lambda(x_\lambda).$$

In order to show that φ is a quotient mapping, it suffices to prove:
(1) φ is surjective.
(2) $\lim_Y$ is the final structure with respect to the sink $\varphi : (X, \lim_X) \longrightarrow Y$.

(1) is true since $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ are all surjective. For (2), let $\overline{\lim}_Y$ denote the final structure with respect to the sink $\varphi : (X, \lim_X) \longrightarrow Y$. Then

$$\forall \mathcal{G} \in \mathcal{F}_L(Y), \forall y \in Y,\ \overline{\lim}_Y\mathcal{G}(y) = \bigvee_{\varphi^{\Rightarrow}(\mathcal{F})\leqslant\mathcal{G}}\ \bigvee_{\varphi(x)=y} \lim{}_X\mathcal{F}(x).$$

Next, we will show $\lim_Y = \overline{\lim}_Y$.

On one hand, for each $\mathcal{G} \in \mathcal{F}_L(X)$ and $y \in Y$ with $\varphi^{\Rightarrow}(\mathcal{F}) \leqslant \mathcal{G}$ and $\varphi(x) = y$, we have

$$\begin{aligned}
\lim{}_X\mathcal{F}(x) &\leqslant \lim{}_{X_\lambda} p_\lambda^{\Rightarrow}(\mathcal{F})(p_\lambda(x)) \\
&\leqslant \lim{}_{Y_\lambda}\varphi_\lambda^{\Rightarrow}(p_\lambda^{\Rightarrow}(\mathcal{F}))(\varphi_\lambda(p_\lambda(x))) \\
&= \lim{}_{Y_\lambda} q_\lambda^{\Rightarrow}(\varphi^{\Rightarrow}(\mathcal{F}))(q_\lambda(\varphi(x))) \\
&\leqslant \lim{}_{Y_\lambda} q_\lambda^{\Rightarrow}(\mathcal{G})(q_\lambda(y))
\end{aligned}$$

for all $\lambda \in \Lambda$. Then it follows that $\lim_X\mathcal{F}(x) \leqslant \bigwedge_{\lambda\in\Lambda}\lim_{Y_\lambda} q_\lambda^{\Rightarrow}(\mathcal{G})(q_\lambda(y)) = \lim_Y\mathcal{G}(y)$. Thus, we obtain $\overline{\lim}_Y\mathcal{G}(y) \leqslant \lim_Y\mathcal{G}(y)$.

On the other hand, for each $\mathcal{G} \in \mathcal{F}_L(X)$ and $y \in Y$, by the completely distributive law, we have

$$\begin{aligned}
\lim_Y \mathcal{G}(y) &= \bigwedge_{\lambda\in\Lambda} \lim_{Y_\lambda} q_\lambda^{\Rightarrow}(\mathcal{G})(q_\lambda(y)) \\
&= \bigwedge_{\lambda\in\Lambda} \bigvee_{\varphi_\lambda^{\Rightarrow}(\mathcal{F}_\lambda)\leqslant q_\lambda^{\Rightarrow}(\mathcal{G})} \bigvee_{\varphi_\lambda(x_\lambda)=q_\lambda(y)} \lim_{X_\lambda}\mathcal{F}_\lambda(x_\lambda) \\
&= \bigvee_{\phi\in\prod_{\lambda\in\Lambda}\Phi_\lambda} \bigwedge_{\lambda\in\Lambda} \bigvee_{\varphi_\lambda(x_\lambda)=q_\lambda(y)} \lim_{X_\lambda}\phi(\lambda)(x_\lambda) \quad (\text{where } \Phi_\lambda = \{\mathcal{F}_\lambda \in \mathcal{F}_L(X_\lambda) \mid \varphi_\lambda^{\Rightarrow}(\mathcal{F}_\lambda) \leqslant q_\lambda^{\Rightarrow}(\mathcal{G})\}) \\
&= \bigvee_{\phi\in\prod_{\lambda\in\Lambda}\Phi_\lambda} \bigvee_{\psi\in\prod_{\lambda\in\Lambda}\Psi_\lambda} \bigwedge_{\lambda\in\Lambda} \lim_{X_\lambda}\phi(\lambda)(\psi(\lambda)) \quad (\text{where } \Psi_\lambda = \{x_\lambda \in X_\lambda \mid \varphi_\lambda(x_\lambda) = q_\lambda(y)\}) \\
&\leqslant \bigvee_{\phi\in\prod_{\lambda\in\Lambda}\Phi_\lambda} \bigvee_{\psi\in\prod_{\lambda\in\Lambda}\Psi_\lambda} \bigwedge_{\lambda\in\Lambda} \lim_{X_\lambda} p_\lambda^{\Rightarrow}\left(\prod_{\lambda\in\Lambda}\phi(\lambda)\right)\left(p_\lambda\left(\prod_{\lambda\in\Lambda}\psi(\lambda)\right)\right) \\
&= \bigvee_{\phi\in\prod_{\lambda\in\Lambda}\Phi_\lambda} \bigvee_{\psi\in\prod_{\lambda\in\Lambda}\Psi_\lambda} \lim_X \prod_{\lambda\in\Lambda}\phi(\lambda)\left(\prod_{\lambda\in\Lambda}\psi(\lambda)\right) \\
&\leqslant \bigvee_{\varphi^{\Rightarrow}(\mathcal{F})\leqslant\mathcal{G}} \bigvee_{\varphi(x)=y} \lim_X \mathcal{F}(x) \\
&= \overline{\lim}_Y \mathcal{G}(y),
\end{aligned}$$

where the last inequality holds from

$$\varphi\left(\prod_{\lambda\in\Lambda}\psi(\lambda)\right) = \prod_{\lambda\in\Lambda}\varphi_\lambda(\psi(\lambda)) = \prod_{\lambda\in\Lambda} q_\lambda(y) = y$$

and

$$\begin{aligned}
\varphi^{\Rightarrow}\left(\prod_{\lambda\in\Lambda}\phi(\lambda)\right) &= \left(\prod_{\lambda\in\Lambda}\varphi_\lambda\right)^{\Rightarrow}\left(\prod_{\lambda\in\Lambda}\phi(\lambda)\right) \\
&= \prod_{\lambda\in\Lambda}\varphi_\lambda^{\Rightarrow}(\phi(\lambda)) \quad (\text{by Proposition 3.6 }) \\
&\leqslant \prod_{\lambda\in\Lambda} q_\lambda^{\Rightarrow}(\mathcal{G}) \quad (\text{since } \phi(\lambda) \in \phi_\lambda) \\
&\leqslant \mathcal{G}. \quad (\text{by Proposition 2.4(1)})
\end{aligned}$$

As a consequence, we obtain $\lim_Y = \overline{\lim}_Y$, as desired. □

By Theorems 3.4 and 3.7, we obtain the main result in this section.

Theorem 3.8. *Let L be a completely distributive lattice. Then the category L-***FYC** *is a strong topological universe.*

4. *L*-Fuzzifying Kent Convergence Spaces

In [11], Jäger proposed the concept of stratified *L*-Kent convergence structures by means of stratified *L*-filters and established its relationship with stratified *L*-generalized convergence structures. In this section, we will adopt a similar way to propose the concept of *L*-fuzzifying Kent convergence spaces and study its relationship with *L*-fuzzifying convergence spaces. Moreover, we will show that the category of *L*-fuzzifying Kent convergence spaces is also a strong topological universe.

Definition 4.1. An L-fuzzifying convergence structure $\lim : \mathcal{F}_L(X) \longrightarrow L^X$ satisfying the following condition

$$\text{(LFYK)} \quad \forall \mathcal{F} \in \mathcal{F}_L(X),\ \forall x \in X,\ \lim \mathcal{F}(x) = \lim(\mathcal{F} \wedge [x])(x)$$

is called an L-fuzzifying Kent convergence structure. The pair $(X, \lim)$ is called an L-fuzzifying Kent convergence space. Let L-**FYKC** denote the full subcategory of L-**FYC** consisting of all L-fuzzifying Kent convergence spaces.

Next let us explore deeper relationship between L-**FYKC** and L-**FYC**.

Theorem 4.2. *The category L-**FYKC** is a full and bireflective subcategory of L-**FYC**.*

Proof. For an L-fuzzifying convergence space $(X, \lim)$, we define

$$\lim{}^{*}\mathcal{F}(x) = \bigvee_{\mathcal{F} \geqslant \mathcal{G} \wedge [x]} \lim \mathcal{G}(x).$$

Then we claim that $id_X : (X, \lim) \longrightarrow (X, \lim^*)$ is the L-**FYKC**-bireflector. For this it suffices to prove:

(1) $\lim^*$ is an L-fuzzifying Kent convergence structure on X.
(2) For $(Y, \lim^Y) \in |L\text{-}\mathbf{FYKC}|$ and each mapping $\varphi : X \longrightarrow Y$, the continuity of $\varphi : (X, \lim) \longrightarrow (Y, \lim^Y)$ implies the continuity of $\varphi : (X, \lim^*) \longrightarrow (Y, \lim^Y)$.

(1) (LFY1) and (LFY2) are obvious. For (LFYK), we first have

$$\{\mathcal{G} : \mathcal{F} \geqslant \mathcal{G} \wedge [x]\} = \{\mathcal{G} : \mathcal{F} \wedge [x] \geqslant \mathcal{G} \wedge [x]\}.$$

Thus it follows that

$$\lim{}^{*}\mathcal{F}(x) = \bigvee_{\mathcal{F} \geqslant \mathcal{G} \wedge [x]} \lim \mathcal{G}(x) \leqslant \bigvee_{\mathcal{F} \wedge [x] \geqslant \mathcal{G} \wedge [x]} \lim \mathcal{G}(x) = \lim{}^{*}(\mathcal{F} \wedge [x])(x).$$

That is, $\lim^*\mathcal{F}(x) = \lim^*(\mathcal{F} \wedge [x])(x)$.

(2) Since $\varphi : (X, \lim) \longrightarrow (Y, \lim^Y)$ is continuous, it follows that for each $\mathcal{F} \in \mathcal{F}_L(X)$ and each $x \in X$,

$$\lim \mathcal{F}(x) \leqslant \lim{}^{Y} \varphi^{\Rightarrow}(\mathcal{F})(\varphi(x)).$$

Then

$$\begin{aligned}
\lim{}^{*}\mathcal{F}(x) &= \bigvee_{\mathcal{F} \geqslant \mathcal{G} \wedge [x]} \lim \mathcal{G}(x) \\
&\leqslant \bigvee_{\varphi^{\Rightarrow}(\mathcal{F}) \geqslant \varphi^{\Rightarrow}(\mathcal{G}) \wedge [\varphi(x)]} \lim{}^{Y} \varphi^{\Rightarrow}(\mathcal{G})(\varphi(x)) \\
&= \bigvee_{\varphi^{\Rightarrow}(\mathcal{F}) \geqslant \varphi^{\Rightarrow}(\mathcal{G}) \wedge [\varphi(x)]} \lim{}^{Y} (\varphi^{\Rightarrow}(\mathcal{G}) \wedge [\varphi(x)])(\varphi(x)) \\
&\leqslant \lim{}^{Y} \varphi^{\Rightarrow}(\mathcal{F})(\varphi(x)).
\end{aligned}$$

The continuity of $\varphi : (X, \lim^*) \longrightarrow (Y, \lim^Y)$ is proved, as desired. □

By Theorem 4.2, we obtain the following result.

Corollary 4.3. *The category L-**FYKC** is closed under formation of subspaces and product spaces in L-**FYC**.*

Theorem 4.4. *The category L-**FYKC** is a full and bicoreflective subcategory of L-**FYC**.*

Proof. For an L-fuzzifying convergence space $(X, \lim)$, we define

$$\lim_*\mathcal{F}(x) = \lim(\mathcal{F} \wedge [x])(x).$$

Then we claim that $id_X : (X, \lim_*) \longrightarrow (X, \lim)$ is the L-**FYKC**-bicoreflector. For this it suffices to prove:

(1) $\lim_*$ is an L-fuzzifying Kent convergence structure on X.
(2) For $(Y, \lim_Y) \in |L\text{-}\mathbf{FYKC}|$ and each mapping $\varphi : Y \longrightarrow X$, the continuity of $\varphi : (Y, \lim_Y) \longrightarrow (X, \lim)$ implies the continuity of $\varphi : (Y, \lim_Y) \longrightarrow (X, \lim_*)$.

(1) Obvious.
(2) Since $(Y, \lim_Y) \in |L\text{-}\mathbf{FYKC}|$, it follows that for each $\mathcal{G} \in \mathcal{F}_L(Y)$ and each $y \in Y$,

$$\lim_Y\mathcal{G}(y) = \lim_Y(\mathcal{G} \wedge [y])(y).$$

By the continuity of φ, we have

$$\lim_Y(\mathcal{G} \wedge [y])(y) \leqslant \lim(\varphi^{\Rightarrow}(\mathcal{G}) \wedge [\varphi(y)])(\varphi(y)) = \lim_*\varphi^{\Rightarrow}(\mathcal{G})(\varphi(y)).$$

This implies $\lim_Y\mathcal{G}(y) \leqslant \lim_*\varphi^{\Rightarrow}(\mathcal{G})(\varphi(y))$. The continuity of $\varphi : (Y, \lim_Y) \longrightarrow (X, \lim_*)$ is proved, as desired. □

Since L-**FYC** is a topological category and L-**FYKC** is a full and bicoreflective subcategory of L-**FYC**, we obtain

Corollary 4.5. *The category L-**FYKC** is a topological category.*

In order to show that L-**FYKC** is a strong topological universe, the following preparations are necessary.

Definition 4.6. A continuous mapping $\varphi : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ in the category L-**FYC** is called an isomorphism provided that $\varphi : X \longrightarrow Y$ is bijective and that its inverse mapping $\psi : (Y, \lim_Y) \longrightarrow (X, \lim_X)$ is continuous. We say that an L-fuzzifying convergence space $(X, \lim_X)$ is isomorphic to an L-fuzzifying convergence space $(Y, \lim_Y)$ if there exists an isomorphism between them.

In the following, we say that a subcategory **D** of a category **C** is isomorphism-closed in **C** if each **C**-object C that is isomorphic to a **D**-object must be a **D**-object.

Lemma 4.7. *The category L-**FYKC** is a full and isomorphism-closed subcategory of L-**FYC**.*

Proof. Let $\varphi : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ be an isomorphism in L-**FYC** and $(X, \lim_X)$ be an L-fuzzifying Kent convergence space. To verify $(Y, \lim_Y)$ is an L-fuzzifying Kent convergence space, it is necessary to show that $\lim_Y$ satisfies (LFYK). Now let ψ denote the inverse mapping of φ. Then for $\mathcal{F} \in \mathcal{F}_L(Y)$, by the continuity of φ and ψ, we have

$$\begin{aligned}
\lim_Y\mathcal{F}(y) &\leqslant \lim_X\psi^{\Rightarrow}(\mathcal{F})(\psi(y)) \\
&\leqslant \lim_Y\varphi^{\Rightarrow}(\psi^{\Rightarrow}(\mathcal{F}))(\varphi(\psi(y))) \\
&= \lim_Y(\varphi \circ \psi)^{\Rightarrow}(\mathcal{F})(\varphi(\psi(y))) = \lim_Y\mathcal{F}(y).
\end{aligned}$$

Hence, $\lim_Y\mathcal{F}(y) = \lim_X\psi^{\Rightarrow}(\mathcal{F})(\psi(y))$. Further, since $(X, \lim_X)$ is an L-fuzzifying Kent convergence space, we obtain

$$\begin{aligned}
\lim_Y(\mathcal{F} \wedge [y])(y) &= \lim_X(\psi^{\Rightarrow}(\mathcal{F} \wedge [y]))(\psi(y)) \\
&= \lim_X(\psi^{\Rightarrow}(\mathcal{F}) \wedge [\psi(y)]))(\psi(y)) \\
&= \lim_X\psi^{\Rightarrow}(\mathcal{F}))(\psi(y)) = \lim_Y\mathcal{F}(y).
\end{aligned}$$

Thus, $\lim_Y$ satisfies (LFYK), as desired. □

Lemma 4.8. ([35]) *Let* **A** *be a topological category.*

(1) *If* **B** *is a bicoreflective (full and isomorphism-closed) subcategory of* **A** *which is closed under formation of finite products in* **A**, *then* **B** *fulfills* (CP1) *whenever* **A** *fulfills* (CP1) *and the power objects in* **B** *arise from the corresponding power objects in* **A** *by applying the bicoreflector.*

(2) *If* **B** *is a bicoreflective (full and isomorphism-closed) subcategory of* **A** *which is closed under formation of subspaces in* **A**, *then* **B** *fulfills* (CP2) *whenever* **A** *fulfills* (CP2) *and the one point extensions in* **B** *arise from the corresponding one point extensions in* **A** *by applying the bicoreflector.*

(3) *If* **B** *is a bicoreflective (full and isomorphism-closed) subcategory of* **A** *which is closed under formation of products in* **A**, *then* **B** *fulfills* (CP3) *whenever* **A** *fulfills* (CP3).

By Theorems 3.1, 3.3, 3.7 and Lemmas 4.7, 4.8, we obtain

Theorem 4.9. *The category* L-**FYKC** *is a topological universe.*

Theorem 4.10. *Let* L *be a completely distributive lattice. Then the category* L-**FYKC** *is a strong topological universe.*

By Theorem 4.4 and Lemma 4.8, we can obtain the concrete forms of power objects and one point extensions in L-**FYKC** as follows:

Proposition 4.11. (1) *Let* $(X, \lim_X)$ *and* $(Y, \lim_Y)$ *be* L*-fuzzifying Kent convergence spaces. Then the power structure on* $C(X, Y)$ *is defined as follows:*

$$c\text{-}\lim\mathcal{F}(\varphi) = \bigwedge_{\mathcal{H}\in\mathcal{F}_L(X)}\bigwedge_{x\in X}\Big(\lim{}^X\mathcal{H}(x) \to \lim{}^Y ev^{\Rightarrow}((\mathcal{F}\wedge[\varphi])\times\mathcal{H})(\varphi(x))\Big).$$

(2) *Let* $(X, \lim)$ *be an* L*-fuzzifying Kent convergence space. Then the one point extension* $(X^*, \lim^*)$ *is defined by*

$$\lim{}^*\mathcal{F}(x) = \begin{cases} \top, & x = \infty_X \text{ or } (\mathcal{F}\wedge[x])(\{\infty_X\}) \neq \bot; \\ \lim i^{\Leftarrow}(\mathcal{F}\wedge[x])(x), & x \neq \infty_X \text{ and } (\mathcal{F}\wedge[x])(\{\infty_X\}) = \bot. \end{cases}$$

5. L-Fuzzifying Limit Spaces

In this section, we will adopt the method in [11] to introduce the concept of L-fuzzifying limit spaces and study its relationship with L-fuzzifying Kent convergence spaces. Besides, we will study the categorical properties of L-fuzzifying limit spaces and will show that the category of L-fuzzifying limit spaces is a topological universe.

Definition 5.1. An L-fuzzifying convergence structure $\lim : \mathcal{F}_L(X) \longrightarrow L^X$ satisfying the following condition

$$\text{(LFYL)} \quad \forall\mathcal{F}, \mathcal{G}\in\mathcal{F}_L(X), \forall x\in X, \lim(\mathcal{F}\wedge\mathcal{G})(x) = \lim\mathcal{F}(x)\wedge\lim\mathcal{G}(x)$$

is called an L-fuzzifying limit structure. The pair $(X, \lim)$ is called an L-fuzzifying limit space. Let L-**FYLC** denote the full subcategory of L-**FYKC** consisting of L-fuzzifying limit spaces.

The axiom (LFYL) implies the axiom (LFYK), hence every L-fuzzifying limit space is an L-fuzzifying Kent convergence space, i.e., the category L-**FYLC** is a full subcategory of L-**FYKC**. Moreover, we have the following result.

Theorem 5.2. *The category* L-**FYLC** *is a full and bireflective subcategory of* L-**FYKC**.

Proof. For an L-fuzzifying Kent convergence space $(X, \lim)$, we define

$$\lim{}^*\mathcal{F}(x) = \bigvee_{n\in\mathbb{N}}\Big\{\wedge_{i=1}^n \lim\mathcal{F}_i(x) : \mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n \in \mathcal{F}_L(X) \ s.t.\ \wedge_{i=1}^n \mathcal{F}_i(x) \leqslant \mathcal{F}\Big\}.$$

Then we claim that $id_X : (X, \lim) \longrightarrow (X, \lim^*)$ is the L-**FYLC**-bireflector.

For this it suffices to prove:

(1) $\lim^*$ is an L-fuzzifying limit structure on X.
(2) For $(Y, \lim_Y) \in |L$-**FYLC**$|$ and each mapping $\varphi : X \longrightarrow Y$, the continuity of $\varphi : (X, \lim) \longrightarrow (Y, \lim_Y)$ implies the continuity of $\varphi : (X, \lim^*) \longrightarrow (Y, \lim_Y)$.

(1) (LFY1) and (LFY2) are obvious. For (LFYL) we use the distributivity of finite meets over arbitrary joins. For $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L(X)$ and $x \in X$, we find

$$\begin{aligned}
& \lim^*\mathcal{F}(x) \wedge \lim^*\mathcal{G}(x) \\
= & \bigvee_{n\in\mathbb{N}} \left\{\wedge_{i=1}^n \lim \mathcal{F}_i(x) : \wedge_{i=1}^n \mathcal{F}_i \leqslant \mathcal{F}\right\} \wedge \bigvee_{m\in\mathbb{N}} \left\{\wedge_{k=1}^m \lim \mathcal{G}_k(x) : \wedge_{k=1}^m \mathcal{G}_k \leqslant \mathcal{G}\right\} \\
= & \bigvee_{n,m\in\mathbb{N}} \left\{\wedge_{i=1}^n \lim \mathcal{F}_i(x) \wedge \wedge_{k=1}^m \lim \mathcal{G}_k(x) : \wedge_{i=1}^n \mathcal{F}_i \leqslant \mathcal{F}, \wedge_{k=1}^m \mathcal{G}_k \leqslant \mathcal{G}\right\} \\
= & \bigvee_{n\in\mathbb{N}} \left\{\wedge_{i=1}^n \lim \mathcal{H}_i(x) : \wedge_{i=1}^n \mathcal{H}_i \leqslant \mathcal{F} \wedge \mathcal{G}\right\} \\
= & \lim^*(\mathcal{F} \wedge \mathcal{G})(x).
\end{aligned}$$

Thus, we obtain $\lim^*(\mathcal{F} \wedge \mathcal{G})(x) = \lim^*\mathcal{F}(x) \wedge \lim^*\mathcal{G}(x)$.

(2) Since $\varphi : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ is continuous, it follows that for each $\mathcal{F} \in \mathcal{F}_L(X)$ and each $x \in X$,

$$\lim\mathcal{F}(x) \leqslant \lim_Y \varphi^{\Rightarrow}(\mathcal{F})(\varphi(x)).$$

Then

$$\begin{aligned}
\lim^*\mathcal{F}(x) & = \bigvee_{n\in\mathbb{N}} \left\{\wedge_{i=1}^n \lim\mathcal{F}_i(x) : \wedge_{i=1}^n \mathcal{F}_i \leqslant \mathcal{F}\right\} \\
& \leqslant \bigvee_{n\in\mathbb{N}} \left\{\wedge_{i=1}^n \lim_Y \varphi^{\Rightarrow}(\mathcal{F}_i)(\varphi(x)) : \varphi^{\Rightarrow}(\wedge_{i=1}^n \mathcal{F}_i) \leqslant \varphi^{\Rightarrow}(\mathcal{F})\right\} \\
& = \bigvee_{n\in\mathbb{N}} \{\lim_Y(\wedge_{i=1}^n \varphi^{\Rightarrow}(\mathcal{F}_i))(\varphi(x)) : \varphi^{\Rightarrow}(\wedge_{i=1}^n \mathcal{F}_i) \leqslant \varphi^{\Rightarrow}(\mathcal{F})\} \\
& = \bigvee_{n\in\mathbb{N}} \left\{\lim_Y \varphi^{\Rightarrow}(\wedge_{i=1}^n \mathcal{F}_i)(\varphi(x)) : \varphi^{\Rightarrow}(\wedge_{i=1}^n \mathcal{F}_i) \leqslant \varphi^{\Rightarrow}(\mathcal{F})\right\} \\
& \leqslant \lim_Y \varphi^{\Rightarrow}(\mathcal{F})(\varphi(x)).
\end{aligned}$$

The continuity of $\varphi : (X, \lim^*) \longrightarrow (Y, \lim_Y)$ is proved, as desired. □

Since L-**FYKC** is a topological category and L-**FYLC** is a full and bireflective subcategory of L-**FYKC**, we obtain

Corollary 5.3. *The category L-**FYLC** is a topological category.*

Lemma 5.4. ([35]) *Let **A** be a topological category.*

(1) *If **B** is a bireflective (full and isomorphism-closed) subcategory of **A** which is closed under formation of power objects in **A**, then **B** fulfills* (CP1) *whenever **A** fulfills* (CP1).

(2) *If **B** is a bireflective (full and isomorphism-closed) subcategory of **A** which is closed under formation of one point extensions in **A**, then **B** fulfills* (CP2) *whenever **A** fulfills* (CP2).

Lemma 5.5. *The category L-**FYLC** is a full and isomorphism-closed subcategory of L-**FYKC**.*

Proof. The proof is similar to that of Lemma 4.7, so we omit it. □

Lemma 5.6. *The category L-**FYLC** is closed under formation of power objects in L-**FYKC**.*

Proof. Let $(X, \lim^X)$ and $(Y, \lim^Y)$ be L-fuzzifying limit spaces and let $(C(X,Y), c\text{-}\lim)$ be the power space in L-**FYKC**. Next we show $c\text{-}\lim$ is an L-fuzzifying limit structure on $C(X,Y)$. It suffices to show that $c\text{-}\lim$ satisfies (LFY1), (LFY2) and (LFYL). (LFY1) and (LFY2) are obvious.

(LFYL) Take each $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L(C(X,Y))$ and $\varphi \in C(X,Y)$. Then by Proposition 4.11, we have

$$\begin{aligned}
& c\text{-}\lim \mathcal{F}(\varphi) \wedge c\text{-}\lim \mathcal{G}(\varphi) \\
= & \bigwedge_{\mathcal{H}\in\mathcal{F}_L(X)} \bigwedge_{x\in X} \Big(\lim_X \mathcal{H}(x) \to \lim_Y ev^{\Rightarrow}((\mathcal{F} \wedge [\varphi]) \times \mathcal{H})(\varphi(x))\Big) \\
& \quad \wedge \bigwedge_{\mathcal{K}\in\mathcal{F}_L(X)} \bigwedge_{y\in X} \Big(\lim_X \mathcal{K}(x) \to \lim_Y ev^{\Rightarrow}((\mathcal{G} \wedge [\varphi]) \times \mathcal{K})(\varphi(y))\Big) \\
\leqslant & \bigwedge_{\mathcal{H}\in\mathcal{F}_L(X)} \bigwedge_{x\in X} \Big((\lim_X \mathcal{H}(x) \to \lim_Y ev^{\Rightarrow}((\mathcal{F} \wedge [\varphi]) \times \mathcal{H})(\varphi(x))) \\
& \quad \wedge (\lim_X \mathcal{H}(x) \to \lim_Y ev^{\Rightarrow}((\mathcal{G} \wedge [\varphi]) \times \mathcal{H})(\varphi(x)))\Big) \\
= & \bigwedge_{\mathcal{H}\in\mathcal{F}_L(X)} \bigwedge_{x\in X} \Big(\lim_X \mathcal{H}(x) \to \Big(\lim_Y ev^{\Rightarrow}((\mathcal{F} \wedge [\varphi]) \times \mathcal{H})(\varphi(x)) \\
& \qquad \wedge \lim_Y ev^{\Rightarrow}((\mathcal{G} \wedge [\varphi]) \times \mathcal{H})(\varphi(x))\Big)\Big) \\
= & \bigwedge_{\mathcal{H}\in\mathcal{F}_L(X)} \bigwedge_{x\in X} \Big(\lim_X \mathcal{H}(x) \to \lim_Y (ev^{\Rightarrow}((\mathcal{F} \wedge [\varphi]) \wedge (\mathcal{G} \wedge [\varphi])) \times \mathcal{H})(\varphi(x))\Big) \\
= & \bigwedge_{\mathcal{H}\in\mathcal{F}_L(X)} \bigwedge_{x\in X} \Big(\lim_X \mathcal{H}(x) \to \lim_Y (ev^{\Rightarrow}((\mathcal{F} \wedge \mathcal{G} \wedge [\varphi]) \times \mathcal{H}))(\varphi(x))\Big) \\
= & c\text{-}\lim(\mathcal{F} \wedge \mathcal{G})(\varphi),
\end{aligned}$$

as desired. □

Since L-**FYKC** is Cartesian closed, it follows from Lemmas 5.4, 5.5 and 5.6 that

Theorem 5.7. *The category L-**FYLC** is Cartesian closed.*

Lemma 5.8. *Let $\mathcal{F}$, $\mathcal{G} \in \mathcal{F}_L(Y)$ and let $\varphi : X \longrightarrow Y$ be a mapping. Then $\varphi^{\Leftarrow}(\mathcal{F} \wedge \mathcal{G}) = \varphi^{\Leftarrow}(\mathcal{F}) \wedge \varphi^{\Leftarrow}(\mathcal{G})$ whenever $\varphi^{\Leftarrow}(\mathcal{F})$ and $\varphi^{\Leftarrow}(\mathcal{G})$ exist.*

Proof. By Proposition 2.3, it is easy to prove that $\varphi^{\Leftarrow}(\mathcal{F} \wedge \mathcal{G})$ exists whenever $\varphi^{\Leftarrow}(\mathcal{F})$ and $\varphi^{\Leftarrow}(\mathcal{G})$ exist. We only need to show that $\varphi^{\Leftarrow}(\mathcal{F} \wedge \mathcal{G}) = \varphi^{\Leftarrow}(\mathcal{F}) \wedge \varphi^{\Leftarrow}(\mathcal{G})$. The inequality $\varphi^{\Leftarrow}(\mathcal{F} \wedge \mathcal{G}) \leqslant \varphi^{\Leftarrow}(\mathcal{F}) \wedge \varphi^{\Leftarrow}(\mathcal{G})$ holds obviously. Conversely, for each $A \in 2^X$, we have

$$\begin{aligned}
(\varphi^{\Leftarrow}(\mathcal{F}) \wedge \varphi^{\Leftarrow}(\mathcal{G}))(A) &= \bigvee_{\varphi^{\leftarrow}(B)\subseteq A} \mathcal{F}(B) \wedge \bigvee_{\varphi^{\leftarrow}(C)\subseteq A} \mathcal{G}(C) \\
&= \bigvee_{\varphi^{\leftarrow}(B)\subseteq A} \bigvee_{\varphi^{\leftarrow}(C)\subseteq A} (\mathcal{F}(B) \wedge \mathcal{G}(C)) \\
&= \bigvee_{\varphi^{\leftarrow}(B\cup C)\subseteq A} (\mathcal{F}(B) \wedge \mathcal{G}(C)) \\
&\leqslant \bigvee_{\varphi^{\leftarrow}(D)\subseteq A} (\mathcal{F}(D) \wedge \mathcal{G}(D)) = \varphi^{\Leftarrow}(\mathcal{F} \wedge \mathcal{G})(A).
\end{aligned}$$

This proves $\varphi^{\Leftarrow}(\mathcal{F} \wedge \mathcal{G}) = \varphi^{\Leftarrow}(\mathcal{F}) \wedge \varphi^{\Leftarrow}(\mathcal{G})$. □

Lemma 5.9. *The category L-**FYLC** is closed under formation of one point extensions in L-**FYKC**.*

Proof. Let $(X, \lim)$ be an L-fuzzifying limit space and $(X^*, \lim^*)$ be the one point extension of $(X, \lim)$ in L-**FYKC**. By Proposition 4.11, we have

$$\lim{}^*\mathcal{F}(x) = \begin{cases} \top, & x = \infty_X \text{ or } (\mathcal{F} \wedge [x])(\{\infty_X\}) \neq \bot; \\ \lim i^{\Leftarrow}(\mathcal{F} \wedge [x])(x), & x \neq \infty_X \text{ and } (\mathcal{F} \wedge [x])(\{\infty_X\}) = \bot. \end{cases}$$

Next we show that $\lim^*$ is an L-fuzzifying limit structure on X^*. It suffices to prove that $\lim^*$ satisfies (LFYL). If $x = \infty_X$, then $\lim^*(\mathcal{F} \wedge \mathcal{G})(x) = \top = \lim^*\mathcal{F}(x) \wedge \lim^*\mathcal{G}(x)$. If $x \neq \infty_X$, then $(\mathcal{F} \wedge [x])(\{\infty_X\}) = (\mathcal{G} \wedge [x])(\{\infty_X\}) = \bot$. By Proposition 2.3, we know $i^{\Leftarrow}(\mathcal{F} \wedge [x])$ and $i^{\Leftarrow}(\mathcal{G} \wedge [x])$ exist. Further, by Lemma 5.8, we obtain $i^{\Leftarrow}(\mathcal{F} \wedge \mathcal{G} \wedge [x]) = i^{\Leftarrow}(\mathcal{F} \wedge [x]) \wedge i^{\Leftarrow}(\mathcal{G} \wedge [x])$. Hence, it follows that

$$\begin{aligned} \lim{}^*\mathcal{F}(x) \wedge \lim{}^*\mathcal{G}(x) &= \lim i^{\Leftarrow}(\mathcal{F} \wedge [x])(x) \wedge \lim i^{\Leftarrow}(\mathcal{G} \wedge [x])(x) \\ &= \lim(i^{\Leftarrow}(\mathcal{F} \wedge [x]) \wedge i^{\Leftarrow}(\mathcal{G} \wedge [x]))(x) \\ &= \lim i^{\Leftarrow}(\mathcal{F} \wedge \mathcal{G} \wedge [x]))(x) \\ &= \lim{}^*(\mathcal{F} \wedge \mathcal{G})(x). \end{aligned}$$

This proves $\lim^*$ is an L-fuzzifying limit structure on X^*. □

Since L-**FYKC** is extensional, it follows from Lemmas 5.4, 5.5 and 5.9 that

Theorem 5.10. *The category L-**FYLC** is extensional.*

Finally, we get the main result in this section.

Theorem 5.11. *The category L-**FYLC** is a topological universe.*

6. Conclusions

In this paper, we mainly demonstrated the categorical properties of several categories. Concretely, we showed the categories of L-fuzzifying convergence spaces and L-fuzzifying Kent convergence spaces possess Cartesian-closedness, extensionality and the productivity of quotient mappings. Hence they both are strong topological universe in the sense of [35]. Further, we showed that the category of L-fuzzifying limit spaces is Cartesian closed and extensional, whence it is a topological universe. Obviously, there is a problem that has not been solved in this paper. That is, we are not sure if the category of L-fuzzifying limit spaces satisfies (CP3). In the future, we will consider if the quotient mappings in the category of L-fuzzifying limit spaces is productive.

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References

[1] J.M. Fang, Stratified L-ordered convergence structures, Fuzzy Sets Syst. 161 (2010) 2130–2149.
[2] J.M. Fang, Relationships between L-ordered convergence structures and strong L-topologies, Fuzzy Sets Syst. 161 (2010) 2923–2944.
[3] J.M. Fang, Lattice-valued semiuniform convergence spaces, Fuzzy Sets Syst. 195 (2012) 33–57.
[4] J.M. Fang, Stratified L-ordered quasiuniform limit spaces, Fuzzy Sets Syst. 227 (2013) 51–73.
[5] J.M. Fang, Lattice-valued preuniform convergence spaces, Fuzzy Sets Syst. 251 (2014) 52–70.
[6] P.V. Flores, R.N. Mohapatra, G. Richardson, Lattice-valued spaces: Fuzzy convergence, Fuzzy Sets Syst. 157 (2006) 2706–2714.
[7] M. Güloğlu, D. Coker, Convergence in I-fuzzy topological spaces, Fuzzy Sets Syst. 151 (2005) 615–623.
[8] U. Höhle, A.P. Šostak, Axiomatic foundations of fixed-basis fuzzy topology, in: U. Höhle, S.E. Rodabaugh (Eds.), Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory, Handbook Series, vol.3, Kluwer Academic Publishers, Boston, Dordrecht, London, 1999, pp. 123–173.

[9] U. Höhle, Characterization of L-topologies by L-valued neighborhoods, chapter 5, in: Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory, The Handbooks of Fuzzy Sets Series, (U. Höhle, S. E. Rodabaugh, Eds.), Vol. 3, Kluwer Academic Publishers, Boston, Dordrecht, London, 1999, pp.389–432.
[10] G. Jäger, A category of L-fuzzy convergence spaces, Quaest. Math. 24 (2001) 501–517.
[11] G. Jäger, Subcategories of lattice-valued convergence spaces, Fuzzy Sets Syst. 156 (2005) 1–24.
[12] G. Jäger, Pretopological and topological lattice-valued convergence spaces, Fuzzy Sets Syst. 158 (2007) 424–435.
[13] G. Jäger, Stratified LMN-convergence tower spaces, Fuzzy Sets Syst. 282 (2016) 62–73.
[14] G. Jäger, Connectedness and local connectedness for lattice-valued convergence spaces, Fuzzy Sets Syst. 300 (2016) 134–146.
[15] L.Q. Li, Q. Jin, On adjunctions between **Lim**, **S***L***-Top**, and **S***L***-Lim**, Fuzzy Sets Syst. 182 (2011) 66–78.
[16] L.Q. Li, Q. Jin, On stratified L-convergence spaces: Pretopological axioms and diagonal axioms, Fuzzy Sets Syst. 204 (2012) 40–52.
[17] L.Q. Li, Q. Jin, p-Topologicalness and p-regularity for lattice-valued convergence spaces, Fuzzy Sets Syst. 238 (2014) 26–45.
[18] L.Q. Li, Q. Jin, K. Hu, On stratified L-convergence spaces: Fischer's diagonal axiom, Fuzzy Sets Syst. 267 (2015) 31–40.
[19] L.Q. Li, Q. Jin, G.W. Meng, et al., The lower and upper ptopological (p-regular) modifications for lattice-valued convergence spaces, Fuzzy Sets Syst. 282 (2016) 47–61.
[20] R. Lowen, Convergence in fuzzy topological spaces, Gen. Top. Appl. 10 (1979) 147–160.
[21] K.C. Min, Fuzzy limit spaces , Fuzzy Sets Syst. 32 (1989) 343–357.
[22] B. Pang, J.M. Fang, L-fuzzy Q-convergence structures, Fuzzy Sets Syst. 182 (2011) 53–65.
[23] B. Pang, Further study on L-fuzzy Q-convergence structures, Iran. J. Fuzzy Syst. 10:5 (2013) 149–166.
[24] B. Pang, On (L, M)-fuzzy convergence spaces, Fuzzy Sets Syst. 238 (2014) 46–70.
[25] B. Pang, Enriched (L, M)-fuzzy convergence spaces, J. Intell. Fuzzy Syst. 27 (2014) 93–103.
[26] B. Pang, The category of stratified L-filter spaces, Fuzzy Sets Syst. 247 (2014) 108–126.
[27] B. Pang, Stratified L-ordered filter spaces, Quaest. Math. 40 (2017) 661–678.
[28] B. Pang, Degrees of separation properties in stratified L-generalized convergence spaces using residual implication, Filomat 31 (2017) 6293–6305.
[29] B. Pang, F.-G. Shi, Degrees of compactness of (L, M)-fuzzy convergence spaces and its applications, Fuzzy Sets Syst. 251 (2014) 1–22.
[30] B. Pang, F.-G. Shi, Subcategories of the category of L-convex spaces, Fuzzy Sets Syst. 313 (2017) 61–74.
[31] B. Pang, F.-G. Shi, Strong inclusion orders between L-subsets and its applications in L-convex spaces, Quaest. Math. 2018, DOI: 10.2989/16073606.2018.1436613.
[32] B. Pang, Z.-Y. Xiu, Lattice-valued interval operators and its induced lattice-valued convex structures, IEEE T. Fuzzy Syst. 26:3 (2018) 1525-1534.
[33] B. Pang, Z.-Y. Xiu, Stratified L-prefilter convergence structures in stratified L-topological spaces, Soft Comput. 2018, DOI:10.1007/s00500-018-3040-1.
[34] B. Pang, Y. Zhao, Characterizations of L-convex spaces, Iran. J. Fuzzy Syst. 13:4 (2016) 51–61.
[35] G. Preuss, Foundations of Topology–An Approach to Convenient Topology, Kluwer Academic Publisher, Dordrecht, Boston, London, 2002.
[36] Z.-Y. Xiu, B. Pang, M-fuzzifying cotopological spaces and M-fuzzifying convex spaces as M-fuzzifying closure spaces, J. Intell. Fuzzy Syst. 33 (2017) 613–620.
[37] Z.-Y. Xiu, B. Pang, Base axioms and subbase axioms in M-fuzzifying convex spaces, Iran. J. Fuzzy Syst. 15:2 (2018) 75-87.
[38] L. Xu, Characterizations of fuzzifying topologies by some limit structures, Fuzzy Sets Syst. 123 (2001) 169–176.
[39] X.-F. Yang, S.-G. Li, Completion of stratified (L, M)-filter tower spaces, Fuzzy Sets Syst. 210 (2013) 22–38.
[40] W. Yao, On many-valued stratified L-fuzzy convergence spaces, Fuzzy Sets Syst. 159 (2008) 2503–2519.
[41] W. Yao, On L-fuzzifying convergence spaces, Iran. J. Fuzzy Syst. 6:1 (2009) 63–80.