Filomat 32:11 (2018), 3957–3991
https://doi.org/10.2298/FIL1811957L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

New Existence Results for Solutions of BVPs for Higher Order IFDEs Involving Riemann-Liouville Type Hadamard Fractional Derivatives

Yuji Liu[a], Xiaohui Yang[b]

[a] *Department of Mathematics, Guangdong University of Finance and Economics, Guangzhou 510320, P.R. China*
[b] *Department of Computer, Guangdong Police College, Guangzhou, 510230, P.R. China*

Abstract. In this article, we present a method for converting boundary value problems for impulsive fractional differential systems involving the Riemann-Liouville type Hadamard fractional derivatives to integral systems. The existence results for solutions of this kind of boundary value problems are established. Our analysis relies on the well known fixed point theorem. Some comments on recent published papers are made at the end of the paper.

1. Introduction

Fractional differential equations have many applications in modeling of physical, industrial and chemical processes. In its turn, mathematical aspects of fractional differential equations and methods of their solutions were discussed by many authors, see the text books [3, 11].

It has been noticed that most of the work on the topic is based on Riemann-Liouville and Caputo type fractional differential equations. Besides Riemann-Liouville and Caputo derivatives, there is another kind of fractional derivatives in the literature due to Hadamard [11], which is known as Hadamard derivative and differs from the preceding ones in the sense that its definition involves logarithmic function of arbitrary exponent. It is imperative to note that the study of Hadamard type initial and boundary value problems is at its initial phase and needs further attention [20].

In [18, 23], authors established the existence of solutions for a class of nonlinear impulsive Hadamard fractional differential equations with initial condition of the form

$$\begin{cases} {}^{rh}D_{1^+}^{\alpha}x(t) = f(t, x(t)), t \in (1, e] \setminus \{t_1, t_2, \cdots, t_m\}, \\ \Delta^* x(t_i) = {}^{h}J_{1^+}^{1-\alpha}x(t_i^+) - {}^{h}J_{1^+}^{1-\alpha}x(t_i^-) = p_i, i = 1, 2, \cdots, m, \\ {}^{h}J_{1^+}^{1-\alpha}x(1) = u_0, \end{cases} \tag{1.1}$$

2010 *Mathematics Subject Classification*. 34K37, 34K45, 34B37; Secondary 34B15, 34B10, 92D25, 34A37, 34K15

Keywords. impulsive multi-term fractional differential equation, Riemann-Liouville type Hadamard fractional derivatives, Picard iterative method, boundary value problem, fixed point theorem

Received: 30 October 2017; Accepted: 12 April 2018

Communicated by Jelena Manojlović

Research supported by Natural Science Foundation of Guangdong province (No. S2011010001900) and Natural science research project for colleges and universities of Guangdong Province (No: 2014KTSCX126), the Foundation for High-level talents in Guangdong Higher Education Project, the foundations of Guangzhou science and technology project 201707010425 and 201804010350

Email address: yuji_liu@126.com (Xiaohui Yang)

where ${}^{rh}D^{\alpha}_{1^+}$ is the left-side Riemann-Liouville type Hadamard derivative of order $\alpha \in (0,1)$ with the starting point 1 and ${}^{h}J^{1-\alpha}_{1^+}$ denotes left-side Hadamard fractional integral of order $1-\alpha$, $1 = t_0 < t_1 < \cdots < t_m < t_{m+1} = e$, $u_0, p_i \in \mathbf{R}(i = 1, 2, \cdots, m)$, $f : [1, e] \times \mathbf{R} \mapsto \mathbf{R}$ is a continuous function.

In [22], Zhang and Wang Studied the existence and finite-time stability for the following impulsive fractional differential equation

$$\begin{cases} {}^{rh}D^{\alpha}_{1^+}x(t) = f(t, x(t), \max_{\xi\in[\beta t,t]} x(\xi)), t \in (1, e] \setminus \{t_1, t_2, \cdots, t_m\}, \\ \Delta^* x(t_i) = {}^{h}J^{1-\alpha}_{1^+}x(t_i^+) - {}^{h}J^{1-\alpha}_{1^+}x(t_i^-) = a_i x(t_i) + b_i, i = 1, 2, \cdots, m, \\ {}^{h}J^{1-\alpha}_{1^+}x(1) = u_0, \end{cases}$$

with the initial condition $x(t) = \Phi(t), t \in [\beta, 1]$, where ${}^{rh}D^{\alpha}_{1^+}$ is the left-side Riemann-Liouville type Hadamard derivative of order $\alpha \in (0,1)$ with the starting point 1 and ${}^{h}J^{1-\alpha}_{1^+}$ denotes left-side Hadamard fractional integral of order $1-\alpha$, $\beta \in (0,1)$, $1 = t_0 < t_1 < \cdots < t_m < t_{m+1} = e$, $u_0, p_i \in \mathbf{R}(i = 1, 2, \cdots, m)$, $\Phi : [\beta, 1] \to \mathbf{R}$ and $f : [1, e] \times \mathbf{R} \mapsto \mathbf{R}$ are continuous functions.

In [21], Zhang studied the following impulsive system with Hadamard fractional derivative:

$$\begin{cases} {}^{rh}D^{\alpha}_{a^+}x(t) = f(t, x(t)), t \in (a, T], t \neq t_k, \bar{t}_l, k = 1, 2, \cdots, m, l = 1, 2, \cdots, n, \\ \Delta {}^{h}J^{2-\alpha}_{a^+}x(t_i) = {}^{h}J^{2-\alpha}_{a^+}x(t_i^+) - {}^{h}J^{2-\alpha}_{a^+}x(t_i^-) = \Delta_i(x(t_i)), i = 1, 2, \cdots, m, \\ \Delta {}^{rh}D^{\alpha-1}_{a^+}x(\bar{t}_l) = {}^{rh}D^{\alpha-1}_{a^+}x(\bar{t}_l^+) - {}^{rh}D^{\alpha-1}_{a^+}x(\bar{t}_l^-) = \overline{\Delta}_l(x(\bar{t}_l)), l = 1, 2, \cdots, n, \\ {}^{h}J^{2-\alpha}_{a^+}x(a) = x_2, \ {}^{rh}D^{\alpha-1}_{a^+}x(a) = x_1 \end{cases} \tag{1.2}$$

and its special case:

$$\begin{cases} {}^{rh}D^{\alpha}_{a^+}x(t) = f(t, x(t)), t \in (a, T], t \neq t_k, k = 1, 2, \cdots, m, \\ \Delta {}^{h}J^{2-\alpha}_{a^+}x(t_i) = {}^{h}J^{2-\alpha}_{a^+}x(t_i^+) - {}^{h}J^{2-\alpha}_{a^+}x(t_i^-) = \Delta_i(x(t_i)), i = 1, 2, \cdots, m, \\ \Delta {}^{rh}D^{\alpha-1}_{a^+}x(t_i) = {}^{rh}D^{\alpha-1}_{a^+}x(\bar{t}_i^+) - {}^{rh}D^{\alpha-1}_{a^+}x(\bar{t}_i^-) = \overline{\Delta}_l(x(t_i)), i = 1, 2, \cdots, m, \\ {}^{h}J^{2-\alpha}_{a^+}x(a) = x_2, \ {}^{rh}D^{\alpha-1}_{a^+}x(a) = x_1 \end{cases} \tag{1.3}$$

where ${}^{rh}D^{\alpha}_{1^+}$ is the left-side Riemann-Liouville type Hadamard derivative of order $\alpha \in (1,2)$ with the starting point $a > 0$ and ${}^{h}J^{2-\alpha}_{1^+}$ denotes left-side Hadamard fractional integral of order $2-\alpha$, $a = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $a = \bar{t}_0 < \bar{t}_1 < \cdots < \bar{t}_n < \bar{t}_{n+1} = T$, $x_1, x_2 \in \mathbf{R}$, $f : [a, T] \times \mathbf{R} \mapsto \mathbf{R}$ is a continuous function.

In [6], Liu studied the following boundary value problem for impulsive higher order fractional differential equation involving the Riemann-Liouville type Hadamard fractional derivatives

$$\begin{cases} {}^{rh}D^{\alpha}_{1^+}x(t) - \lambda x(t) = p(t) f(t, x(t)), a.e., t \in (t_k, t_{k+1}], k \in \mathbf{N}_0^m, \\ \Delta {}^{h}I^{n-\alpha}_{1^+}x(t_k) = I_n(t_k, x(t_k)), k \in \mathbf{N}_1^m, \\ \Delta {}^{rh}D^{\alpha-\nu}_{1^+}x(t_k) = I_\nu(t_k, x(t_k)), k \in \mathbf{N}_1^m, \nu \in \mathbf{N}_1^{n-1}, \\ {}^{rh}D^{n-\nu}_{1^+}x(1) = 0, \ x(e) = 0, \ \nu \in \mathbf{N}_1^{n-1}, \end{cases} \tag{1.4}$$

where $\alpha \in (n-1, n)$, $\lambda \in \mathbf{R}$, ${}^{h}I^{*}_{1^+}$ and ${}^{rh}D^{*}_{1^+}$ are the Hadamard fractional integral and the Riemann-Liouville Hadamard fractional derivative respectively, m is a positive integer, denote $\mathbf{N}_a^b = \{a, a+1, a+2, \cdots, b\}$ for integers a, b with $a < b$, $1 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = e$, $p \in C^0(1, e)$ and there exist constants $\sigma > -1, \tau \in (\max\{-\alpha, -n-\sigma\}, 0]$ such that $|p(t)| \leq (\ln t)^\sigma (1-\ln t)^\tau$ on $(1, e)$, $f : (1, e) \times \mathbf{R} \mapsto \mathbf{R}$ is a Carathéodory function, $I_i : \{t_i\} \times \mathbf{R} \mapsto \mathbf{R}$ are discrete Carathéodory functions.

This paper is motivated by [6, 9, 18, 21, 23], we consider the following boundary value problem for impulsive Riemann-Liouville Hadamard fractional differential equation

$$\begin{cases} {}^{rh}D_{1^+}^{\alpha} x(t) - \lambda\, {}^{rh}D_{1^+}^{\beta} x(t) = h(t) f(t, x(t)), a.e., t \in (t_k, t_{k+1}], k \in \mathbf{N}_0^m, \\ \Delta {}^{h}I_{1^+}^{n-\alpha} x(t_k) = I_n(t_k, x(t_k)), k \in \mathbf{N}_1^m, \\ \Delta {}^{rh}D_{1^+}^{\alpha-\nu} x(t_k) = I_\nu(t_k, x(t_k)), k \in \mathbf{N}_1^m, \nu \in \mathbf{N}_{p+1}^{n-1}, \\ \Delta[{}^{rh}D_{1^+}^{\alpha-p} x - A\, {}^{h}I_{1^+}^{p-\beta} x](t_k) = I_p(t_k, x(t_k)), k \in \mathbf{N}_1^m, \\ \Delta[{}^{rh}D_{1^+}^{\alpha-\nu} x - A\, {}^{h}I_{1^+}^{\beta-\nu} x](t_k) = I_\nu(t_k, x(t_k)), k \in \mathbf{N}_1^m, \nu \in \mathbf{N}_1^{p-1}, \\ {}^{rh}D_{1^+}^{\alpha-\nu} x(1) = 0, \nu \in \mathbf{N}_{p+1}^{n-1}, \\ [{}^{rh}D_{1^+}^{\alpha-p} x - A\, {}^{h}I_{1^+}^{p-\beta} x](1) = 0, \\ [{}^{rh}D_{1^+}^{\alpha-\nu} x - A\, {}^{h}I_{1^+}^{\beta-\nu} x](1) = 0, \nu \in \mathbf{N}_1^{p-1},\ x(e) = 0, \end{cases} \tag{1.5}$$

where n, p are positive integers and $\alpha \in (n-1, n), \beta \in (p-1, p), \beta < \alpha, \lambda \in \mathbf{R}$, ${}^{h}I_{1^+}^{*}$ and ${}^{rh}D_{1^+}^{*}$ are the Hadamard fractional integral and the Riemann-Liouville Hadamard fractional derivative respectively, m is a positive integer, denote $\mathbf{N}_a^b = \{a, a+1, a+2, \cdots, b\}$ for integers a, b with $a < b$, $1 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = e$, $h \in C^0(1, e)$ and there exist constants $\sigma > -1, \tau \in (\max\{-\alpha, -n-\sigma\}, 0]$ such that $|h(t)| \leq (\ln t)^\sigma (1-\ln t)^\tau$ on $(1, e)$, $f : (1, e) \times \mathbf{R} \mapsto \mathbf{R}$ is a Carathéodory function, $I_i : \{t_i\} \times \mathbf{R} \mapsto \mathbf{R}$ are discrete Carathéodory functions. .

A function $u : (1, e] \mapsto \mathbf{R}$ is called a solution of BVP(1.5) if

$$u|_{(t_k, t_{k+1}]} \in C^0(t_k, t_{k+1}], k \in \mathbf{N}_0^m, \lim_{t \to t_k^+} (\ln t - \ln t_k)^{n-\alpha} u(t) \text{ are finite}, k \in \mathbf{N}_0^m$$

and all equations in (1.5) are satisfied. The first purpose of this paper is to give continuous general solutions of the following Riemann-Liouville Hadamard fractional differential equation

$${}^{rh}D_{1^+}^{\alpha} x(t) - \lambda\, {}^{rh}D_{1^+}^{\beta} x(t) = p(t), a.e., t \in (1, e].$$

The second purpose of this paper is to give piecewise continuous general solutions of the following impulsive Riemann-Liouville Hadamard fractional differential equation

$${}^{rh}D_{1^+}^{\alpha} x(t) - \lambda\, {}^{rh}D_{1^+}^{\beta} x(t) = p(t), a.e., t \in (t_k, t_{k+1}], k \in \mathbf{N}_0^m.$$

The third purpose of this paper is to transform BVP(1.5) to an equivalent integral equation and to establish existence results for solutions of BVP(1.5).

The remainder of the paper is organized as follows: In Section 2, we present some preliminary results. In Section 3, the existence results for solutions of BVP(1.5). Some examples are given in the final section.

2. Preliminary results

In this section, we present some necessary definitions from the fractional calculus theory which can be found in the literatures [3, 11]. Let $a < b$. Denote $L^1(a, b)$ the set of all integrable functions on (a, b), $C^0(a, b]$ the set of all continuous functions on $(a, b]$. For $\phi \in L^1(a, b)$, denote $\|\phi\|_1 = \int_a^b |\phi(s)| ds$. For $\phi \in C^0[a, b]$, denote $\|\phi\|_0 = \max\limits_{t\in[a,b]} |\phi(t)|$. Let the Gamma and Beta functions $\Gamma(\alpha)$, $\mathbf{B}(p,q)$ and the Mitag-Leffler function $E_{\alpha,\delta}(x)$ be defined by

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1}e^{-x}dx, \ \ \mathbf{B}(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx, \ \ \mathbf{E}_{\alpha,\delta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha+\delta)}, \alpha, p, q, \delta > 0.$$

Definition 2.1[11]. The left Hadamard fractional integral of order $\alpha > 0$ of a function $h : (1,e] \mapsto \mathbf{R}$ is given by ${}^{h}I_{1^+}^{\alpha}h(t) = \frac{1}{\Gamma(\alpha)} \int_1^t (\ln t - \ln s)^{\alpha-1}h(s)\frac{ds}{s}$, $t > 1$ provided that the right-hand side exists.

Definition 2.2[11]. The left Riemann-Liouville Hadamard fractional derivative of order $\alpha > 0$ of a function $h : (1,e] \mapsto \mathbf{R}$ is given by ${}^{rh}D_{1^+}^{\alpha}h(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_1^t \frac{h(s)}{(t-s)^{\alpha-n+1}}\frac{ds}{s}$, $t > 1$ where $n-1 < \alpha < n$, provided that the right-hand side exists.

Remark 2.1. It is known that if the traditional derivative (integer order derivative) $x^{(n)}(t)$ exists, $x(t), x'(t), \cdots, x^{(n-1)}(t)$ are continuous. Motivated by this fact, it follows certainly for $a > 0$ that if ${}^{rh}D_{a^+}^{\alpha}h(t)$ exists ($\alpha \in (n-1,n)$, $x \in C(a,b)$, $\lim\limits_{t\to a^+}(\ln t - \ln a)^{n-\alpha}x(t)$ exists and ${}^{rh}D_{a^+}^{\alpha-(n-1)}x$, ${}^{rh}D_{a^+}^{\alpha-(n-2)}x$, $\cdots$, ${}^{rh}D_{a^+}^{\alpha-1}x$ are continuous on $[a,t]$.

Definition 2.3. $h : (1,e) \times \mathbf{R} \mapsto \mathbf{R}$ is called a Carathéodory function if

(i) $t \mapsto h\left(t, (\ln t - \ln t_i)^{\alpha-n}x\right)$ is integrable on (t_i, t_{i+1}) for every $x \in \mathbf{R}$,

(ii) $x \mapsto h\left(t, (\ln t - \ln t_i)^{\alpha-n}x\right)$ is continuous on $\mathbf{R}$ for each $t \in (t_i, t_{i+1}]$($i \in \mathbf{N}_0^m$),

(iii) for each $r > 0$, there exists $M_r > 0$ such that $|x| \le r$ implies that

$$|h\left(t, (\ln t - \ln t_i)^{\alpha-n}x\right)| \le M_r, t \in (t_i, t_{i+1}), i \in \mathbf{N}_0^m.$$

Definition 2.4. $I : \{t_i : i \in \mathbf{N}_1^m\} \times \mathbf{R} \mapsto \mathbf{R}$ is a discrete Carathéodory function if

(i)$x \mapsto I\left(t_i, (\ln t_i - \ln t_{i-1})^{\alpha-n}x\right)$ is continuous on $\mathbf{R}$ for each $i \in \mathbf{N}_1^m$,

(ii) for each $r > 0$, there exists $M_{I,r} > 0$ such that $|x| \le r$ implies that

$$|I\left(t_i, (\ln t_i - \ln t_{i-1})^{\alpha-n}x\right)| \le M_{I,r}, i \in \mathbf{N}_1^m.$$

Let n be a positive integer, $\alpha \in (n-1,n)$, $1 = t_0 < t_1 < \cdots < t_m < t_{m+1} = e$. Denote

$$PC_{n-\alpha}(1,e] =$$

$$\left\{x : (1,e] \mapsto \mathbf{R} : x|_{(t_k,t_{k+1}]} \in C^0(t_k, t_{k+1}],\ \lim_{t\to t_k^+}(\ln t - \ln t_k)^{n-\alpha}x(t) \text{ are finite}, i \in \mathbf{N}_0^m\right\}.$$

Define

$$\|x\| = \max\left\{\sup_{t\in(t_k,t_{k+1}]} (\ln t - \ln t_k)^{n-\alpha}|x(t)| : k \in \mathbf{N}_0^m\right\}, x \in PC_{n-\alpha}(1,e].$$

Then $PC_{n-\alpha}(1,e]$ is a Banach space.

We firstly, by the Picard iterative method, give an exact expression of solutions of the following linear fractional differential equation

$$^{rh}D_{1^+}^{\alpha}x(t) - A\,^{rh}D_{1^+}^{\beta}x(t) = h(t), a.e., t \in (1,e], \tag{2.1}$$

where n,p are two positive integers, $\alpha \in (n-1,n)$, $\beta \in (0,\alpha)$ with $p-1 < \beta < p$, $A \in \mathbf{R}$, $h \in C(1,e)$ and there exist constants $\sigma > -1, \tau \in \max\{-\alpha+n-1, -1-\sigma\}, 0]$ such that $|h(t)| \le (\ln t)^{\sigma}(1-\ln t)^{\tau}$ for all $t \in (1,e)$, n,p are positive integers. We give the exact expression of continuous general solutions of (2.1).

We secondly consider the impulsive linear fractional differential equation

$$^{rh}D_{1^+}^{\alpha}x(t) - A\,^{rh}D_{1^+}^{\beta}x(t) = h(t), a.e., t \in (t_i, t_{i+1}], i \in \mathbf{N}_0^m, \tag{2.2}$$

where $1 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = e$, n, p are two positive integers, $\alpha \in (n-1, n)$, $\beta \in (0, \alpha)$ with $\beta \in (p-1, p)$, $A \in \mathbf{R}$, $h \in C(1, e)$ and there exist constants $\sigma > -1, \tau \in \max\{-\alpha + n - 1, -1 - \sigma\}, 0]$ such that $|h(t)| \le (\ln t)^{\sigma}(1 - \ln t)^{\tau}$ for all $t \in (1, e)$, n, p are positive integers. We given the exact expression of piecewise general continuous solutions of (2.2).

We note that if a function $x : (1, e] \mapsto \mathbf{R}$ is a continuous solution of (2.1), we can get that

$$[{}^{rh}D_{1^+}^{\alpha-i}x - A\,{}^{rh}D_{1^+}^{\beta-i}x] \in C[1,e], i \in \mathbf{N}_1^{p-1}, \ \ [{}^{rh}D_{1^+}^{\alpha-p} - A\,{}^{h}I_{1^+}^{p-\beta}x] \in C[1,e],$$

$$ {}^{rh}D_{1^+}^{\alpha-i}x \in C[1,e], \ i \in \mathbf{N}_{p+1}^{n-1}, \ {}^{h}I_{1^+}^{n-\alpha}x \in C[1,e]$$

and x satisfies (2.1).

We also note that if a function $x : (1, e] \mapsto \mathbf{R}$ is a piecewise continuous solution of (2.2), we can get that

$$[{}^{rh}D_{1^+}^{\alpha-i}x - A\,{}^{rh}D_{1^+}^{\beta-i}x]|_{(t_j,t_{j+1}]} \in C(t_j, t_{j+1}], i \in \mathbf{N}_1^{p-1}, j \in \mathbf{N}_0^m,$$

$$\lim_{t \to t_j^+}[{}^{rh}D_{1^+}^{\alpha-i}x - A\,{}^{rh}D_{1^+}^{\beta-i}x](t) \text{ is finite}, i \in \mathbf{N}_1^{P-1}, j \in \mathbf{N}_0^m,$$

$$[{}^{rh}D_{1^+}^{\alpha-p} - A\,{}^{h}I_{1^+}^{p-\beta}x]_{(t_j,t_{j+1}]} \in C(t_j, t_{j+1}], j \in \mathbf{N}_0^m,$$

$$\lim_{t \to t_j^+}[{}^{rh}D_{1^+}^{\alpha-p} - A\,{}^{h}I_{1^+}^{p-\beta}x](t) \text{ is finite}, \ j \in \mathbf{N}_0^m,$$

$${}^{rh}D_{1^+}^{\alpha-i}x_{(t_j,t_{j+1}]} \in C(t_j, t_{j+1}], \ i \in \mathbf{N}_{p+1}^{n-1}, j \in \mathbf{N}_0^m,$$

$$\lim_{t \to t_j^+} {}^{rh}D_{1^+}^{\alpha-i}x(t) \text{ is finite}, \ j \in \mathbf{N}_0^m, i \in \mathbf{N}_{p+1}^{n-1},$$

$${}^{h}I_{1^+}^{n-\alpha}x_{(t_j,t_{j+1}]} \in C(t_j, t_{j+1}], \ \lim_{t \to t_j^+} {}^{h}I_{1^+}^{n-\alpha}x(t) \text{ is finite }, j \in \mathbf{N}_0^m$$

and x satisfies (2.2).

We firstly give an exact expression of solutions of (2.1) satisfying the following initial conditions The initial conditions are as follows:

$$\begin{aligned} &[{}^{rh}D_{1^+}^{\alpha-i}x - A\,{}^{rh}D_{1^+}^{\beta-i}x]|_{t=1} = x_i, \ i \in \mathbf{N}_1^{p-1}, \\ &[{}^{rh}D_{1^+}^{\alpha-p}x - A\,{}^{h}I_{0^+}^{p-\beta}x]|_{t=1} = x_p, \\ &{}^{rh}D_{1^+}^{\alpha-i}x(1) = x_i, \ i \in \mathbf{N}_{p+1}^{n-1}, \ \ {}^{h}I_{1^+}^{n-\alpha}x(1) = x_n \end{aligned} \tag{2.3}$$

where $x_i \in \mathbf{R}(i \in \mathbf{N}_1^n)$.

We choose the following Picard function sequence:

$$\phi_0(t) = \sum_{i=1}^{n} \frac{x_i}{\Gamma(\alpha-i+1)}(\ln t)^{\alpha-i} + \int_1^t \frac{(\ln t - \ln s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\frac{ds}{s}, \ t \in (1, e],$$

$$\phi_i(t) = \phi_0(t) + \frac{A}{\Gamma(\alpha-\beta)} \int_1^t (\ln t - \ln s)^{\alpha-\beta-1}\phi_{i-1}(s)\frac{ds}{s}, t \in (1, e], i = 1, 2, \cdots.$$

Claim 2.1. $t \mapsto (\ln t)^{n-\alpha}\phi_i(t)$ is continuous on $[1, e]$.

In fact, one sees that

$$\left|\int_1^t \frac{(\ln t-\ln s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\frac{ds}{s}\right| \le \int_1^t \frac{(\ln t-\ln s)^{\alpha-1}}{\Gamma(\alpha)} (\ln s)^\sigma (1-\ln s)^\tau \frac{ds}{s}$$

$$\le \int_1^t \frac{(\ln t-\ln s)^{\alpha+\tau-1}}{\Gamma(\alpha)} (\ln s)^\sigma \frac{ds}{s} = (\ln t)^{\alpha+\sigma+\tau} \int_0^1 \frac{(1-w)^{\alpha+\tau-1}}{\Gamma(\alpha)} w^\sigma dw.$$

Then $t \mapsto (\ln t)^{n-\alpha}\phi_0(t)$ is continuous on $[1,e]$ by $\tau > -n-\sigma$. By mathematical induction method, we know that $t \mapsto (\ln t)^{n-\alpha}\phi_i(t)$ is continuous on $[1,e]$.

Claim 2.2. $\{t \mapsto (\ln t)^{n-\alpha}\phi_i(t)\}$ is convergent uniformly on $[1,e]$.

In fact, by Claim 2.1, we have $\|\phi_0\|_{n-\alpha} = \sup_{t\in(1,e]} (\ln t)^{n-\alpha}|\phi_0(t)| < +\infty$. Then

$$(\ln t)^{n-\alpha}|\phi_1(t)-\phi_0(t)| = (\ln t)^{n-\alpha}\left|\frac{A}{\Gamma(\alpha-\beta)}\int_1^t (\ln t-\ln s)^{\alpha-\beta-1}\phi_0(s)\frac{ds}{s}\right|$$

$$\le \frac{|A|}{\Gamma(\alpha-\beta)}(\ln t)^{n-\alpha}\int_1^t (\ln t-\ln s)^{\alpha-\beta-1}|\phi_0(s)|\frac{ds}{s}$$

$$\le \frac{|A|}{\Gamma(\alpha-\beta)}(\ln t)^{n-\alpha}\int_1^t (\ln t-\ln s)^{\alpha-\beta-1}(\ln s)^{\alpha-n}\frac{ds}{s}\sup_{t\in[1,e]}(\ln t)^{n-\alpha}|\phi_0(t)|$$

$$= \frac{|A|\|\phi_0\|_{n-\alpha}}{\Gamma(\alpha-\beta)}(\ln t)^{n-\alpha}(\ln t)^{2\alpha-\beta-n}\int_0^1 (1-w)^{\alpha-\beta-1}w^{\alpha-n}dw$$

$$= \frac{|A|\|\phi_0\|_{n-\alpha}\mathbf{B}(\alpha-\beta,\alpha-n+1)}{\Gamma(\alpha-\beta)}(\ln t)^{\alpha-\beta}.$$

Similarly we have

$$(\ln t)^{n-\alpha}|\phi_2(t)-\phi_1(t)| = (\ln t)^{n-\alpha}\left|\frac{A}{\Gamma(\alpha-\beta)}\int_1^t (\ln t-\ln s)^{\alpha-\beta-1}[\phi_1(s)-\phi_0(s)]\frac{ds}{s}\right|$$

$$\le \frac{|A|}{\Gamma(\alpha-\beta)}(\ln t)^{n-\alpha}\int_1^t (\ln t-\ln s)^{\alpha-\beta-1}|\phi_1(s)-\phi_0(s)|\frac{ds}{s}$$

$$\le \frac{|A|}{\Gamma(\alpha-\beta)}(\ln t)^{n-\alpha}\int_1^t (\ln t-\ln s)^{\alpha-\beta-1}(\ln s)^{\alpha-n}\frac{|A|\|\phi_0\|_{n-\alpha}\mathbf{B}(\alpha-\beta,\alpha-n+1)}{\Gamma(\alpha-\beta)}(\ln s)^{\alpha-\beta}\frac{ds}{s}$$

$$= \frac{|A|\|\phi_0\|_{n-\alpha}\mathbf{B}(\alpha-\beta,\alpha-n+1)}{\Gamma(\alpha-\beta)}\frac{|A|\mathbf{B}(\alpha-\beta,2\alpha-\beta-n+1)}{\Gamma(\alpha-\beta)}(\ln t)^{2\alpha-2\beta}.$$

By mathematical induction method, we can get

$$(\ln t)^{n-\alpha}|\phi_{i+1}(t)-\phi_i(t)| = (\ln t)^{n-\alpha}\left|\frac{A}{\Gamma(\alpha-\beta)}\int_1^t (\ln t-\ln s)^{\alpha-\beta-1}[\phi_i(s)-\phi_{i-1}(s)]\frac{ds}{s}\right|$$

$$\le \|\phi_0\|_{n-\alpha}\left[\prod_{j=0}^{i}\frac{|A|\mathbf{B}(\alpha-\beta,(j+1)\alpha-j\beta-n+1)}{\Gamma(\alpha-\beta)}\right](\ln t)^{(i+1)(\alpha-\beta)}, i=0,1,2,3,\cdots.$$

It follows for $t \in [1,e]$ that

$$(\ln t)^{n-\alpha}|\phi_{i+1}(t)-\phi_i(t)| \le \|\phi_0\|_{n-\alpha}\left[\prod_{j=0}^{i}\frac{|A|\mathbf{B}(\alpha-\beta,(j+1)\alpha-j\beta-n+1)}{\Gamma(\alpha-\beta)}\right], i=0,1,2,\cdots.$$

Consider

$$\sum_{i=0}^{\infty} u_i =: \sum_{i=0}^{\infty}\|\phi_0\|_{n-\alpha}\left[\prod_{j=0}^{i}\frac{|A|\mathbf{B}(\alpha-\beta,(j+1)\alpha-j\beta-n+1)}{\Gamma(\alpha-\beta)}\right].$$

It is easy to see that

$$\frac{u_{i+1}}{u_i} = \frac{|A|\mathbf{B}(\alpha-\beta,(i+2)\alpha-(i+1)\beta-n+1)}{\Gamma(\alpha-\beta)} = \frac{|A|}{\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-\beta-1}s^{(i+2)\alpha-(i+1)\beta-n}ds$$

For each $\epsilon > 0$, choose $\delta \in (0,1)$ such that $\int_\delta^1 (1-s)^{\alpha-\beta-1}s^{\alpha-n}ds < \epsilon$. Then

$$\int_\delta^1 (1-s)^{\alpha-\beta-1}s^{(i+2)\alpha-(i+1)\beta-n}ds \le \int_\delta^1 (1-s)^{\alpha-\beta-1}s^{\alpha-n}ds < \epsilon.$$

Choose N sufficiently large such that $\frac{\delta^{(i+1)(\alpha-\beta)+\alpha-n}}{\alpha-\beta} < \epsilon$ for all $i > N$. Then

$$\begin{aligned}
&\int_0^1 (1-s)^{\alpha-\beta-1}s^{(i+2)\alpha-(i+1)\beta-n}ds \\
&= \int_0^\delta (1-s)^{\alpha-\beta-1}s^{(i+2)\alpha-(i+1)\beta-n}ds + \int_\delta^1 (1-s)^{\alpha-\beta-1}s^{(i+2)\alpha-(i+1)\beta-n}ds \\
&\le \int_0^\delta (1-s)^{\alpha-\beta-1}ds\delta^{(i+1)(\alpha-\beta)+\alpha-n} + \int_\delta^1 (1-s)^{\alpha-\beta-1}s^{\alpha-n}ds \\
&\le \frac{\delta^{(i+1)(\alpha-\beta)+\alpha-n}}{\alpha-\beta} + \epsilon < 2\epsilon, i > N.
\end{aligned}$$

It follows that $\lim_{i\to+\infty}\frac{u_{i+1}}{u_i} = 0$. Then $\sum_{i=0}^{\infty} u_i$ is convergent. Hence

$$(\ln t)^{n-\alpha}\phi_0(t) + \sum_{i=0}^{\infty}(\ln t)^{n-\alpha}\left[\phi_{i+1}(t) - \phi_i(t)\right]$$

is uniformly convergent. Hence $\{t \mapsto (\ln t)^{n-\alpha}\phi_i(t)\}$ is uniformly convergent on $[1,e]$.

Claim 2.3. $\phi(t) = \lim_{i\to+\infty}\phi_i(t)$ defined on $(1,e]$ is a unique continuous solution of the integral equation

$$x(t) = \sum_{i=1}^{n}\frac{x_i}{\Gamma(\alpha-i+1)}(\ln t)^{\alpha-i} + \int_1^t \frac{(\ln t-\ln s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\frac{ds}{s} + A\int_1^t \frac{(\ln t-\ln u)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}x(u)\frac{du}{u}, t \in (1,e]. \tag{2.4}$$

From Claim 2.2, we have $\lim_{i\to+\infty}(\ln t)^{n-\alpha}\phi_i(t) = (\ln t)^{n-\alpha}\phi(t)$ uniformly on $(1,e]$. Then for $t \in (1,e]$, we have

$$\begin{aligned}
\phi(t) &= \lim_{i\to+\infty}(\ln t)^{n-\alpha}\phi_i(t) \\
&= \sum_{i=1}^{n}\frac{x_i}{\Gamma(\alpha-i+1)}(\ln t)^{\alpha-i} + \int_1^t \frac{(\ln t-\ln s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\frac{ds}{s} + \frac{A}{\Gamma(\alpha-\beta)}\int_1^t (\ln t-\ln s)^{\alpha-\beta-1}\lim_{i\to+\infty}\phi_{i-1}(s)\frac{ds}{s} \\
&= \sum_{i=1}^{n}\frac{x_i}{\Gamma(\alpha-i+1)}(\ln t)^{\alpha-i} + \int_1^t \frac{(\ln t-\ln s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\frac{ds}{s} + \frac{A}{\Gamma(\alpha-\beta)}\int_1^t (\ln t-\ln s)^{\alpha-\beta-1}\phi(s)\frac{ds}{s}.
\end{aligned}$$

Hence ϕ is a solution of (2.4).

Suppose that ψ is also a solution of (2.4) such that $\lim_{t\to1}(\ln t)^{n-\alpha}\psi(t)$ is finite. We will prove that $\phi(t) \equiv \psi(t)$ on $(1,e]$. Then

$$\begin{aligned}
&(\ln t)^{n-\alpha}|\psi(t) - \phi_0(t)| = (\ln t)^{n-\alpha}\left|\frac{A}{\Gamma(\alpha-\beta)}\int_1^t (\ln t-\ln s)^{\alpha-\beta-1}\psi(s)\frac{ds}{s}\right| \\
&\le \frac{|A|}{\Gamma(\alpha-\beta)}(\ln t)^{n-\alpha}\int_1^t (\ln t-\ln s)^{\alpha-\beta-1}|\psi(s)|\frac{ds}{s} \\
&\le \frac{|A|}{\Gamma(\alpha-\beta)}(\ln t)^{n-\alpha}\int_1^t (\ln t-\ln s)^{\alpha-\beta-1}(\ln s)^{\alpha-n}\frac{ds}{s}\sup_{t\in[1,e]}(\ln t)^{n-\alpha}|\psi(t)| \\
&= \frac{|A|\|\psi\|_{n-\alpha}}{\Gamma(\alpha-\beta)}(\ln t)^{n-\alpha}(\ln t)^{2\alpha-\beta-n}\int_0^1 (1-w)^{\alpha-\beta-1}w^{\alpha-n}dw \\
&= \frac{|A|\|\psi\|_{n-\alpha}\mathbf{B}(\alpha-\beta,\alpha-n+1)}{\Gamma(\alpha-\beta)}(\ln t)^{\alpha-\beta}.
\end{aligned}$$

Similarly we have

$$(\ln t)^{n-\alpha}|\psi(t)-\phi_1(t)| = (\ln t)^{n-\alpha}\left|\frac{A}{\Gamma(\alpha-\beta)}\int_1^t(\ln t-\ln s)^{\alpha-\beta-1}[\psi(s)-\phi_0(s)]\frac{ds}{s}\right|$$

$$\leq \frac{|A|}{\Gamma(\alpha-\beta)}(\ln t)^{n-\alpha}\int_1^t(\ln t-\ln s)^{\alpha-\beta-1}|\psi(s)-\phi_0(s)|\frac{ds}{s}$$

$$\leq \frac{|A|}{\Gamma(\alpha-\beta)}(\ln t)^{n-\alpha}\int_1^t(\ln t-\ln s)^{\alpha-\beta-1}(\ln s)^{\alpha-n}\frac{|A|\|\psi\|_{n-\alpha}\mathbf{B}(\alpha-\beta,\alpha-n+1)}{\Gamma(\alpha-\beta)}(\ln s)^{\alpha-\beta}\frac{ds}{s}$$

$$= \frac{|A|\|\psi\|_{n-\alpha}\mathbf{B}(\alpha-\beta,\alpha-n+1)}{\Gamma(\alpha-\beta)}\frac{|A|\mathbf{B}(\alpha-\beta,2\alpha-\beta-n+1)}{\Gamma(\alpha-\beta)}(\ln t)^{2\alpha-2\beta}.$$

By mathematical induction method, we can get

$$(\ln t)^{n-\alpha}|\psi(t)-\phi_i(t)| = (\ln t)^{n-\alpha}\left|\frac{A}{\Gamma(\alpha-\beta)}\int_1^t(\ln t-\ln s)^{\alpha-\beta-1}[\psi(s)-\phi_{i-1}(s)]\frac{ds}{s}\right|$$

$$\leq \|\psi\|_{n-\alpha}\left[\prod_{j=0}^{i}\frac{|A|\mathbf{B}(\alpha-\beta,(j+1)\alpha-j\beta-n+1)}{\Gamma(\alpha-\beta)}\right](\ln t)^{(i+1)(\alpha-\beta)}, i=0,1,2,3,\cdots.$$

It follows for $t\in[1,e]$ that

$$(\ln t)^{n-\alpha}|\psi(t)-\phi_i(t)| \leq \|\psi\|_{n-\alpha}\left[\prod_{j=0}^{i}\frac{|A|\mathbf{B}(\alpha-\beta,(j+1)\alpha-j\beta-n+1)}{\Gamma(\alpha-\beta)}\right], i=0,1,2,\cdots.$$

Similarly to the proof of Claim 2.2, we know that

$$\sum_{i=0}^{\infty}u_i =: \sum_{i=0}^{\infty}\|\psi\|_{n-\alpha}\left[\prod_{j=0}^{i}\frac{|A|\mathbf{B}(\alpha-\beta,(j+1)\alpha-j\beta-n+1)}{\Gamma(\alpha-\beta)}\right]$$

is convergent. Hence

$$\lim_{i\to+\infty}\|\psi\|_{n-\alpha}\left[\prod_{j=0}^{i}\frac{|A|\mathbf{B}(\alpha-\beta,(j+1)\alpha-j\beta-n+1)}{\Gamma(\alpha-\beta)}\right]=0.$$

Then $\lim_{i\to+\infty}[\psi(t)-\phi_i(t)]=0$. So $\psi(t)\equiv\phi(t)$.

Claim 2.4. Suppose that x is a continuous solution of (2.1) satisfying (2.3). Then x is a solution of the integral equation (2.4).

Proof. Since x is a solution of (2.1) satisfying (2.3), we have $x\in C(1,e]$ and $\lim_{t\to0^+}(\ln t)^{n-\alpha}x(t)$. Since $\alpha-\beta+p-n\geq 0$, we know that $\lim_{t\to1^+}(\ln t)^{p-\beta}x(t)$ are finite. Then

$${}^hI_{1^+}^{n-\alpha}x(1) = \lim_{t\to1^+}\int_1^t\frac{(\ln t-\ln s)^{n-\alpha-1}}{\Gamma(n-\alpha)}x(s)\frac{ds}{s} = \lim_{t\to1^+}\int_1^t\frac{(\ln t-\ln s)^{n-\alpha-1}}{\Gamma(n-\alpha)}(\ln s)^{\alpha-n}(\ln s)^{n-\alpha}x(s)\frac{ds}{s}$$

$$= \lim_{t\to1^+}\int_1^t\frac{(\ln t-\ln s)^{n-\alpha-1}}{\Gamma(n-\alpha)}(\ln s)^{\alpha-n}\frac{ds}{s}\xi^{n-\alpha}x(\xi)\ (\text{where } \xi\in(1,t))$$

$$= \lim_{t\to1^+}\int_0^1\frac{(1-w)^{n-\alpha-1}}{\Gamma(n-\alpha)}w^{\alpha-n}dw\,\xi^{n-\alpha}x(\xi) = \Gamma(\alpha-n+1)\lim_{\xi\to1^+}\xi^{n-\alpha}x(\xi) \text{ is finite.}$$

Similarly we know that ${}^{h}I_{1^+}^{p-\beta}x(1)$ is finite. It follows from (2.1) and (2.108) in [11] (page 70) that

$$x(t) = \sum_{i=1}^{n-1} \frac{{}^{rh}D_{1^+}^{\alpha-i}x(1)}{\Gamma(\alpha-i+1)}(\ln t)^{\alpha-i} + \frac{{}^{h}I_{1^+}^{n-\alpha}x(1)}{\Gamma(\alpha-n+1)}(\ln t)^{\alpha-n} + \int_1^t \frac{(\ln t-\ln s)^{\alpha-1}}{\Gamma(\alpha)}[h(s) + A\,{}^{rh}D_{1^+}^{\beta}x(s)]\frac{ds}{s}$$

$$= \sum_{i=1}^{n-1} \frac{{}^{rh}D_{1^+}^{\alpha-i}x(1)}{\Gamma(\alpha-i+1)}(\ln t)^{\alpha-i} + \frac{{}^{h}I_{1^+}^{n-\alpha}x(1)}{\Gamma(\alpha-n+1)}(\ln t)^{\alpha-n} + \int_1^t \frac{(\ln t-\ln s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\frac{ds}{s} \tag{2.5}$$

$$+A\int_1^t \frac{(\ln t-\ln s)^{\alpha-1}}{\Gamma(\alpha)}\,{}^{rh}D_{1^+}^{\beta}x(s)\frac{ds}{s}.$$

Since $x \in C(1,e]$ and $\lim_{t\to 1^+}(\ln t)^{n-\alpha}x(t)$ is finite, there exists a constant $M > 0$ such that $(\ln t)^{n-\alpha}|x(t)| \leq M$ for all $t \in (1,e]$. For $t > 1$, $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ with $1 + \epsilon_1 + \epsilon_2 + \epsilon_3 \in (1,t)$ and $\epsilon = \max\{\epsilon_1, \epsilon_2, \epsilon_3\}$, we have

$$\left|\int_{1+\epsilon_2+\epsilon_3}^{t-\epsilon_1}(\ln t - \ln s)^{\alpha-p}\int_{1+\epsilon_3}^{s-\epsilon_2}(\ln s - \ln u)^{p-\beta-1}x(u)\frac{du}{u}\frac{ds}{s}\right|$$

$$\leq \int_{1+\epsilon_2+\epsilon_3}^{t-\epsilon_1}(\ln t - \ln s)^{\alpha-p}\int_{1+\epsilon_3}^{s-\epsilon_2}(\ln s - \ln u)^{p-\beta-1}(\ln u)^{\alpha-n}M\frac{du}{u}\frac{ds}{s}$$

$$= M\int_{1+\epsilon_2+\epsilon_3}^{t-\epsilon_1}(\ln t - \ln s)^{\alpha-p}(\ln s)^{\alpha-\beta-n+p}\int_{\frac{\ln(1+\epsilon_3)}{\ln s}}^{\frac{\ln(s-\epsilon_2)}{\ln s}}(1-w)^{p-\beta-1}w^{\alpha-n}dw\frac{ds}{s}$$

$$\leq M\int_{1+\epsilon_2+\epsilon_3}^{t-\epsilon_1}(\ln t - \ln s)^{\alpha-p}(\ln s)^{\alpha-\beta-n+p}\mathbf{B}(p-\beta, \alpha-n+1)\frac{ds}{s}$$

$$= M(\ln t)^{2\alpha-\beta-n}\int_{\frac{\ln(1+\epsilon_2+\epsilon_3)}{\ln t}}^{\frac{\ln(t-\epsilon_1)}{\ln t}}(1-w)^{\alpha-p}w^{\alpha-\beta-n+p}dw\mathbf{B}(p-\beta, \alpha-n+1)$$

$$\leq M(\ln t)^{2\alpha-\beta-n}\mathbf{B}(\alpha-p+1, \alpha-\beta-n+p+1)\mathbf{B}(p-\beta, \alpha-n+1)$$

and

$$\left|\int_{1+\epsilon_3}^{t-\epsilon_1-\epsilon_2}\int_{u+\epsilon_2}^{t-\epsilon_1}(\ln t - \ln s)^{\alpha-p}(\ln s - \ln u)^{p-\beta-1}\frac{ds}{s}x(u)\frac{du}{u}\right|$$

$$\leq \int_{1+\epsilon_3}^{t-\epsilon_1-\epsilon_2}\int_{u+\epsilon_2}^{t-\epsilon_1}(\ln t - \ln s)^{\alpha-p}(\ln s - \ln u)^{p-\beta-1}\frac{ds}{s}(\ln u)^{\alpha-n}M\frac{du}{u}$$

$$= M\int_{1+\epsilon_3}^{t-\epsilon_1-\epsilon_2}(\ln t - \ln u)^{\alpha-\beta}\int_{\frac{\ln(1+\epsilon_2)}{\ln(t-u)}}^{\frac{\ln(t-u-\epsilon_1)}{\ln(t-u)}}(1-w)^{\alpha-p}w^{p-\beta-1}dw(\ln u)^{\alpha-n}\frac{du}{u}$$

$$\leq M\int_{\ln(1+\epsilon_3)}^{\ln(t-\epsilon_1-\epsilon_2)}(\ln t - \ln u)^{\alpha-\beta}(\ln u)^{\alpha-n}\frac{du}{u}\mathbf{B}(\alpha-p+1, p-\beta)$$

$$= M(\ln t)^{2\alpha-\beta-n+1}\int_{\frac{\ln(1+\epsilon_3)}{\ln t}}^{\frac{\ln(t-\epsilon_1-\epsilon_2)}{\ln t}}(1-w)^{\alpha-\beta}w^{\alpha-n}dw\mathbf{B}(\alpha-p+1, p-\beta)$$

$$\leq M(\ln t)^{2\alpha-\beta-n+1}\mathbf{B}(\alpha-\beta+1, \alpha-n+1)\mathbf{B}(\alpha-p+1, p-\beta).$$

Then

$$\int_1^t(\ln t - \ln s)^{\alpha-p}\int_1^s(\ln s - \ln u)^{p-\beta-1}x(u)\frac{du}{u}\frac{ds}{s}$$

$$= \lim_{\epsilon\to 0^+}\int_{1+\epsilon_2+\epsilon_3}^{t-\epsilon_1}(\ln t - \ln s)^{\alpha-p}\int_{1+\epsilon_3}^{s-\epsilon_2}(\ln s - \ln u)^{p-\beta-1}x(u)\frac{du}{u}\frac{ds}{s} \tag{2.6}$$

$$= \lim_{\epsilon\to 0^+}\int_{1+\epsilon_3}^{t-\epsilon_1-\epsilon_2}\int_{u+\epsilon_2}^{t-\epsilon_1}(\ln t - \ln s)^{\alpha-p}(s-u)^{p-\beta-1}\frac{ds}{s}x(u)\frac{du}{u}$$

$$= \int_1^t\int_u^t(\ln t - \ln s)^{\alpha-p}(\ln s - \ln u)^{p-\beta-1}\frac{ds}{s}x(u)\frac{du}{u}.$$

From Remark 2.1, $x \in C(1,e]$, $\lim_{t\to 1^+}(\ln t)^{n-\alpha}x(t)$ and $\lim_{t\to 1^+}(\ln t)^{p-\beta}x(t)$ are finite, ${}^{rh}D_{1^+}^{\alpha-i}x(i \in \mathbf{N}_1^{n-1})$ and ${}^{rh}D_{1^+}^{\beta-j}x(j \in \mathbf{N}_1^{p-1})$ are continuous on $[1,e]$. One sees

$$\int_1^t \frac{(\ln t-\ln s)^{\alpha-1}}{\Gamma(\alpha)} {}^{rh}D_{1^+}^{\beta}x(s)\frac{ds}{s} = \int_1^t \frac{(\ln t-\ln s)^{\alpha-1}}{\Gamma(\alpha)}\left[\int_1^s \frac{(\ln s-\ln u)^{p-\beta-1}}{\Gamma(p-\beta)}x(u)\frac{du}{u}\right]^{(p)}\frac{ds}{s}$$

$$= \int_1^t \frac{(\ln t-\ln s)^{\alpha-1}}{\Gamma(\alpha)} d\left[\int_1^s \frac{(\ln s-\ln u)^{p-\beta-1}}{\Gamma(p-\beta)}x(u)\frac{du}{u}\right]^{(p-1)}$$

$$= \frac{(\ln t-\ln s)^{\alpha-1}\left[\int_1^s(\ln s-\ln u)^{p-\beta-1}x(u)\frac{du}{u}\right]^{(p-1)}\Big|_1^t + (\alpha-1)\int_1^t(\ln t-\ln s)^{\alpha-2}\left[\int_1^s(\ln s-\ln u)^{p-\beta-1}x(u)\frac{du}{u}\right]^{(p-1)}\frac{ds}{s}}{\Gamma(\alpha)\Gamma(p-\beta)}$$

$$= -\frac{(\ln t)^{\alpha-1}}{\Gamma(\alpha)} {}^{rh}D_{1^+}^{\beta-1}x(1) + \frac{\int_1^t(\ln t-\ln s)^{\alpha-2}\left[\int_1^s(\ln s-\ln u)^{p-\beta-1}x(u)\frac{du}{u}\right]^{(p-1)}\frac{ds}{s}}{\Gamma(\alpha-1)\Gamma(p-\beta)}$$

$$= \cdots\cdots\cdots$$

$$= -\sum_{i=1}^{p-1}\frac{(\ln t)^{\alpha-i}}{\Gamma(\alpha-i+1)} {}^{rh}D_{1^+}^{\beta-i}x(1) + \frac{\int_1^t(\ln t-\ln s)^{\alpha-p}\left[\int_1^s(\ln s-\ln u)^{p-\beta-1}x(u)\frac{du}{u}\right]'\frac{ds}{s}}{\Gamma(\alpha-p+1)\Gamma(p-\beta)}$$

$$= -\sum_{i=1}^{p-1}\frac{(\ln t)^{\alpha-i}}{\Gamma(\alpha-i+1)} {}^{rh}D_{1^+}^{\beta-i}x(1) + \left[\frac{\int_1^t(\ln t-\ln s)^{\alpha-p+1}\left[\int_1^s(\ln s-\ln u)^{p-\beta-1}x(u)\frac{du}{u}\right]'\frac{ds}{s}}{\Gamma(\alpha-p+2)\Gamma(p-\beta)}\right]_t'$$

$$= -\sum_{i=1}^{p-1}\frac{(\ln t)^{\alpha-i}}{\Gamma(\alpha-i+1)} {}^{rh}D_{1^+}^{\beta-i}x(1) \text{ (here use (2.6))}$$

$$+t\left[\frac{(\ln t-\ln s)^{\alpha-p+1}\int_1^s(\ln s-\ln u)^{p-\beta-1}x(u)\frac{du}{u}\Big|_1^t + (\alpha-p+1)\int_1^t(\ln t-\ln s)^{\alpha-p}\int_1^s(\ln s-\ln u)^{p-\beta-1}x(u)\frac{du}{u}\frac{ds}{s}}{\Gamma(\alpha-p+2)\Gamma(p-\beta)}\right]_t'$$

$$= -\sum_{i=1}^{p-1}\frac{(\ln t)^{\alpha-i}}{\Gamma(\alpha-i+1)} {}^{rh}D_{1^+}^{\beta-i}x(1) - \frac{(\ln t)^{\alpha-p}}{\Gamma(\alpha-p+1)} {}^{h}I_{1^+}^{p-\beta}x(1)$$

$$+t\left[\frac{\int_1^t\int_u^t(\ln t-\ln s)^{\alpha-p}(\ln s-\ln u)^{p-\beta-1}\frac{ds}{s}x(u)\frac{du}{u}}{\Gamma(\alpha-p+1)\Gamma(p-\beta)}\right]_t'$$

$$= -\sum_{i=1}^{p-1}\frac{(\ln t)^{\alpha-i}}{\Gamma(\alpha-i+1)} {}^{rh}D_{1^+}^{\beta-i}x(1) - \frac{(\ln t)^{\alpha-p}}{\Gamma(\alpha-p+1)} {}^{h}I_{1^+}^{p-\beta}x(1)$$

$$+t\left[\frac{\int_1^t(\ln t-\ln u)^{\alpha-\beta}\int_0^1(1-w)^{\alpha-p}w^{p-\beta-1}dwx(u)\frac{du}{u}}{\Gamma(\alpha-p+1)\Gamma(p-\beta)}\right]_t'$$

$$= -\sum_{i=1}^{p-1}\frac{{}^{rh}D_{1^+}^{\beta-i}x(1)}{\Gamma(\alpha-i+1)}(\ln t)^{\alpha-i} - \frac{{}^{h}I_{1^+}^{p-\beta}x(1)}{\Gamma(\alpha-p+1)}(\ln t)^{\alpha-p} + \int_1^t\frac{(\ln t-\ln u)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}x(u)\frac{du}{u}.$$

Note $\beta < \alpha$ and $p \le n$. It follows from (2.5) that

$$x(t) = \sum_{i=1}^{n-1}\frac{{}^{rh}D_{1^+}^{\alpha-i}x(1)}{\Gamma(\alpha-i+1)}(\ln t)^{\alpha-i} + \frac{{}^{h}I_{1^+}^{n-\alpha}x(1)}{\Gamma(\alpha-n+1)}(\ln t)^{\alpha-n} + \int_1^t\frac{(\ln t-\ln s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\frac{ds}{s}$$

$$-A\sum_{i=1}^{p-1}\frac{{}^{rh}D_{1^+}^{\beta-i}x(1)}{\Gamma(\alpha-i+1)}(\ln t)^{\alpha-i} + A\frac{{}^{h}I_{1^+}^{p-\beta}x(1)}{\Gamma(\alpha-p+1)}(\ln t)^{\alpha-p} + A\int_1^t\frac{(\ln t-\ln u)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}x(u)\frac{du}{u}$$

$$= \sum_{i=1}^{p-1} \frac{{}^{rh}D_{1^+}^{\alpha-i}x(1) - A\,{}^{rh}D_{1^+}^{\beta-i}x(1)}{\Gamma(\alpha-i+1)}(\ln t)^{\alpha-i} + \frac{{}^{rh}D_{1^+}^{\alpha-p}x(1) - A\,{}^{h}I_{1^+}^{p-\beta}x(1)}{\Gamma(\alpha-p+1)}(\ln t)^{\alpha-p}$$

$$+ \sum_{i=p+1}^{n-1} \frac{{}^{rh}D_{1^+}^{\alpha-i}x(1)}{\Gamma(\alpha-i+1)}(\ln t)^{\alpha-i} + \frac{{}^{h}I_{1^+}^{n-\alpha}x(1)}{\Gamma(\alpha-n+1)}(\ln t)^{\alpha-n} + \int_1^t \frac{(\ln t - \ln s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\frac{ds}{s}$$

$$+ A\int_1^t \frac{(\ln t - \ln u)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} x(u)\frac{du}{u}$$

$$= \sum_{i=1}^{n} \frac{x_i}{\Gamma(\alpha-i+1)}(\ln t)^{\alpha-i} + \int_1^t \frac{(\ln t - \ln s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\frac{ds}{s} + A\int_1^t \frac{(\ln t - \ln u)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} x(u)\frac{du}{u}.$$

Then x is a solution of (2.4). The proof is completed. □

Lemma 2.1. Suppose $\alpha - \beta + p - n \geq 0$. Then x is a continuous solution of (2.1) satisfying (2.3) if and only if

$$x(t) = \sum_{\nu=1}^{n} x_\nu (\ln t)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(A(\ln t)^{\alpha-\beta})$$
$$+ \int_1^t (\ln t - \ln s)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(A(\ln t - \ln s)^{\alpha-\beta})h(s)\frac{ds}{s}, t \in (1, e]. \quad (2.7)$$

Proof. We divide the proof into two steps:

Step 1. Suppose that x is a solution of (2.1) satisfying (2.3). We prove that x satisfies (2.7).

In fact, by Claim 2.4, x is a solution of (2.4). Claim 2.3 implies $x(t) = \lim_{i\to+\infty} \phi_i(t)$ is the unique solution of (2.4).

On the other hand, we have

$$\phi_i(t) = \phi_0(t) + \frac{A}{\Gamma(\alpha-\beta)}\int_1^t (\ln t - \ln s)^{\alpha-\beta-1}\phi_{i-1}(s)\frac{ds}{s}$$

$$= \phi_0(t) + \frac{A}{\Gamma(\alpha-\beta)}\int_1^t (\ln t - \ln s)^{\alpha-\beta-1}\left(\phi_0(s) + \frac{A)}{\Gamma(\alpha-\beta)}\int_1^s (\ln s - \ln u)^{\alpha-\beta-1}\phi_{i-2}(u)\frac{du}{u}\right)\frac{ds}{s}$$

$$= \phi_0(t) + \frac{A}{\Gamma(\alpha-\beta)}\int_1^t (\ln t - \ln s)^{\alpha-\beta-1}\phi_0(s)ds$$

$$+ \frac{A^2}{\Gamma(\alpha-\beta)^2}\int_1^t \int_u^t (\ln t - \ln s)^{\alpha-\beta-1}(\ln s - \ln u)^{\alpha-\beta-1}\frac{ds}{s}\phi_{i-2}(u)\frac{du}{u}$$

$$= \phi_0(t) + \frac{A}{\Gamma(\alpha-\beta)}\int_1^t (\ln t - \ln s)^{\alpha-\beta-1}\phi_0(s)\frac{ds}{s}$$

$$+ \frac{A^2}{\Gamma(\alpha-\beta)^2}\int_1^t (\ln t - \ln u)^{2\alpha-2\beta-1}\int_0^1 (1-w)^{\alpha-\beta-1}w^{\alpha-\beta-1}dw\phi_{i-2}(u)\frac{du}{u}$$

$$= \phi_0(t) + \frac{A}{\Gamma(\alpha-\beta)}\int_1^t (\ln t - \ln s)^{\alpha-\beta-1}\phi_0(s)\frac{ds}{s} + \frac{A^2}{\Gamma(2\alpha-2\beta)}\int_1^t (\ln t - \ln u)^{2\alpha-2\beta-1}\phi_{i-2}(u)\frac{du}{u}$$

$$= \cdots\cdots\cdots$$

$$= \phi_0(t) + \sum_{j=1}^{i} \frac{A^j}{\Gamma(j\alpha - j\beta)} \int_1^t (\ln t - \ln u)^{j\alpha - j\beta - 1} \phi_0(u) \frac{du}{u}$$

$$= \sum_{\nu=1}^{n} \frac{x_\nu}{\Gamma(\alpha-\nu+1)} (\ln t)^{\alpha-\nu} + \int_1^t \frac{(\ln t - \ln s)^{\alpha-1}}{\Gamma(\alpha)} h(s) \frac{ds}{s}$$

$$+ \sum_{j=1}^{i} \frac{A^j}{\Gamma(j\alpha - j\beta)} \int_1^t (\ln t - \ln s)^{j\alpha - j\beta - 1} \left[\sum_{\nu=1}^{n} \frac{x_\nu}{\Gamma(\alpha-\nu+1)} (\ln s)^{\alpha-\nu} + \int_1^s \frac{(\ln s - \ln u)^{\alpha-1}}{\Gamma(\alpha)} h(u) \frac{du}{u} \right] \frac{ds}{s}$$

$$= \sum_{\nu=1}^{n} \frac{x_\nu}{\Gamma(\alpha-\nu+1)} (\ln t)^{\alpha-\nu} + \int_1^t \frac{(\ln t - \ln s)^{\alpha-1}}{\Gamma(\alpha)} h(s) \frac{ds}{s}$$

$$+ \sum_{j=1}^{i} \frac{A^j}{\Gamma(j\alpha - j\beta)} \int_1^t (\ln t - \ln s)^{j\alpha - j\beta - 1} \sum_{\nu=1}^{n} \frac{x_\nu}{\Gamma(\alpha-\nu+1)} (\ln s)^{\alpha-\nu} \frac{ds}{s}$$

$$+ \sum_{j=1}^{i} \frac{A^j}{\Gamma(j\alpha - j\beta)} \int_1^t \int_u^t (\ln t - \ln s)^{j\alpha - j\beta - 1} \frac{(\ln s - \ln u)^{\alpha-1}}{\Gamma(\alpha)} \frac{ds}{s} h(u) \frac{du}{u}$$

$$= \sum_{\nu=1}^{n} \frac{x_\nu}{\Gamma(\alpha-\nu+1)} (\ln t)^{\alpha-\nu} + \int_1^t \frac{(\ln t - \ln s)^{\alpha-1}}{\Gamma(\alpha)} h(s) \frac{ds}{s}$$

$$+ \sum_{j=1}^{i} \frac{A^j}{\Gamma(j\alpha - j\beta)} \sum_{\nu=1}^{n} \frac{x_\nu}{\Gamma(\alpha-\nu+1)} (\ln t)^{j\alpha - j\beta + \alpha - \nu} \int_0^1 (1-w)^{j\alpha - j\beta - 1} w^{\alpha-\nu} dw$$

$$+ \sum_{j=1}^{i} \frac{A^j}{\Gamma(j\alpha - j\beta)} \int_1^t (\ln t - \ln u)^{j\alpha - j\beta + \alpha - 1} \int_0^1 (1-w)^{j\alpha - j\beta - 1} \frac{w^{\alpha-1}}{\Gamma(\alpha)} dw h(u) \frac{du}{u}$$

$$= \sum_{\nu=1}^{n} \frac{x_\nu}{\Gamma(\alpha-\nu+1)} (\ln t)^{\alpha-\nu} + \sum_{\nu=1}^{n} x_\nu \sum_{j=1}^{i} \frac{A^j}{\Gamma((j+1)\alpha - j\beta - \nu + 1)} (\ln t)^{(j+1)\alpha - j\beta - \nu}$$

$$+ \int_1^t \frac{(\ln t - \ln s)^{\alpha-1}}{\Gamma(\alpha)} h(s) \frac{ds}{s} + \sum_{j=1}^{i} \frac{A^j}{\Gamma((j+1)\alpha - j\beta)} \int_1^t (\ln t - \ln u)^{(j+1)\alpha - j\beta - 1} h(u) \frac{du}{u}$$

$$= \sum_{\nu=1}^{n} x_\nu \sum_{j=0}^{i} \frac{A^j}{\Gamma((j+1)\alpha - j\beta - \nu + 1)} (\ln t)^{(j+1)\alpha - j\beta - \nu}$$

$$+ \sum_{j=0}^{i} \frac{A^j}{\Gamma((j+1)\alpha - j\beta)} \int_1^t (\ln t - \ln u)^{(j+1)\alpha - j\beta - 1} h(u) \frac{du}{u}.$$

Thus

$$x(t) = \lim_{i \to +\infty} \left[\sum_{\nu=1}^{n} x_\nu \sum_{j=0}^{i} \frac{A^j}{\Gamma((j+1)\alpha - j\beta - \nu + 1)} (\ln t)^{(j+1)\alpha - j\beta - \nu} \right.$$

$$\left. + \sum_{j=0}^{i} \frac{A^j}{\Gamma((j+1)\alpha - j\beta)} \int_1^t (\ln t - \ln u)^{(j+1)\alpha - j\beta - 1} h(u) \frac{du}{u} \right]$$

$$= \sum_{\nu=1}^{n} x_\nu (\ln t)^{\alpha-\nu} \mathbf{E}_{\alpha-\beta, \alpha-\nu+1}(A(\ln t)^{\alpha-\beta}) + \int_1^t (\ln t - \ln s)^{\alpha-1} \mathbf{E}_{\alpha-\beta,\alpha}(A(\ln t - \ln s)^{\alpha-\beta}) h(s) \frac{ds}{s}.$$

So x satisfies (2.7).

Step 2. Suppose that x satisfies (2.7). We prove that x is a solution of (2.1) satisfying (2.3). Since $h \in C(1,e)$ and $|h(t)| \leq (\ln t)^{\sigma}(1-\ln t)^{\tau}$ for $t \in (1,e)$ with $\sigma > -1, \tau \in (-n-\sigma, 0]$, we know

$$\left|\int_1^t (\ln t - \ln s)^{i-1}\mathbf{E}_{\alpha-\beta,i}(A(\ln t - \ln s)^{\alpha-\beta})h(s)\frac{ds}{s}\right|$$

$$\leq \int_1^t (\ln t - \ln s)^{i-1}\mathbf{E}_{\alpha-\beta,i}(|A|)(\ln s)^{\sigma}(1-\ln s)^{\tau}\frac{ds}{s} \leq \int_1^t (\ln t - \ln s)^{i+\tau-1}(\ln s)^{\sigma}\frac{ds}{s}\mathbf{E}_{\alpha-\beta,i}(|A|)$$

$$= (\ln t)^{i+\sigma+\tau}\int_0^1 (1-w)^{i+l-1}w^k dw\mathbf{E}_{\alpha-\beta,i}(|A|).$$

Then $t \to \int_1^t (\ln t - \ln s)^{i-1}\mathbf{E}_{\alpha-\beta,i}(A(\ln t - \ln s)^{\alpha-\beta})h(s)\frac{ds}{s}$ is continuous on $[1,e]$ for all $i \in \mathbf{N}_1^n$.

Similarly we have

$$\left|\int_1^t (\ln t - \ln s)^{\alpha-\beta+i-1}\mathbf{E}_{\alpha-\beta,\alpha-\beta+i}(A(\ln t - \ln s)^{\alpha-\beta})h(s)\frac{ds}{s}\right|$$

$$\leq \int_1^t (\ln t - \ln s)^{\alpha-\beta+i-1}\mathbf{E}_{\alpha-\beta,\alpha-\beta+i}(|A|)(\ln s)^{\sigma}(1-\ln s)^{\tau}\frac{ds}{s}$$

$$\leq \int_1^t (\ln t - \ln s)^{\alpha-\beta+i+\tau-1}(\ln s)^{\sigma}\frac{ds}{s}\mathbf{E}_{\alpha-\beta,\alpha-\beta+i}(|A|)$$

$$= (\ln t)^{\alpha-\beta+i+\sigma+\tau}\int_0^1 (1-w)^{\alpha-\beta+i+\tau-1}w^{\sigma}dw\mathbf{E}_{\alpha-\beta,\alpha-\beta+i}(|A|).$$

Then $t \to \int_1^t (\ln t - \ln s)^{\alpha-\beta+i-1}\mathbf{E}_{\alpha-\beta,\alpha-\beta+i}(A(\ln t - \ln s)^{\alpha-\beta})h(s)\frac{ds}{s}$ is continuous on $[1,e]$ for all $i \in \mathbf{N}_1^p$.

For $i \in \mathbf{N}_1^{n-1}$, we have

$$^{rh}D_{1^+}^{\alpha-i}x(t) = \frac{\left(t\frac{d}{dt}\right)^{n-i}\left[\int_1^t (\ln t-\ln s)^{n-\alpha-1}x(s)\frac{ds}{s}\right]}{\Gamma(n-\alpha)}$$

$$= \frac{\left(t\frac{d}{dt}\right)^{n-i}\left[\int_1^t (\ln t-\ln s)^{n-\alpha-1}\left(\sum_{\nu=1}^n x_\nu s^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(A(\ln s)^{\alpha-\beta})+\int_1^s (\ln s-\ln u)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(A(\ln s-\ln u)^{\alpha-\beta})h(u)\frac{du}{u}\right)ds\right]}{\Gamma(n-\alpha)}$$

$$= \frac{\left(t\frac{d}{dt}\right)^{n-i}\left[\sum_{\nu=1}^n x_\nu \int_1^t (\ln t-\ln s)^{n-\alpha-1}\sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha-\nu+1)}(\ln s)^{\chi(\alpha-\beta)+\alpha-\nu}\frac{ds}{s}\right]}{\Gamma(n-\alpha)}$$

$$+\frac{\left(t\frac{d}{dt}\right)^{n-i}\left[\int_1^t (\ln t-\ln s)^{n-\alpha-1}\int_1^s \sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha)}(\ln s-\ln u)^{\chi(\alpha-\beta)+\alpha-1}h(u)\frac{du}{u}\frac{ds}{s}\right]}{\Gamma(n-\alpha)}$$

$$= \frac{\left(t\frac{d}{dt}\right)^{n-i}\left[\sum_{\nu=1}^n x_\nu \sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha-\nu+1)}(\ln t)^{\chi(\alpha-\beta)+n-\nu}\int_0^1 (1-w)^{n-\alpha-1}w^{\chi(\alpha-\beta)+\alpha-\nu}dw\right]}{\Gamma(n-\alpha)}$$

$$+\frac{\left(t\frac{d}{dt}\right)^{n-i}\left[\sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha)}\int_1^t\int_u^t (\ln t-\ln s)^{n-\alpha-1}(\ln s-\ln u)^{\chi(\alpha-\beta)+\alpha-1}\frac{ds}{s}h(u)\frac{du}{u}\right]}{\Gamma(n-\alpha)}$$

$$= \left(t\frac{d}{dt}\right)^{n-i}\left[\sum_{\nu=1}^n x_\nu \sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+n-\nu+1)}(\ln t)^{\chi(\alpha-\beta)+n-\nu}\right]$$

$$+\frac{\left(t\frac{d}{dt}\right)^{n-i}\left[\sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha)}\int_1^t (\ln t-\ln u)^{\chi(\alpha-\beta)+n-1}\int_0^1 (1-w)^{n-\alpha-1}w^{\chi(\alpha-\beta)+\alpha-1}dwh(u)\frac{du}{u}\right]}{\Gamma(n-\alpha)}$$

$$= \left(t\frac{d}{dt}\right)^{n-i}\left[\sum_{\nu=1}^n x_\nu \sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+n-\nu+1)}(\ln t)^{\chi(\alpha-\beta)+n-\nu}\right]$$

$$+\left(t\frac{d}{dt}\right)^{n-i}\left[\sum_{\chi=0}^{\infty}\frac{A^{\chi}}{\Gamma(\chi(\alpha-\beta)+n)}\int_1^t(\ln t-\ln u)^{\chi(\alpha-\beta)+n-1}h(u)\frac{du}{u}\right]$$

$$=\sum_{\nu=1}^{i}x_{\nu}\frac{1}{\Gamma(i-\nu+1)}(\ln t)^{i-\nu}+\sum_{\nu=1}^{n}x_{\nu}\sum_{\chi=1}^{\infty}\frac{A^{\chi}}{\Gamma(\chi(\alpha-\beta)+i-\nu+1)}(\ln t)^{\chi(\alpha-\beta)+i-\nu}$$

$$+\sum_{\chi=0}^{\infty}\frac{A^{\chi}}{\Gamma(\chi(\alpha-\beta)+i)}\int_1^t(\ln t-\ln u)^{\chi(\alpha-\beta)+i-1}h(u)\frac{du}{u}.$$

Then

$$\begin{aligned}{}^{rh}D_{1^+}^{\alpha-i}x(t)&=\sum_{\nu=1}^{i}x_{\nu}\frac{1}{\Gamma(i-\nu+1)}(\ln t)^{i-\nu}+\sum_{\nu=1}^{n}x_{\nu}\sum_{\chi=1}^{\infty}\frac{A^{\chi}}{\Gamma(\chi(\alpha-\beta)+i-\nu+1)}(\ln t)^{\chi(\alpha-\beta)+i-\nu}\\&+\sum_{\chi=0}^{\infty}\frac{A^{\chi}}{\Gamma(\chi(\alpha-\beta)+i)}\int_1^t(\ln t-\ln u)^{\chi(\alpha-\beta)+i-1}h(u)\frac{du}{u}.\end{aligned}\tag{2.8}$$

Similarly we get for $j\in\mathbf{N}_1^{p-1}$ that

$${}^{rh}D_{1^+}^{\beta-j}x(t)=\frac{\left(t\frac{d}{dt}\right)^{p-j}\left[\int_1^t(t-s)^{p-\beta-1}x(s)ds\right]}{\Gamma(p-\beta)}$$

$$=\frac{\left(t\frac{d}{dt}\right)^{p-j}\left[\int_1^t(\ln t-\ln s)^{p-\beta-1}\left(\sum_{\nu=1}^{n}x_{\nu}(\ln s)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(A(\ln s)^{\alpha-\beta})+\int_1^s(\ln s-\ln u)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(A(\ln s-\ln u)^{\alpha-\beta})h(u)\frac{du}{u}\right)\frac{ds}{s}\right]}{\Gamma(p-\beta)}$$

$$=\frac{\left(t\frac{d}{dt}\right)^{p-j}\left[\sum_{\nu=1}^{n}x_{\nu}\int_1^t(\ln t-\ln s)^{p-\beta-1}\sum_{\chi=0}^{\infty}\frac{A^{\chi}}{\Gamma(\chi(\alpha-\beta)+\alpha-\nu+1)}(\ln s)^{\chi(\alpha-\beta)+\alpha-\nu}\frac{ds}{s}\right]}{\Gamma(p-\beta)}$$

$$+\frac{\left(t\frac{d}{dt}\right)^{p-j}\left[\int_1^t(\ln t-\ln s)^{p-\beta-1}\int_1^s\sum_{\chi=0}^{\infty}\frac{A^{\chi}}{\Gamma(\chi(\alpha-\beta)+\alpha)}(\ln s-\ln u)^{\chi(\alpha-\beta)+\alpha-1}h(u)\frac{du}{u}\frac{ds}{s}\right]}{\Gamma(p-\beta)}$$

$$=\frac{\left(t\frac{d}{dt}\right)^{p-j}\left[\sum_{\nu=1}^{n}x_{\nu}\sum_{\chi=0}^{\infty}\frac{A^{\chi}}{\Gamma(\chi(\alpha-\beta)+\alpha-\nu+1)}(\ln t)^{\chi(\alpha-\beta)+p-\beta+\alpha-\nu}\int_0^1(1-w)^{p-\beta-1}w^{\chi(\alpha-\beta)+\alpha-\nu}dw\right]}{\Gamma(p-\beta)}$$

$$+\frac{\left(t\frac{d}{dt}\right)^{p-j}\left[\sum_{\chi=0}^{\infty}\frac{A^{\chi}}{\Gamma(\chi(\alpha-\beta)+\alpha)}\int_1^t\int_u^t(\ln t-\ln s)^{p-\beta-1}(\ln s-\ln u)^{\chi(\alpha-\beta)+\alpha-1}\frac{ds}{s}h(u)\frac{du}{u}\right]}{\Gamma(p-\beta)}$$

$$=\left(t\frac{d}{dt}\right)^{p-j}\left[\sum_{\nu=1}^{n}x_{\nu}\sum_{\chi=0}^{\infty}\frac{A^{\chi}}{\Gamma(\chi(\alpha-\beta)+p-\beta+\alpha-\nu+1)}(\ln t)^{\chi(\alpha-\beta)+p-\beta+\alpha-\nu}\right]$$

$$+\frac{\left(t\frac{d}{dt}\right)^{p-j}\left[\sum_{\chi=0}^{\infty}\frac{A^{\chi}}{\Gamma(\chi(\alpha-\beta)+\alpha)}\int_1^t(\ln t-\ln u)^{\chi(\alpha-\beta)+p-\beta+\alpha-1}\int_0^1(1-w)^{p-\beta-1}w^{\chi(\alpha-\beta)+\alpha-1}dwh(u)\frac{du}{u}\right]}{\Gamma(p-\beta)}$$

$$= \left(t\frac{d}{dt}\right)^{p-j}\left[\sum_{\nu=1}^{n} x_\nu \sum_{\chi=0}^{\infty} \frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+p-\beta+\alpha-\nu+1)}(\ln t)^{\chi(\alpha-\beta)+p-\beta+\alpha-\nu}\right]$$

$$+\left(t\frac{d}{dt}\right)^{p-j}\left[\sum_{\chi=0}^{\infty} \frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+p-\beta+\alpha)}\int_1^t (\ln t-\ln u)^{\chi(\alpha-\beta)+p-\beta+\alpha-1}h(u)\frac{du}{u}\right]$$

$$= \left(t\frac{d}{dt}\right)^{p-j}\left[\sum_{\nu=1}^{n} x_\nu \sum_{\chi=0}^{\infty} \frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha-\beta+p-\nu+1)}(\ln t)^{\chi(\alpha-\beta)+\alpha-\beta+p-\nu}\right]$$

$$+\sum_{\chi=0}^{\infty} \frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha-\beta+j)}\int_1^t (\ln t-\ln u)^{\chi(\alpha-\beta)+\alpha-\beta+j-1}h(u)\frac{du}{u}.$$

It follows that

$$\begin{aligned} {}^{rh}D_{1^+}^{\beta-i}x(t) &= \sum_{\nu=1}^{n} x_\nu \sum_{\chi=0}^{\infty} \frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha-\beta+i-\nu+1)}(\ln t)^{\chi(\alpha-\beta)+\alpha-\beta+i-\nu} \\ &+\sum_{\chi=0}^{\infty} \frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha-\beta+i)}\int_1^t (\ln t-\ln u)^{\chi(\alpha-\beta)+\alpha-\beta+i-1}h(u)\frac{du}{u}. \end{aligned} \tag{2.9}$$

From (2.8) and (2.9), we get for $i \in \mathbf{N}_1^{p-1}$ that

$$\begin{aligned} [{}^{rh}D_{1^+}^{\alpha-i}x - A\,{}^{rh}D_{1^+}^{\beta-i}x](t) &= \sum_{\nu=1}^{i} x_\nu \frac{1}{\Gamma(i-\nu+1)}(\ln t)^{i-\nu} \\ &+\sum_{\nu=1}^{n} x_\nu \sum_{\chi=1}^{\infty} \frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+i-\nu+1)}(\ln t)^{\chi(\alpha-\beta)+i-\nu} \\ &+\sum_{\chi=0}^{\infty} \frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+i)}\int_1^t (\ln t-\ln u)^{\chi(\alpha-\beta)+i-1}h(u)\frac{du}{u} \\ &-A\left[\sum_{\nu=1}^{n} x_\nu \sum_{\chi=0}^{\infty} \frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha-\beta+i-\nu+1)}(\ln t)^{\chi(\alpha-\beta)+\alpha-\beta+i-\nu}\right. \\ &\left.+\sum_{\chi=0}^{\infty} \frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha-\beta+i)}\int_1^t (\ln t-\ln u)^{\chi(\alpha-\beta)+\alpha-\beta+i-1}h(u)\frac{du}{u}\right] \\ &= \sum_{\nu=1}^{i} x_\nu \frac{1}{\Gamma(i-\nu+1)}(\ln t)^{i-\nu} + \int_1^t \frac{(\ln t-\ln u)^{i-1}}{\Gamma(i)}h(u)\frac{du}{u}. \end{aligned} \tag{2.10}$$

We have

$$\begin{aligned} {}^{h}I_{1^+}^{n-\alpha}x(t) &= \sum_{\nu=1}^{n} x_\nu \sum_{\chi=0}^{\infty} \frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+n-\nu+1)}(\ln t)^{\chi(\alpha-\beta)+n-\nu} \\ &+\sum_{\chi=0}^{\infty} \frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+n)}\int_1^t (\ln t-\ln u)^{\chi(\alpha-\beta)+n-1}h(u)\frac{du}{u} \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} {}^{h}I_{1^+}^{p-\beta}x(t) &= \sum_{\nu=1}^{n} x_\nu \sum_{\chi=0}^{\infty} \frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+p-\beta+\alpha-\nu+1)}(\ln t)^{\chi(\alpha-\beta)+p-\beta+\alpha-\nu} \\ &+\sum_{\chi=0}^{\infty} \frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+p-\beta+\alpha)}\int_1^t (\ln t-\ln u)^{\chi(\alpha-\beta)+p-\beta+\alpha-1}h(u)\frac{du}{u}. \end{aligned} \tag{2.12}$$

It follows for $i \in \mathbf{N}_1^{p-1}$ from (2.8), (2.12) that

$$
\begin{aligned}
&[{}^{rh}D_{1^+}^{\alpha-p}x - A^h I_{1^+}^{p-\beta}x](t) = \sum_{\nu=1}^{p} x_\nu \frac{1}{\Gamma(n-\nu+1)}(\ln t)^{p-\nu} \\
&\sum_{\nu=1}^{n} x_\nu \sum_{\chi=1}^{\infty} \frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+p-\nu+1)}(\ln t)^{\chi(\alpha-\beta)+p-\nu} \\
&+\sum_{\chi=0}^{\infty} \frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+p)} \int_1^t (\ln t - \ln u)^{\chi(\alpha-\beta)+p-1} h(u)\frac{du}{u} \\
&-A\left[\sum_{\nu=1}^{n} x_\nu \sum_{\chi=0}^{\infty} \frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+p-\beta+\alpha-\nu+1)}(\ln t)^{\chi(\alpha-\beta)+p-\beta+\alpha-\nu}\right. \\
&\left.+\sum_{\chi=0}^{\infty} \frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+p-\beta+\alpha)} \int_1^t (\ln t - \ln u)^{\chi(\alpha-\beta)+p-\beta+\alpha-1} h(u)\frac{du}{u}\right] \\
&= \sum_{\nu=1}^{p} x_\nu \frac{1}{\Gamma(p-\nu+1)}(\ln t)^{p-\nu} + \int_1^t \frac{(\ln t - \ln u)^{p-1}}{\Gamma(p)} h(u)\frac{du}{u}.
\end{aligned}
\tag{2.13}
$$

For $i \in \mathbf{N}_{p+1}^{n-1}$, we have $\alpha - \beta + i - \nu \geq \alpha - \beta + p + 1 - (n-1) = \alpha - \beta + p - n + 2 \geq 0$. From (2.8), we get

$$
\begin{aligned}
&{}^{rh}D_{1^+}^{\alpha-i}x(t) = \sum_{\nu=1}^{i} x_\nu \frac{1}{\Gamma(i-\nu+1)}(\ln t)^{i-\nu} + A\sum_{\nu=1}^{n} x_\nu (\ln t)^{\alpha-\beta+i-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\beta+i-\nu+1}(A(\ln t)^{\alpha-\beta}) \\
&+\int_1^t (\ln t - \ln u)^{i-1}\mathbf{E}_{\alpha-\beta,i}(A(\ln t - \ln u)^{\alpha-\beta})h(u)\frac{du}{u}.
\end{aligned}
\tag{2.14}
$$

One sees easily from (2.10), (2.13), (2.14) and (2.11) that

$$
\begin{aligned}
&[{}^{rh}D_{1^+}^{\alpha-i}x - A^{rh}D_{1^+}^{\beta-i}x] \in C[1,e], i \in \mathbf{N}_1^{p-1}, \quad [{}^{rh}D_{1^+}^{\alpha-p} - A^h I_{1^+}^{p-\beta}x] \in C[1,e], \\
&{}^{rh}D_{1^+}^{\alpha-i}x \in C[1,e], \ i \in \mathbf{N}_{p+1}^{n-1}, \ {}^h I_{1^+}^{n-\alpha}x \in C[1,e]
\end{aligned}
$$

and

$$
\begin{aligned}
&[{}^{rh}D_{1^+}^{\alpha-i}x - A^{rh}D_{1^+}^{\beta-i}x](1) = x_i, i \in \mathbf{N}_1^{p-1}, \quad [{}^{rh}D_{1^+}^{\alpha-p} - A^h I_{1^+}^{p-\beta}x](1) = x_p, \\
&{}^{rh}D_{1^+}^{\alpha-i}x(1) = x_i, \ i \in \mathbf{N}_{p+1}^{n-1}, \ {}^h I_{1^+}^{n-\alpha}x(1) = x_n.
\end{aligned}
$$

We now by direct computation get

$$
\begin{aligned}
&{}^{rh}D_{1^+}^{\alpha}x(t) = \sum_{\nu=1}^{n} x_\nu \sum_{\chi=1}^{\infty} \frac{A^\chi}{\Gamma(\chi(\alpha-\beta)-\nu+1)}(\ln t)^{\chi(\alpha-\beta)-\nu} \\
&+h(t) + \sum_{\chi=1}^{\infty} \frac{A^\chi}{\Gamma(\chi(\alpha-\beta))} \int_1^t (\ln t - \ln u)^{\chi(\alpha-\beta)-1} h(u)\frac{du}{u}
\end{aligned}
$$

and

$$
\begin{aligned}
&{}^{rh}D_{1^+}^{\beta}x(t) = \sum_{\nu=1}^{n} x_\nu \sum_{\chi=0}^{\infty} \frac{A^\chi}{\Gamma(\chi(\alpha-\beta)-\beta+\alpha-\nu+1)}(\ln t)^{\chi(\alpha-\beta)-\beta+\alpha-\nu} \\
&+\sum_{\chi=0}^{\infty} \frac{A^\chi}{\Gamma(\chi(\alpha-\beta)-\beta+\alpha)} \int_1^t (\ln t - \ln u)^{\chi(\alpha-\beta)-\beta+\alpha-1} h(u)\frac{du}{u}.
\end{aligned}
$$

It follows that ${}^{rh}D^{\alpha}_{1^+}x(t) - A^{rh}D^{\beta}_{1^+}x(t) = h(t), t \in (1, e]$.

From above discussion, we know from that x is a solution of (2.1) satisfies (2.3). The proof is completed. □

Remark 2.2. Consider the following fractional differential equation:

$$ {}^{rh}D^{\frac{3}{2}}_{1^+}x(t) - {}^{rh}D^{\frac{3}{4}}_{1^+}x(t) = \ln t, t \in (1, e]. $$

Corresponding to (2.1), $\alpha = \frac{3}{2}, \beta = \frac{3}{4}, A = 1$ and $h(t) = \ln t$. By Lemma 2.1, it has solutions

$$ x(t) = x_1(\ln t)^{\frac{1}{2}}\mathbf{E}_{3/4,3/2}(\ln t^{3/4}) + x_2(\ln t)^{-\frac{1}{2}}\mathbf{E}_{3/4,1/2}((\ln t)^{3/4}) $$

$$ + \int_1^t(\ln t - \ln s)^{\frac{1}{2}}\mathbf{E}_{3/4,3/2}((\ln t - \ln s)^{3/4}) \ln s\frac{ds}{s}, t \in (1, e]. $$

One can get

$$ {}^{rh}D^{\alpha-1}_{1^+}x(t) = {}^{rh}D^{\frac{1}{2}}_{1^+}x(t) = x_1 + x_1(\ln t)^{3/4}\mathbf{E}_{3/4,7/4}((\ln t)^{3/4}) + x_2(\ln t)^{-\frac{1}{4}}\mathbf{E}_{3/4,3/4}((\ln t)^{3/4}) $$

$$ + \int_1^t \mathbf{E}_{3/4,1}((\ln t - \ln u)^{3/4})h(u)\frac{du}{u}, t \in (1, e], $$

$$ I^{1-\beta}_{0^+}x(t) = I^{\frac{1}{4}}_{0^+}x(t) = x_1t^{\frac{3}{4}}\mathbf{E}_{3/4,7/4}((t^{3/4}) + x_2t^{-\frac{1}{4}}\mathbf{E}_{3/4,3/4}((t^{3/4}) $$

$$ + \int_0^t(t - u)^{\frac{3}{4}}\mathbf{E}_{3/4,7/4}((t - s)^{3/4})sds, t \in (0, 1], $$

and

$$ {}^{h}I^{2-\alpha}_{1^+}x(t) = {}^{h}I^{\frac{1}{2}}_{1^+}x(t) = x_1 \ln t\mathbf{E}_{3/4,2}((\ln t)^{3/4}) + x_2\mathbf{E}_{3/4,1}((\ln t)^{3/4}) $$

$$ + \int_1^t(\ln t - \ln u)\mathbf{E}_{3/42}((t - u)^{3/4})h(u)\frac{du}{u}, t \in (1, e]. $$

One finds that $x, {}^{rh}D^{\alpha-1}_{1^+}x, {}^{h}I^{1-\beta}_{1^+}x$ are not continuous on $[1, e]$, but both ${}^{rh}D^{\alpha-1}_{1^+}x - {}^{h}I^{1-\beta}_{1^+}x$ and ${}^{h}I^{2-\alpha}_{1^+}x$ are continuous on $[1, e]$. □

Remark 2.3. Consider the following equation:

$$ {}^{rh}D^{\alpha}_{1^+}x(t) - A^{rh}D^{\beta}_{1^+}x(t) = h(t), t \in (1, e], $$

where $\alpha \in (1, 2), \beta \in (0, \alpha - 1]$, $h \in C(1, e)$ and $|h(t)| \leq (\ln t)^k(1 - \ln t)^l$ for all $t \in (1, e)$, $k > -1$ and $l \in \left(\max\{-1 - k, -\frac{1}{2}\}, 0\right]$, $A \in \mathbf{R}$.

By Lemma 2.1, it has solutions

$$ x(t) = \sum_{\nu=1}^{2} x_\nu(\ln t)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(A(\ln t)^{\alpha-\beta}) $$

$$ + \int_1^t(\ln t - \ln s)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(A(] \ln t - \ln s)^{\alpha-\beta})h(s)\frac{ds}{s}, t \in (1, e]. $$

One can get

$$ {}^{rh}D^{\alpha-1}_{1^+}x(t) = x_1 + \sum_{\nu=1}^{2} x_\nu(\ln t)^{\alpha-\beta+i-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\beta+i-\nu+1}(A(\ln t)^{\alpha-\beta}) $$

$$ + \int_1^t(\ln t - \ln u)^{i-1}\mathbf{E}_{\alpha-\beta,i}(A(\ln t - \ln u)^{\alpha-\beta})h(u)\frac{du}{u}, t \in (1, e], $$

$$^{h}I_{1^+}^{1-\beta}x(t)=\sum_{\nu=1}^{2}x_\nu(\ln t)^{\alpha-\beta+1-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\beta+2-\nu}(A(\ln t^{\alpha-\beta})$$

$$+\int_1^t(\ln t-\ln u)^{\alpha-\beta}\mathbf{E}_{\alpha-\beta,\alpha-\beta+1}(A(\ln t-\ln u)^{\alpha-\beta})h(u)\frac{du}{u},t\in(1,e]$$

and

$$^{h}I_{1^+}^{2-\alpha}x(t)=\sum_{\nu=1}^{2}x_\nu(\ln t)^{2-\nu}\mathbf{E}_{\alpha-\beta,3-\nu}(A(\ln t)^{\alpha-\beta})$$

$$+\int_1^t(\ln t-\ln u)\mathbf{E}_{\alpha-\beta,2}(A(\ln t-\ln u)^{\alpha-\beta})h(u)\frac{du}{u},t\in(1,e].$$

One finds that x is not continuous on $[1,e]$, however $^{rh}D_{1^+}^{\alpha-1}x$, $^{h}I_{1^+}^{1-\beta}x$ and $^{h}I_{1^+}^{2-\alpha}x$ are continuous on $[1,e]$. □

Now we give an exact expression of piecewise continuous solutions of (2.2).

Lemma 2.2. Suppose that $\alpha-\beta+p-n\geq 0$, there exist constants $\sigma>-1,\tau\in(-n-\sigma,0]$ such that $|h(t)|\leq(\ln t)^\sigma(1-\ln t)^\tau$ for all $t\in(1,e)$. Then x is a piecewise continuous solution of (2.2) if and only if there exist constants $c_{j,\nu}\in\mathbf{R}(j\in\mathbf{N}_0^m,\nu\in\mathbf{N}_1^n)$ such that

$$x(t)=\sum_{j=0}^{\tau}\sum_{\nu=1}^{n}c_{j,\nu}(\ln t-\ln t_j)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(A(\ln t-\ln t_j)^{\alpha-\beta})$$
$$+\int_1^t(\ln t-\ln s)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(A(\ln t-\ln s)^{\alpha-\beta})h(s)\frac{ds}{s},t\in(t_\tau,t_{\tau+1}],\tau\in\mathbf{N}_0^m. \tag{2.15}$$

Proof. The proof is divide into two steps:

Step 1. Suppose that x satisfies (2.15). We prove that x is a piecewise solution of (2.2).

Similar to Step 2 in the proof of Lemma 2.1, we get that $t\to\int_1^t(\ln t-\ln s)^{i-1}\mathbf{E}_{\alpha-\beta,i}(A(\ln t-\ln s)^{\alpha-\beta})h(s)\frac{ds}{s}$ is continuous on $[1,e]$ for $i\in\mathbf{N}_1^n$ and $t\to\int_1^t(\ln t-\ln s)^{\alpha-\beta+i-1}\mathbf{E}_{\alpha-\beta,\alpha-\beta+i}(A(\ln t-\ln s)^{\alpha-\beta})h(s)\frac{ds}{s}$ is continuous on $[1,e]$ for $i\in\mathbf{N}_1^p$.

For $t>0$, $\epsilon_1,\epsilon_2,\epsilon_3>0$ with $1+\epsilon_1+\epsilon_2+\epsilon_3\in(1,t)$ and $\epsilon=\max\{\epsilon_1,\epsilon_2,\epsilon_3\}$, we have

$$\left|\int_{1+\epsilon_2+\epsilon_3}^{t-\epsilon_1}(\ln t-\ln s)^{n-\alpha-1}\int_{1+\epsilon_3}^{s-\epsilon_2}(\ln s-\ln u)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(A(\ln s-\ln u)^{\alpha-\beta})h(u)\frac{du}{u}\frac{ds}{s}\right|$$

$$\leq\sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha)}\int_{1+\epsilon_2+\epsilon_3}^{t-\epsilon_1}\int_{1+\epsilon_3}^{s-\epsilon_2}(\ln t-\ln s)^{n-\alpha-1}(\ln s-\ln u)^{\chi(\alpha-\beta)+\alpha-1}(\ln u)^\sigma(1-\ln u)^\tau\frac{du}{u}\frac{ds}{s}$$

$$\leq\sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha)}\int_{1+\epsilon_2+\epsilon_3}^{t-\epsilon_1}(\ln t-\ln s)^{n-\alpha-1}(\ln s)^{\chi(\alpha-\beta)+\tau+\sigma+\alpha}\int_{\frac{\ln(1+\epsilon_3)}{\ln s}}^{\frac{\ln(s-\epsilon_2)}{\ln s}}(1-w)^{\chi(\alpha-\beta)+\tau+\alpha-1}w^\sigma dw\frac{ds}{s}$$

$$\leq\sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha)}\int_{1+\epsilon_2+\epsilon_3}^{t-\epsilon_1}(\ln t-\ln s)^{n-\alpha-1}(\ln s)^{\chi(\alpha-\beta)+\tau+\sigma+\alpha}\frac{ds}{s}\mathbf{B}(\alpha+\tau,\sigma+1)$$

$$\leq\sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha)}(\ln t)^{\chi(\alpha-\beta)+\sigma+\tau+n}\mathbf{B}(n-\alpha,\alpha+\sigma+\tau+1)\mathbf{B}(\alpha+\tau,\sigma+1)$$

$$=t^{n+\sigma+\tau}\mathbf{E}_{\alpha-\beta,\alpha}(A(\ln t)^{\alpha-\beta})\mathbf{B}(n-\alpha,\alpha+\sigma+\tau+1)\mathbf{B}(\alpha+\tau,\sigma+1)$$

and

$$\Big|\int_{1+\epsilon_3}^{t-\epsilon_2-\epsilon_1}\int_{u+\epsilon_2}^{t-\epsilon_1}(\ln t-\ln s)^{n-\alpha-1}(\ln s-\ln u)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(A(\ln s-\ln u)^{\alpha-\beta})\frac{ds}{s}h(u)\frac{du}{u}\Big|$$

$$\leq\sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha)}\int_{1+\epsilon_3}^{t-\epsilon_2-\epsilon_1}\int_{u+\epsilon_2}^{t-\epsilon_1}(\ln t-\ln s)^{n-\alpha-1}(\ln s-\ln u)^{\chi(\alpha-\beta)+\alpha-1}\frac{ds}{s}(\ln u)^{\sigma}(1-\ln u)^{\tau}\frac{du}{u}$$

$$\leq\sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha)}\int_{1+\epsilon_3}^{t-\epsilon_2-\epsilon_1}(\ln t-\ln u)^{\chi(\alpha-\beta)+n+\tau-1}\int_{\frac{\ln(u+\epsilon_2)}{\ln t-\ln u}}^{\frac{\ln(t-\epsilon_1)-\ln u}{\ln t-\ln u}}(1-w)^{n-\alpha-1}w^{\chi(\alpha-\beta)+\tau+\alpha-1}dwu^{\sigma}\frac{du}{u}$$

$$\leq\sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha)}\int_{1+\epsilon_3}^{t-\epsilon_2-\epsilon_1}(\ln t-\ln u)^{\chi(\alpha-\beta)+n+\tau-1}u^{\sigma}\frac{du}{u}\mathbf{B}(n-\alpha,\tau+\alpha)$$

$$\leq\sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha)}(\ln t)^{\chi(\alpha-\beta)+\sigma+n+\tau}\mathbf{B}(n+\tau,\sigma+1)\mathbf{B}(n-\alpha,\tau+\alpha)$$

$$=(\ln t)^{n+\sigma+\tau}\mathbf{E}_{\alpha-\beta,\alpha}(A(\ln t)^{\alpha-\beta})\mathbf{B}(n+\tau,\sigma+1)\mathbf{B}(n-\alpha,\tau+\alpha).$$

Then

$$\int_1^t(\ln t-\ln s)^{n-\alpha-1}\int_1^s(\ln s-\ln u)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(A(\ln s-\ln u)^{\alpha-\beta})h(u)\frac{du}{u}\frac{ds}{s}$$

$$=\lim_{\epsilon\to0^+}\int_{1+\epsilon_2+\epsilon_3}^{t-\epsilon_1}(\ln t-\ln s)^{n-\alpha-1}\int_{1+\epsilon_3}^{s-\epsilon_2}(\ln s-\ln u)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(A(\ln s-\ln u)^{\alpha-\beta})h(u)\frac{du}{u}\frac{ds}{s}$$

$$=\lim_{\epsilon\to0^+}\int_{1+\epsilon_3}^{t-\epsilon_2-\epsilon_1}\int_{u+\epsilon_2}^{t-\epsilon_1}(\ln t-\ln s)^{n-\alpha-1}(\ln s-\ln u)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(A(\ln s-\ln u)^{\alpha-\beta})\frac{ds}{s}h(u)\frac{du}{u}$$

$$=\int_1^t\int_u^t(\ln t-\ln s)^{n-\alpha-1}(\ln s-\ln u)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(A(\ln s-\ln u)^{\alpha-\beta})\frac{ds}{s}h(u)\frac{du}{u}.$$

By Definition 2.1 and (2.11), we have for $t\in(t_\sigma,t_{\sigma+1}]$ and $i\in\mathbf{N}_1^{n-1}$ that

$$^{rh}D_{1^+}^{\alpha-i}x(t)=\frac{\left(t\frac{d}{dt}\right)^{n-i}\left[\int_1^t(\ln t-\ln s)^{n-\alpha-1}x(s)\frac{ds}{s}\right]}{\Gamma(n-\alpha)}$$

$$=\frac{\left[\left(t\frac{d}{dt}\right)^{n-i}\sum_{\tau=0}^{\sigma-1}\int_{t_\tau}^{t_{\tau+1}}(\ln t-\ln s)^{n-\alpha-1}x(s)\frac{ds}{s}+\int_{t_\sigma}^{t}(\ln t-\ln s)^{n-\alpha-1}x(s)\frac{ds}{s}\right]}{\Gamma(n-\alpha)}$$

$$=\frac{\left(t\frac{d}{dt}\right)^{n-i}\left[\sum_{\tau=0}^{\sigma-1}\int_{t_\tau}^{t_{\tau+1}}(\ln t-\ln s)^{n-\alpha-1}\sum_{j=0}^{\tau}\sum_{v=1}^{n}c_{j,v}(\ln s-\ln t_j)^{\alpha-v}\mathbf{E}_{\alpha-\beta,\alpha-v+1}(A(\ln s-\ln t_j)^{\alpha-\beta})\frac{ds}{s}\right]}{\Gamma(n-\alpha)}$$

$$+\frac{\left(t\frac{d}{dt}\right)^{n-i}\left[\int_{t_\sigma}^{t}(\ln t-\ln s)^{n-\alpha-1}\sum_{j=0}^{\sigma}\sum_{v=1}^{n}c_{j,v}(\ln s-\ln t_j)^{\alpha-v}\mathbf{E}_{\alpha-\beta,\alpha-v+1}(A(\ln s-\ln t_j)^{\alpha-\beta})ds\right]}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\int_1^t(\ln t-\ln s)^{n-\alpha-1}\int_1^s(\ln s-\ln u)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(A(\ln s-\ln u)^{\alpha-\beta})h(u)\frac{du}{u}\frac{ds}{s}\right]}{\Gamma(n-\alpha)}$$

by changing the order of the sum and integral respectively

$$= \frac{\left(t\frac{d}{dt}\right)^{n-i}\left[\sum_{j=0}^{\sigma-1}\sum_{\tau=j}^{\sigma-1}\sum_{\nu=1}^{n} c_{j,\nu}\int_{t_\tau}^{t_{\tau+1}}(\ln t-\ln s)^{n-\alpha-1}(\ln s-\ln t_j)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(A(\ln s-\ln t_j)^{\alpha-\beta})\frac{ds}{s}\right]}{\Gamma(n-\alpha)}$$

$$+\frac{\left(t\frac{d}{dt}\right)^{n-i}\left[\sum_{j=0}^{\sigma}\sum_{\nu=1}^{n} c_{j,\nu}\int_{t_\sigma}^{t}(\ln t-\ln s)^{n-\alpha-1}(\ln s-\ln t_j)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(A(\ln s-\ln t_j)^{\alpha-\beta})\frac{ds}{s}\right]}{\Gamma(n-\alpha)}$$

$$+\frac{\left(t\frac{d}{dt}\right)^{n-i}\left[\int_1^t\int_u^t(\ln t-\ln s)^{n-\alpha-1}(\ln s-\ln u)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(A(\ln s-\ln u)^{\alpha-\beta})\frac{ds}{s}h(u)\frac{du}{u}\right]}{\Gamma(n-\alpha)}$$

$$= \frac{\left(t\frac{d}{dt}\right)^{n-i}\left[\sum_{j=0}^{\sigma}\sum_{\nu=1}^{n} c_{j,\nu}\int_{t_j}^{t}(\ln t-\ln s)^{n-\alpha-1}(\ln s-\ln t_j)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(A(\ln s-\ln t_j)^{\alpha-\beta})\frac{ds}{s}\right]}{\Gamma(n-\alpha)}$$

$$+\frac{\left(t\frac{d}{dt}\right)^{n-i}\left[\int_1^t\int_u^t(\ln t-\ln s)^{n-\alpha-1}(\ln s-\ln u)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(A(\ln s-\ln u)^{\alpha-\beta})\frac{ds}{s}h(u)\frac{du}{u}\right]}{\Gamma(n-\alpha)}$$

$$= \frac{\left(t\frac{d}{dt}\right)^{n-i}\left[\sum_{j=0}^{\sigma}\sum_{\nu=1}^{n} c_{j,\nu}\int_{t_j}^{t}(\ln t-\ln s)^{n-\alpha-1}(\ln s-\ln t_j)^{\alpha-\nu}\sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha-\nu+1)}(\ln s-\ln t_j)^{\chi(\alpha-\beta)}\frac{ds}{s}\right]}{\Gamma(n-\alpha)}$$

$$+\frac{\left(t\frac{d}{dt}\right)^{n-i}\left[\int_1^t\int_u^t(\ln t-\ln s)^{n-\alpha-1}(\ln s-\ln u)^{\alpha-1}\sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha)}(\ln s-\ln u)^{\chi(\alpha-\beta)}\frac{ds}{s}h(u)\frac{du}{u}\right]}{\Gamma(n-\alpha)}$$

$$= \frac{\left(t\frac{d}{dt}\right)^{n-i}\left[\sum_{j=0}^{\sigma}\sum_{\nu=1}^{n} c_{j,\nu}\sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha-\nu+1)}\int_{t_j}^{t}(\ln t-\ln s)^{n-\alpha-1}(\ln s-\ln t_j)^{\chi(\alpha-\beta)+\alpha-\nu}\frac{ds}{s}\right]}{\Gamma(n-\alpha)}$$

$$+\frac{\left(t\frac{d}{dt}\right)^{n-i}\left[\sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha)}\int_1^t\int_u^t(\ln t-\ln s)^{n-\alpha-1}(\ln s-\ln u)^{\chi(\alpha-\beta)+\alpha-1}\frac{ds}{s}h(u)\frac{du}{u}\right]}{\Gamma(n-\alpha)} \text{ by } \frac{\ln s-\ln t_j}{\ln t-\ln t_j}=w,\ \frac{\ln s-\ln u}{\ln t-\ln u}=w$$

$$= \frac{\left(t\frac{d}{dt}\right)^{n-i}\left[\sum_{j=0}^{\sigma}\sum_{\nu=1}^{n} c_{j,\nu}\sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha-\nu+1)}(\ln t-\ln t_j)^{\chi(\alpha-\beta)+n-\nu}\int_0^1(1-w)^{n-\alpha-1}w^{\chi(\alpha-\beta)+\alpha-\nu}dw\right]}{\Gamma(n-\alpha)}$$

$$+\frac{\left(t\frac{d}{dt}\right)^{n-i}\left[\sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha)}\int_1^t(\ln t-\ln u)^{\chi(\alpha-\beta)+n-1}\int_0^1(1-w)^{n-\alpha-1}w^{\chi(\alpha-\beta)+\alpha-1}dw\,h(u)\frac{du}{u}\right]}{\Gamma(n-\alpha)}$$

$$= \left(t\frac{d}{dt}\right)^{n-i}\left[\sum_{j=0}^{\sigma}\sum_{\nu=1}^{n} c_{j,\nu}\sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+n-\nu+1)}(\ln t-\ln t_j)^{\chi(\alpha-\beta)+n-\nu}\right]$$

$$+\left(t\frac{d}{dt}\right)^{n-i}\left[\sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+n)}\int_1^t(\ln t-\ln u)^{\chi(\alpha-\beta)+n-1}h(u)\frac{du}{u}\right] \text{ by } \mathbf{B}(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

$$= \sum_{j=0}^{\sigma}\sum_{\nu=1}^{i} c_{j,\nu}\frac{1}{\Gamma(i-\nu+1)}(\ln t-\ln t_j)^{i-\nu}+\sum_{j=0}^{\sigma}\sum_{\nu=1}^{n} c_{j,\nu}\sum_{\chi=1}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+i-\nu+1)}(t-t_j)^{\chi(\alpha-\beta)+i-\nu}$$

$$+\sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+i)}\int_1^t(\ln t-\ln u)^{\chi(\alpha-\beta)+i-1}h(u)\frac{du}{u},\ i\in\mathbf{N}_1^{n-1}.$$

Similarly we get

$$ {}^{rh}D_{1^+}^{\beta-i}x(t)=\sum_{j=0}^{\sigma}\sum_{\nu=1}^{n}c_{j,\nu}\sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha-\beta+i-\nu+1)}(\ln t-\ln t_j)^{\chi(\alpha-\beta)+\alpha-\beta+i-\nu} $$

$$ +\sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha-\beta+i)}\int_1^t(\ln t-\ln u)^{\chi(\alpha-\beta)+\alpha-\beta+i-1}h(u)\frac{du}{u},\ i\in\mathbf{N}_1^{p-1}. $$

It follows that

$$ \begin{aligned} &[{}^hD_{1^+}^{\alpha-i}x-A{}^hD_{1^+}^{\beta-i}x](t)=\sum_{j=0}^{\sigma}\sum_{\nu=1}^{i}c_{j,\nu}\frac{1}{\Gamma(i-\nu+1)}(\ln t-\ln t_j)^{i-\nu}\\ &+\int_1^t\frac{(\ln t-\ln u)^{i-1}}{\Gamma(i)}h(u)\frac{du}{u},\ t\in(t_\sigma,t_{\sigma+1}],\sigma\in\mathbf{N}_0^m,i\in\mathbf{N}_1^{n-1}. \end{aligned} \tag{2.16} $$

Then $[{}^{rh}D_{1^+}^{\alpha-i}x-A{}^{rh}D_{1^+}^{\beta-i}x]|_{(t_\sigma,t_{\sigma+1}]}\in C(t_\sigma,t_{\sigma+1}](\sigma\in\mathbf{N}_0^m,i\in\mathbf{N}_1^{p-1})$.

By direct computation, we also get for $t\in(t_\sigma,t_{\sigma+1}]$ that

$$ \begin{aligned} &{}^hI_{1^+}^{n-\alpha}x(t)=\sum_{j=0}^{\sigma}\sum_{\nu=1}^{n}c_{j,\nu}\sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+n-\nu+1)}(\ln t-\ln t_j)^{\chi(\alpha-\beta)+n-\nu}\\ &+\sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+n)}\int_1^t(\ln t-\ln u)^{\chi(\alpha-\beta)+n-1}h(u)\frac{du}{u} \end{aligned} \tag{2.17} $$

and

$$ \begin{aligned} &{}^hI_{1^+}^{p-\beta}x(t)=\sum_{j=0}^{\sigma}\sum_{\nu=1}^{n}c_{j,\nu}\sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+p-\beta+\alpha-\nu+1)}(\ln t-\ln t_j)^{\chi(\alpha-\beta)+p-\beta+\alpha-\nu}\\ &+\sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+p-\beta+\alpha)}\int_1^t(\ln t-\ln u)^{\chi(\alpha-\beta)+p-\beta+\alpha-1}h(u)\frac{du}{u}. \end{aligned} \tag{2.18} $$

It follows that ${}^hI_{1^+}^{n-\alpha}x|_{(t_\sigma,t_{\sigma+1}]}\in C(t_\sigma,t_{\sigma+1}]$, ${}^hD_{1^+}^{\alpha-i}x|_{(t_\sigma,t_{\sigma+1}]}\in C(t_\sigma,t_{\sigma+1}](i\in\mathbf{N}_{p+1}^{n-1})$, $[{}^hD_{1^+}^{\alpha-p}x-A{}^hI_{1^+}^{p-\beta}x]|_{(t_\sigma,t_{\sigma+1}]}\in C(t_\sigma,t_{\sigma+1}]$ and the following limits are finite:

$$ \lim_{t\to t_\sigma^+}[{}^{rh}D_{1^+}^{\alpha-i}x-A{}^{rh}D_{1^+}^{\beta-i}x](t),\ i\in\mathbf{N}_1^{p-1},\sigma\in\mathbf{N}_0^m, $$

$$ \lim_{t\to t_\sigma^+}[{}^{rh}D_{1^+}^{\alpha-p}x-A{}^hI_{1^+}^{p-\beta}x](t),\sigma\in\mathbf{N}_0^m, $$

$$ \lim_{t\to t_\sigma^+}{}^{rh}D_{1^+}^{\alpha-i}x(t),i\in\mathbf{N}_{p+1}^{n-1},\sigma\in\mathbf{N}_0^m, $$

$$ \lim_{t\to t_\sigma^+}{}^hI_{1^+}^{n-\alpha}x(t),\sigma\in\mathbf{N}_0^m. $$

Finally, we prove that x satisfies (2.2). From $\alpha\in(n-1,n)$, for $t\in(t_\sigma,t_{\sigma+1}]$, by Definition 2.2, we have similarly to above discussion that

$$ {}^{rh}D_{1^+}^{\alpha}x(t) = \frac{\left(t\frac{d}{dt}\right)^n\left[\int_1^t(\ln t-\ln s)^{n-\alpha-1}x(s)\frac{ds}{s}\right]}{\Gamma(n-\alpha)} $$

$$ = \left(t\frac{d}{dt}\right)^n\left[\sum_{j=0}^{\sigma}\sum_{\nu=1}^{n}c_{j,\nu}\sum_{\chi=0}^{\infty}\frac{A^{\chi}}{\Gamma(\chi(\alpha-\beta)+n-\nu+1)}(\ln t-\ln t_j)^{\chi(\alpha-\beta)+n-\nu}\right] $$

$$ +\left(t\frac{d}{dt}\right)^n\left[\sum_{\chi=0}^{\infty}\frac{A^{\chi}}{\Gamma(\chi(\alpha-\beta)+n)}\int_1^t(\ln t-\ln u)^{\chi(\alpha-\beta)+n-1}h(u)du\right] $$

$$ = \sum_{j=0}^{\sigma}\sum_{\nu=1}^{n}c_{j,\nu}\sum_{\chi=1}^{\infty}\frac{A^{\chi}}{\Gamma(\chi(\alpha-\beta)-\nu+1)}(\ln t-\ln t_j)^{\chi(\alpha-\beta)-\nu} $$

$$ +h(t)+\sum_{\chi=1}^{\infty}\frac{A^{\chi}}{\Gamma(\chi(\alpha-\beta))}\int_1^t(\ln t-\ln u)^{\chi(\alpha-\beta)-1}h(u)\frac{du}{u}. $$

Similarly we have

$$ {}^{rh}D_{1^+}^{\beta}x(t) = \frac{\left(t\frac{d}{dt}\right)^p\left[\int_1^t(\ln t-\ln s)^{p-\beta-1}x(s)\frac{ds}{s}\right]}{\Gamma(p-\beta)} = \frac{\left(t\frac{d}{dt}\right)^p\left[\sum_{\tau=0}^{\sigma-1}\int_{t_\tau}^{t_{\tau+1}}(\ln t-\ln s)^{p-\beta-1}x(s)\frac{ds}{s}+\int_{t_\sigma}^{t}(\ln t-\ln s)^{p-\beta-1}x(s)\frac{ds}{s}\right]}{\Gamma(p-\beta)} $$

$$ = \left(t\frac{d}{dt}\right)^p\left[\sum_{j=0}^{\sigma}\sum_{\nu=1}^{n}c_{j,\nu}\sum_{\chi=0}^{\infty}\frac{A^{\chi}}{\Gamma(\chi(\alpha-\beta)+\alpha-\beta-\nu+2)}(\ln t-\ln t_j)^{\chi(\alpha-\beta)+\alpha-\beta-\nu}\right] $$

$$ +\left(t\frac{d}{dt}\right)^p\left[\sum_{\chi=0}^{\infty}\frac{A^{\chi}}{\Gamma(\chi(\alpha-\beta)+\alpha-\beta+1)}\int_1^t(\ln t-\ln u)^{\chi(\alpha-\beta)+\alpha-\beta}h(u)\frac{du}{u}\right] $$

$$ = \sum_{j=0}^{\sigma}\sum_{\nu=1}^{n}c_{j,\nu}\sum_{\chi=0}^{\infty}\frac{A^{\chi}}{\Gamma(\chi(\alpha-\beta)+\alpha-\beta-\nu+1)}(\ln t-\ln t_j)^{\chi(\alpha-\beta)+\alpha-\beta-\nu-1} $$

$$ +\sum_{\chi=0}^{\infty}\frac{A^{\chi}}{\Gamma(\chi(\alpha-\beta)+\alpha-\beta)}\int_1^t(\ln t-\ln u)^{\chi(\alpha-\beta)+\alpha-\beta-1}h(u)\frac{du}{u}. $$

It is easy to see that ${}^hD_{1^+}^{\alpha}x(t)-A\,{}^hD_{1^+}^{\beta}x(t)=h(t), t\in(t_\sigma,t_{\sigma+1}], \sigma\in\mathbf{N}_0^m$. Then (2.2) is proved. Step 1 is completed.

Step 2. Suppose that x is a piecewise solution of (2.2). We prove that x satisfies (2.15).

Since $\alpha\in(n-1,n)$, $\alpha-\beta+p-n\geq 0$, and $h\in C(1,e)$ and $|h(t)|\leq(\ln t)^{\sigma}(1-\ln t)^{\tau}$ for $t\in(1,e)$, we know by Lemma 2.1, for $t\in(t_0,t_1]$ that there exist constants $c_{0,\nu}\in\mathbf{R}(\nu\in\mathbf{N}_1^n)$ such that

$$ x(t) = \sum_{\nu=1}^{n}c_{0,\nu}(\ln t)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(A(\ln t)^{\alpha-\beta}) $$

$$ +\int_1^t(\ln t-\ln s)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(A(\ln t-\ln s)^{\alpha-\beta})h(s)\frac{ds}{s}, t\in(t_0,t_1]. $$

So we get the expression of x on $(t_0,t_1]$. This fact implies that (2.15) holds for $k=0$.

We will apply the mathematical induction method to prove that (2.15) holds for all $\mu\in\mathbf{N}_0^m$. Suppose that (2.15) holds for $k=0,1,2,\cdots,\sigma$, i.e., there exist constants $c_{j,\nu}\in\mathbf{R}(\nu\in\mathbf{N}_1^n, j\in\mathbf{N}_0^{\sigma})$

$$ x(t) = \sum_{j=0}^{\mu}\sum_{\nu=1}^{n}c_{j,\nu}(\ln t-\ln t_j)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(A(\ln t-\ln t_j)^{\alpha-\beta}) $$

$$ +\int_1^t(\ln t-\ln s)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(A(\ln t-\ln s)^{\alpha-\beta})h(s)\frac{ds}{s}, t\in(t_\mu,t_{\mu+1}], \mu\in\mathbf{N}_0^{\sigma}. $$

In order to get the expression of x on $(t_{\sigma+1}, t_{\sigma+2}]$, we suppose that

$$x(t) = \sum_{j=0}^{\sigma}\sum_{\nu=1}^{n} c_{j,\nu}(\ln t - \ln t_j)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(A(\ln t - \ln t_j)^{\alpha-\beta})$$
$$+ \int_1^t(\ln t - \ln s)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(A(\ln t - \ln s)^{\alpha-\beta})h(s)\frac{ds}{s} + \Phi(t), t \in (t_{\sigma+1}, t_{\sigma+2}]. \tag{2.19}$$

Then for $t \in (t_{\sigma+1}, t_{\sigma+2}]$, we have by Definition 2.2 that

$$^{rh}D_{1^+}^{\alpha}x(t) = \frac{\left(t\frac{d}{dt}\right)^n\left[\int_1^t(\ln t-\ln s)^{n-\alpha-1}x(s)\frac{ds}{s}\right]}{\Gamma(n-\alpha)}$$

$$= \frac{\left(t\frac{d}{dt}\right)^n\left[\sum_{\tau=0}^{\sigma}\int_{t_\tau}^{t_{\tau+1}}(\ln t-\ln s)^{n-\alpha-1}x(s)\frac{ds}{s}+\int_{t_{\sigma+1}}^{t}(\ln t-\ln s)^{n-\alpha-1}x(s)\frac{ds}{s}\right]}{\Gamma(n-\alpha)}$$

$$= {}^{rh}D_{t_{\sigma+1}^+}^{\alpha}\Phi(t) + \frac{\left(t\frac{d}{dt}\right)^n\left[\sum_{\tau=0}^{\sigma}\int_{t_\tau}^{t_{\tau+1}}(\ln t-\ln s)^{n-\alpha-1}\sum_{j=0}^{\tau}\sum_{\nu=1}^{n}c_{j,\nu}(\ln s-\ln t_j)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(A(\ln s-\ln t_j)^{\alpha-\beta})\frac{ds}{s}\right]}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\int_{t_{\sigma+1}}^{t}(\ln t-\ln s)^{n-\alpha-1}\sum_{j=0}^{\sigma}\sum_{\nu=1}^{n}c_{j,\nu}(\ln s-\ln t_j)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(A(\ln s-\ln t_j)^{\alpha-\beta})\frac{ds}{s}\right]}{\Gamma(n-\alpha)}$$

$$+\frac{\left(t\frac{d}{dt}\right)^n\left[\int_1^t(\ln t-\ln s)^{n-\alpha-1}\int_1^s(\ln s-\ln u)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(A(\ln s-\ln u)^{\alpha-\beta})h(u)\frac{du}{u}\frac{ds}{s}\right]}{\Gamma(n-\alpha)} \text{ by similar method used in Step 1}$$

$$= {}^{rh}D_{t_{\sigma+1}^+}^{\alpha}\Phi(t) + \sum_{j=0}^{\sigma}\sum_{\nu=1}^{n}c_{j,\nu}\sum_{\chi=1}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)-\nu+1)}(\ln t - \ln t_j)^{\chi(\alpha-\beta)-\nu}$$

$$+h(t) + \sum_{\chi=1}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta))}\int_1^t(\ln t - \ln u)^{\chi(\alpha-\beta)-1}h(u)\frac{du}{u}.$$

Similarly, we have for $t \in (t_{\sigma+1}, t_{\sigma+2}]$ that

$$^{rh}D_{1^+}^{\beta}x(t) = \frac{\left(t\frac{d}{dt}\right)^p\left[\int_1^t(\ln t-\ln s)^{p-\beta-1}x(s)\frac{ds}{s}\right]}{\Gamma(p-\beta)}$$

$$= \frac{\left(t\frac{d}{dt}\right)^p\left[\sum_{\tau=0}^{\sigma}\int_{t_\tau}^{t_{\tau+1}}(\ln t-\ln s)^{p-\beta-1}x(s)\frac{ds}{s}+\int_{t_{\sigma+1}}^{t}(\ln t-\ln s)^{p-\beta-1}x(s)\frac{ds}{s}\right]}{\Gamma(p-\beta)}$$

$$= {}^{rh}D_{t_{\sigma+1}^+}^{\beta}\Phi(t) + \sum_{j=0}^{\sigma}\sum_{\nu=1}^{n}c_{j,\nu}\sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha-\beta-\nu+1)}(\ln t - \ln t_j)^{\chi(\alpha-\beta)+\alpha-\beta-\nu-1}$$

$$+ \sum_{\chi=0}^{\infty}\frac{A^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha-\beta)}\int_1^t(\ln t - \ln u)^{\chi(\alpha-\beta)+\alpha-\beta-1}h(u)\frac{du}{u}.$$

It follows that ${}^{rh}D_{t_{\sigma+1}^+}^{\alpha}\Phi(t) - A^{rh}D_{t_{\sigma+1}^+}\Phi(t) = 0$ on $(t_{\sigma+1}, t_{\sigma+2}]$. By Lemma 2.1, we know that there exist constants $c_{\sigma+1,\nu} \in \mathbf{R}$ such that $\Phi(t) = \sum_{\nu=1}^{n}c_{\sigma+1,\nu}(\ln t - \ln t_\nu)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(A(\ln t - \ln t_\nu)^{\alpha-\beta})$ on $(t_{\sigma+1}, t_{\sigma+2}]$. Substituting Φ into (2.19), then the expression of x on $(t_{\sigma+1}, t_{\sigma+2}]$ is as follows

$$x(t) = \sum_{j=0}^{\sigma+1}\sum_{\nu=1}^{n} c_{j,\nu}(\ln t - \ln t_j)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(A(\ln t - \ln t_j)^{\alpha-\beta})$$
$$+ \int_1^t(\ln t - \ln s)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(A(\ln t - \ln s)^{\alpha-\beta})h(s)\frac{ds}{s}, t \in (t_{\sigma+1}, t_{\sigma+2}].$$

So (2.15) holds for $k = \sigma + 1$. By the mathematical induction method, (2.15) is proved. The proof of Lemma 2.2 is completed. □

Lemma 2.3. $x \in PC_{n-\alpha}(1, e]$ is a solution of

$$\begin{cases} {}^{rh}D_{1^+}^{\alpha}x(t) - \lambda\, {}^{rh}D_{1^+}^{\beta}x(t) = h(t), a.e., t \in (t_k, t_{k+1}], k \in \mathbf{N}_0^m, \\ \Delta^h I_{1^+}^{n-\alpha}x(t_k) = I_{n,k}, k \in \mathbf{N}_1^m, \\ \Delta^{rh}D_{1^+}^{\alpha-\nu}x(t_k) = I_{\nu,k}, k \in \mathbf{N}_1^m, \nu \in \mathbf{N}_1^{n-1}, \\ {}^{rh}D_{1^+}^{n-\nu}x(1) = 0,\ x(e) = 0,\ \nu \in \mathbf{N}_1^{n-1}, \end{cases} \tag{2.20}$$

if and only if

$$\begin{aligned} x(t) &= \overline{d}_{n0}(\ln t)^{\alpha-n}\mathbf{E}_{\alpha,\alpha-n+1}(\lambda(\ln t)^{\alpha}) \\ &+ \sum_{j=1}^{i}\sum_{\nu=1}^{n} I_\nu(t_j, x(t_j))(\ln t - \ln t_j)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(\lambda(\ln t - \ln t_j)^{\alpha-\beta}) \\ &+ \int_1^t (\ln t - \ln s)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(\lambda(\ln t - \ln s)^{\alpha-\beta})p(s)f(s, x(s))\tfrac{ds}{s},\ t \in (t_i, t_{i+1}], i \in \mathbf{N}_0^m \end{aligned} \tag{2.21}$$

Proof. Let x be a solution of (2.20). From ${}^{rh}D_{1^+}^{\alpha}x(t) - \lambda\, {}^{rh}D_{1^+}^{\beta}x(t) = h(t), a.e., t \in (t_k, t_{k+1}], k \in \mathbf{N}_0^m$ and Lemma 2.2, we get that exist constants $d_{\nu,j} \in \mathbf{R}(j \in \mathbf{N}_0^m, \nu \in \mathbf{N}_1^n)$ such that

$$\begin{aligned} x(t) &= \sum_{j=0}^{\tau}\sum_{\nu=1}^{n} d_{\nu,j}(\ln t - \ln t_j)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(\lambda(\ln t - \ln t_j)^{\alpha-\beta}) \\ &+ \int_1^t (\ln t - \ln s)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(\lambda(\ln t - \ln s)^{\alpha-\beta})h(s)\tfrac{ds}{s}, t \in (t_\tau, t_{\tau+1}], \tau \in \mathbf{N}_0^m. \end{aligned} \tag{2.22}$$

One has for $k \in \mathbf{N}_1^{n-1}$ by direct computation that

$$\begin{aligned} {}^{rh}D_{1^+}^{\alpha-k}x(t) &= \sum_{j=0}^{i}\sum_{\nu=1}^{k} d_{\nu j}(\ln t - \ln t_j)^{k-\nu}\mathbf{E}_{\alpha-\beta,k-\nu+1}(\lambda(\ln t - \ln t_j)^{\alpha-\beta}) \\ &+ \sum_{j=0}^{i}\sum_{\nu=k+1}^{n} d_{\nu j}(\ln t - \ln t_j)^{\alpha+k-\nu}\mathbf{E}_{\alpha-\beta,\alpha+k-\nu+1}(\lambda(\ln t - \ln t_j)^{\alpha-\beta}) \\ &+ \int_1^t (\ln t - \ln s)^{k-1}\mathbf{E}_{\alpha-\beta,k}(\lambda(\ln t - \ln s)^{\alpha-\beta})h(s)\tfrac{ds}{s},\ t \in (t_i, t_{i+1}], i \in \mathbf{N}_0^m, \end{aligned} \tag{2.23}$$

for $k \in \mathbf{N}_1^{p-1}$ that

$$\begin{aligned} {}^{rh}D_{1^+}^{\beta-k}x(t) &= \sum_{j=0}^{i}\sum_{\nu=1}^{n} d_{\nu,j}\sum_{\chi=0}^{\infty}\tfrac{\lambda^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha-\beta+k-\nu+1)}(\ln t - \ln t_j)^{\chi(\alpha-\beta)+\alpha-\beta+k-\nu} \\ &+ \sum_{\chi=0}^{\infty}\tfrac{\lambda^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha-\beta+k)}\int_1^t (\ln t - \ln u)^{\chi(\alpha-\beta)+\alpha-\beta+k-1}h(u)\tfrac{du}{u},\ t \in (t_i, t_{i+1}], i \in \mathbf{N}_0^m, \end{aligned} \tag{2.24}$$

$$\begin{aligned} {}^{h}I_{1^+}^{n-\alpha}x(t) &= \sum_{j=0}^{i}\sum_{\nu=1}^{n} d_{\nu j}(\ln t - \ln t_j)^{n-\nu}\mathbf{E}_{\alpha-\beta,n-\nu+1}(\lambda(\ln t - \ln t_j)^{\alpha-\beta}) \\ &+ \int_1^t (\ln t - \ln s)^{n-1}\mathbf{E}_{\alpha-\beta,n}(\lambda(\ln t - \ln s)^{\alpha-\beta})h(s)\tfrac{ds}{s},\ t \in (t_i, t_{i+1}], i \in \mathbf{N}_0^m. \end{aligned} \tag{2.25}$$

and

$$
{}^{h}I_{1^+}^{p-\beta}x(t) = \sum_{j=0}^{i}\sum_{\nu=1}^{n} d_{\nu,j}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha-\beta+p-\nu+1)}(\ln t-\ln t_j)^{\chi(\alpha-\beta)+\alpha-\beta+p-\nu}
$$
$$
+\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi(\alpha-\beta)+\alpha-\beta+p)}\int_1^t(\ln t-\ln u)^{\chi(\alpha-\beta)+\alpha-\beta+p-1}h(u)\frac{du}{u},\ t\in(t_i,t_{i+1}],i\in\mathbf{N}_0^m. \tag{2.26}
$$

Then for $k\in\mathbf{N}_1^{p-1}$, from (2.23) and (2.24), we have

$$
{}^{rh}D_{1^+}^{\alpha-k}x(t)-A\,{}^{rh}D_{1^+}^{\beta-k}x(t) = \sum_{j=0}^{i}\sum_{\nu=1}^{k} d_{\nu,j}\frac{1}{\Gamma(k-\nu+1)}(\ln t-\ln t_j)^{k-\nu}
$$
$$
+\int_1^t\frac{(\ln t-\ln u)^{k-1}}{\Gamma(k)}h(u)\frac{du}{u},\ t\in(t_i,t_{i+1}],i\in\mathbf{N}_0^m, \tag{2.27}
$$

and from (2.23) and (2.26)($p<n$), we get

$$
{}^{rh}D_{1^+}^{\alpha-p}x(t)-A\,{}^{rh}D_{1^+}^{p-\beta}x(t) = \sum_{j=0}^{i}\sum_{\nu=1}^{p} d_{\nu,j}\frac{1}{\Gamma(p-\nu+1)}(\ln t-\ln t_j)^{p-\nu}
$$
$$
+\int_1^t\frac{(\ln t-\ln u)^{p-1}}{\Gamma(p)}h(u)\frac{du}{u},\ t\in(t_i,t_{i+1}],i\in\mathbf{N}_0^m \tag{2.28}
$$

or from (2.25) and (2.26) ($p=n$), we have

$$
{}^{rh}D_{1^+}^{n-\alpha}x(t)-A\,{}^{rh}D_{1^+}^{p-\beta}x(t) = \sum_{j=0}^{i}\sum_{\nu=1}^{n} d_{\nu,j}\frac{1}{\Gamma(n-\nu+1)}(\ln t-\ln t_j)^{n-\nu}
$$
$$
+\int_1^t\frac{(\ln t-\ln u)^{n-1}}{\Gamma(n)}h(u)\frac{du}{u},\ t\in(t_i,t_{i+1}],i\in\mathbf{N}_0^m. \tag{2.29}
$$

(i) By (2.25) and $\Delta^hI_{1^+}^{n-\alpha}x(t_k)=I_{n,k},k\in\mathbf{N}_1^m$, we get $d_{n,k}=I_{n,k},k\in\mathbf{N}_1^m$.
(ii) By (2.23) and $\Delta^{rh}D_{1^+}^{\alpha-\nu}x(t_k)=I_{\nu,k},k\in\mathbf{N}_1^m,\nu\in\mathbf{N}_{p+1}^{n-1}$, we get $d_{\nu,k}=I_{\nu,k},k\in\mathbf{N}_1^m,\nu\in\mathbf{N}_{p+1}^{n-1}$.
(iii) By (2.28) and $\Delta[{}^{rh}D_{1^+}^{\alpha-p}x-\lambda^hI_{1^+}^{p-\beta}x](t_k)=I_{p,k},k\in\mathbf{N}_1^m$, we get $d_{p,k}=I_{p,k},k\in\mathbf{N}_1^m$.
(iv) By (2.27) and $\Delta[{}^{rh}D_{1^+}^{\alpha-\nu}x-\lambda^hI_{1^+}^{\beta-\nu}x](t_k)=I_{\nu,k},k\in\mathbf{N}_1^m,\nu\in\mathbf{N}_1^{p-1}$, we get $d_{\nu,k}=I_{\nu,k},k\in\mathbf{N}_1^m,\nu\in\mathbf{N}_1^{p-1}$.
(v) By (2.23) and ${}^{rh}D_{1^+}^{\alpha-\nu}x(1)=0,\nu\in\mathbf{N}_{p+1}^{n-1}$, we get $d_{\nu,0}=0$ for $\nu\in\mathbf{N}_{p+1}^{n-1}$.
(vi) By (2.28) and $[{}^{rh}D_{1^+}^{\alpha-p}x-\lambda^hI_{1^+}^{p-\beta}x](1)=0$, we get $d_{p,0}=0$.
(vii) By (2.27) and $[{}^{rh}D_{1^+}^{\alpha-\nu}x-\lambda^hI_{1^+}^{\beta-\nu}x](1)=0,\nu\in\mathbf{N}_1^{p-1}$, we get $d_{\nu,0}=0$ for $\nu\in\mathbf{N}_1^{p-1}$.
(viii) By (2.22) and $x(e)=0$, we get

$$
d_{n,0}\mathbf{E}_{\alpha-\beta,\alpha-n+1}(\lambda)+\sum_{j=1}^{m}\sum_{\nu=1}^{n}I_{\nu,j}(1-\ln t_j)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(\lambda(1-\ln t_j)^{\alpha-\beta})
$$
$$
+\int_1^e(1-\ln s)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(\lambda(1-\ln s)^{\alpha-\beta})h(s)\frac{ds}{s}=0.
$$

It follows that

$$
d_{n,0}=\frac{-1}{\mathbf{E}_{\alpha-\beta,\alpha-n+1}(\lambda)}\Bigg[\sum_{j=1}^{m}\sum_{\nu=1}^{n}I_{\nu,j}(1-\ln t_j)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(\lambda(1-\ln t_j)^{\alpha-\beta})
$$
$$
+\int_1^e(1-\ln s)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(\lambda(1-\ln s)^{\alpha-\beta})h(s)\frac{ds}{s}\Bigg].
$$

Hence

$$x(t) = \frac{-1}{\mathbf{E}_{\alpha-\beta,\alpha-n+1}(\lambda)}\left[\sum_{j=1}^{m}\sum_{\nu=1}^{n} I_{\nu,j}(1-\ln t_j)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(\lambda(1-\ln t_j)^{\alpha-\beta})\right.$$
$$\left.+\int_1^e (1-\ln s)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(\lambda(1-\ln s)^{\alpha-\beta})h(s)\frac{ds}{s}\right](\ln t)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(\lambda(\ln t)^{\alpha-\beta})$$
$$+\sum_{j=1}^{\tau}\sum_{\nu=1}^{n} I_{\nu,j}(\ln t-\ln t_j)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(\lambda(\ln t-\ln t_j)^{\alpha-\beta})$$
$$+\int_1^t (\ln t-\ln s)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(\lambda(\ln t-\ln s)^{\alpha-\beta})h(s)\frac{ds}{s},\ t\in(t_\tau,t_{\tau+1}],\ \tau\in\mathbf{N}_0^m. \tag{2.22}$$

This is just (2.21). On the other hand, we can prove by direction computation that x is a solution of (2.20) if x satisfies (2.20). The proof is omitted. □

Define the operators T on $PC_{n-\alpha}(1,e]$ for $x \in PC_{n-\alpha}(1,e]$ by

$$(Tx)(t) = \frac{-1}{\mathbf{E}_{\alpha-\beta,\alpha-n+1}(\lambda)}\left[\sum_{j=1}^{m}\sum_{\nu=1}^{n} I_\nu(t_j,x(t_j))(1-\ln t_j)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(\lambda(1-\ln t_j)^{\alpha-\beta})\right.$$
$$\left.+\int_1^e (1-\ln s)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(\lambda(1-\ln s)^{\alpha-\beta})h(s)f(s,x(s))\frac{ds}{s}\right](\ln t)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(\lambda(\ln t)^{\alpha-\beta})$$
$$+\sum_{j=1}^{\tau}\sum_{\nu=1}^{n} I_\nu(t_j,x(t_j))(\ln t-\ln t_j)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(\lambda(\ln t-\ln t_j)^{\alpha-\beta})$$
$$+\int_1^t (\ln t-\ln s)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(\lambda(\ln t-\ln s)^{\alpha-\beta})h(s)f(s,x(s))\frac{ds}{s},\ t\in(t_\tau,t_{\tau+1}],\ \tau\in\mathbf{N}_0^m.$$

The proofs of the following two lemmas are standard and are omitted, see [10].

Lemma 2.4. $T : PC_{n-\alpha}(1,e] \mapsto PC_{n-\alpha}(1,e]$ is well defined, x is a solution of BVP(1.5) if and only if x is a fixed point of T, T is completely continuous.

Lemma 2.5(Schauder's fixed point theorem) [7]. Let X be a Banach space and $T : X \mapsto X$ be a completely continuous operator. Suppose $\overline{\Omega}$ is a nonempty closed convex bounded subset of X and $T(\overline{\Omega} \subseteq \overline{\Omega}$. Then there exists $x \in \overline{\Omega}$ such that $x = Tx$.

3. Main results

In this section, we prove the existence of solutions of BVP(1.4) and BVP(1.5) under the assumptions. We need the following assumptions:

(H1) there exists a non-decreasing function $\Phi_f : \mathbf{R} \to [0,+\infty)$ such that

$$\left|f\left(t, \frac{x}{(\ln t-\ln t_i)^{n-\alpha}}\right)\right| \le \Phi(|x|),\ t\in(t_i,t_{i+1}),\ x\in\mathbf{R}, i\in\mathbf{N}_0^m.$$

(H2) there exists a non-decreasing function $\Phi_I : \mathbf{R} \to [0,+\infty)$ such that

$$\left|I_\nu\left(t_i, \frac{x}{(\ln t_i-\ln t_{i-1})^{n-\alpha}}\right)\right| \le \Phi_I(|x|), i\in\mathbf{N}_1^m, \nu\in\mathbf{N}_1^n.$$

Theorem 3.1. Suppose that (a)-(d) and (H3)-(H4) hold. Then BVP(1.4) has at least one solution if

$$\left[\frac{\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(|\lambda|)}{\mathbf{E}_{\alpha-\beta,\alpha-n+1}(\lambda)}\sum_{j=1}^{m}\sum_{\nu=1}^{n}(1-\ln t_j)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(|\lambda|) + m\sum_{\nu=1}^{n}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(|\lambda|)\right]\Phi_I(r)$$
$$+\left[\frac{\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(|\lambda|)}{\mathbf{E}_{\alpha-\beta,\alpha-n+1}(\lambda)}\mathbf{B}(\alpha+\tau,\sigma+1)\mathbf{E}_{\alpha-\beta,\alpha}(|\lambda|) + \mathbf{B}(\alpha+\tau,\sigma+1)\mathbf{E}_{\alpha-\beta,\alpha}(|\lambda|)\right]\Phi_f(r) \le r \tag{3.1}$$

has at least one positive solution r_0.

Let T be defined in Section 2. By Lemma 2.4, $T : X \to X$ is well defined, x is a solution of BVP(1.5) if and only if x is a fixed point of T, T is completely continuous.

Denote $\Omega_r = \{x : x \in X, \|x\| \le r\}$ for $r > 0$. Let r_0 be a positive solution of (3.1). For $x \in \Omega_{r_0}$, then $\|x\|| \le r_0$ and (H1)-(H2) imply that

$$|f(t, x(t))| \le \Phi_f(\|x\|), t \in (t_i, t_{i+1}], i \in \mathbf{N}_0^m,$$

$$|I_\nu(t_i, x(t_i))| \le \Phi_I(\|x\|), i \in \mathbf{N}_1^m, \nu \in \mathbf{N}_0^{n-1}.$$

We get by the definition of T for $t \in (t_\tau, t_{\tau+1}](\tau \in \mathbf{N}_0^m)$ that

$$\begin{aligned}
&|(\ln t - \ln t_\tau)^{n-\alpha}(Tx)(t)|\\
&\le \frac{(\ln t - \ln t_\tau)^{n-\alpha}}{\mathbf{E}_{\alpha-\beta,\alpha-n+1}(\lambda)} \left|\sum_{j=1}^m \sum_{\nu=1}^n I_\nu(t_j, x(t_j))(1-\ln t_j)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(\lambda(1-\ln t_j)^{\alpha-\beta})\right.\\
&\left.+\int_1^e (1-\ln s)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(\lambda(1-\ln s)^{\alpha-\beta})h(s)f(s,x(s))\frac{ds}{s}\right|(\ln t)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(\lambda(\ln t)^{\alpha-\beta})\\
&+(\ln t - \ln t_\tau)^{n-\alpha}\left|\sum_{j=1}^\tau \sum_{\nu=1}^n I_\nu(t_j, x(t_j))(\ln t - \ln t_j)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(\lambda(\ln t - \ln t_j)^{\alpha-\beta})\right.\\
&\left.+\int_1^t (\ln t - \ln s)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(\lambda(\ln t - \ln s)^{\alpha-\beta})h(s)f(s,x(s))\frac{ds}{s}\right|\\
&\le \frac{1}{\mathbf{E}_{\alpha-\beta,\alpha-n+1}(\lambda)}\left[\sum_{j=1}^m \sum_{\nu=1}^n \Phi_I(\|x\|)(1-\ln t_j)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(|\lambda|)\right.\\
&\left.+\int_1^e (1-\ln s)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(|\lambda|)(\ln s)^\sigma(1-\ln s)^\tau\Phi_f(\|x\|)\frac{ds}{s}\right]\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(|\lambda|)\\
&+\sum_{j=1}^\tau \sum_{\nu=1}^n \Phi_I(\|x\|)\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(|\lambda|)\\
&+(\ln t - \ln t_\tau)^{n-\alpha}\int_1^t (\ln t - \ln s)^{\alpha-1}\mathbf{E}_{\alpha-\beta,\alpha}(|\lambda|)(\ln s)^\sigma(1-\ln s)^\tau)^\tau\Phi_f(\|x\|)\frac{ds}{s}\\
&\le \frac{\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(|\lambda|)}{\mathbf{E}_{\alpha-\beta,\alpha-n+1}(\lambda)}\sum_{j=1}^m \sum_{\nu=1}^n (1-\ln t_j)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(|\lambda|)\Phi_I(\|x\|)\\
&+\frac{\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(|\lambda|)}{\mathbf{E}_{\alpha-\beta,\alpha-n+1}(\lambda)}\mathbf{B}(\alpha+\tau,\sigma+1)\mathbf{E}_{\alpha-\beta,\alpha}(|\lambda|)\Phi_f(\|x\|)\\
&+m\sum_{\nu=1}^n \mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(|\lambda|)\Phi_I(\|x\|) + \mathbf{B}(\alpha+\tau,\sigma+1)\mathbf{E}_{\alpha-\beta,\alpha}(|\lambda|)\Phi_f(\|x\|).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|Tx\| \le &\left[\frac{\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(|\lambda|)}{\mathbf{E}_{\alpha-\beta,\alpha-n+1}(\lambda)}\sum_{j=1}^m \sum_{\nu=1}^n (1-\ln t_j)^{\alpha-\nu}\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(|\lambda|) + m\sum_{\nu=1}^n \mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(|\lambda|)\right]\Phi_I(r_0)\\
&+\left[\frac{\mathbf{E}_{\alpha-\beta,\alpha-\nu+1}(|\lambda|)}{\mathbf{E}_{\alpha-\beta,\alpha-n+1}(\lambda)}\mathbf{B}(\alpha+\tau,\sigma+1)\mathbf{E}_{\alpha-\beta,\alpha}(|\lambda|) + \mathbf{B}(\alpha+\tau,\sigma+1)\mathbf{E}_{\alpha-\beta,\alpha}(|\lambda|)\right]\Phi_f(r_0).
\end{aligned}$$

By the assumption (3.1), we have $\|Tx\| \le r_0$ for all $x \in \Omega_{r_0}$. Hence Lemma 2.5 implies that T has at least one fixed point in Ω_{r_0} which is a solution of BVP(1.5). The proof is completed. □

(H3) there exist constants $\sigma, A, B \geq 0$ such that

$$\left| f\left(t, \frac{x}{(\ln t - \ln t_i)^{n-\alpha}}\right)\right| \leq A + B|x|^{\sigma},\ t \in (t_i, t_{i+1}),\ x \in \mathbf{R}, i \in \mathbf{N}_0^m.$$

(H4) there exist constants $\sigma, C, D \geq 0$ such that

$$\left| I_\nu\left(t_i, \frac{x}{(\ln t_i - \ln t_{i-1})^{n-\alpha}}\right)\right| \leq C + D|x|^{\sigma}, i \in \mathbf{N}_1^m, \nu \in \mathbf{N}_1^n.$$

Denote

$$P = \left[\frac{\sum_{j=1}^{m}\sum_{\nu=1}^{n}(1-\ln t_j)^{\alpha-\nu}\mathbf{E}_{\alpha,\alpha-n+1}(|\lambda|)\mathbf{E}_{\alpha,\alpha-\nu+1}(|\lambda|)}{|\mathbf{E}_{\alpha,\alpha-n+1}(\lambda)|} + m\sum_{\nu=1}^{n}\mathbf{E}_{\alpha,\alpha-\nu+1}(|\lambda|)\right]C$$

$$+\left[\frac{\mathbf{B}(\alpha+\tau,\sigma+1)\mathbf{E}_{\alpha,\alpha-n+1}(|\lambda|)\mathbf{E}_{\alpha,\alpha}(|\lambda|)}{|\mathbf{E}_{\alpha,\alpha-n+1}(\lambda)|} + \mathbf{B}(\alpha+\tau,\sigma+1)\mathbf{E}_{\alpha,\alpha}(|\lambda|)\right]A,$$

$$Q = \left[\frac{\sum_{j=1}^{m}\sum_{\nu=1}^{n}(1-\ln t_j)^{\alpha-\nu}\mathbf{E}_{\alpha,\alpha-n+1}(|\lambda|)\mathbf{E}_{\alpha,\alpha-\nu+1}(|\lambda|)}{|\mathbf{E}_{\alpha,\alpha-n+1}(\lambda)|} + m\sum_{\nu=1}^{n}\mathbf{E}_{\alpha,\alpha-\nu+1}(|\lambda|)\right]C$$

$$+\left[\frac{\mathbf{B}(\alpha+\tau,\sigma+1)\mathbf{E}_{\alpha,\alpha-n+1}(|\lambda|)\mathbf{E}_{\alpha,\alpha}(|\lambda|)}{|\mathbf{E}_{\alpha,\alpha-n+1}(\lambda)|} + \mathbf{B}(\alpha+\tau,\sigma+1)\mathbf{E}_{\alpha,\alpha}(|\lambda|)\right]B.$$

Theorem 3.2. Suppose that (a)-(d) and (H3)-(H4) hold. Then BVP(1.4) has at least one solution if
(i) $\sigma\tau \in [0, 1)$ or
(ii) $\sigma = 1$ with $Q < 1$ or
(iii) $\sigma > 1$ with $P^{\sigma-1}Q \leq \frac{(\sigma-1)^{\sigma-1}}{\sigma^\sigma}$.

Proof. In the proof of Theorem 3.1, we choose $\Phi_f(x) = A + Bx^\sigma)$ and $\Phi_I(x) = C + Dx^\sigma$. Then (H1)-(H2) hold. Similar to the proof of Theorem 3.1, we denote $\Omega_r = \{x \in PC_{n-\alpha}(1, e]$. Then for $x \in PC_{n-\alpha}(1, e]$, we have

$$|f(t, x(t))| = \left|f\left(t, (\ln t - \ln t_i)^{\alpha-n}(\ln t - \ln t_i)^{n-\alpha}x(t)\right)\right| \leq A + B\|x\|^\sigma, t \in (t_i, t_{i+1}], i \in \mathbf{N}_0^m,$$

$$|I_\nu(t_i, x(t_i))| = |I_\nu\left(t_i, (\ln t_i - \ln t_{i-1})^{\alpha-n}(\ln t_i - \ln t_{i-1})^{n-\alpha}x(t)\right)| \leq C + D\|x\|^\sigma, i \in \mathbf{N}_1^m, \nu \in \mathbf{N}_1^n.$$

Then (3.1) becomes $P + Qr^\sigma \leq r$.

Case 1. $\sigma \in [0, 1)$. It is easy to see that $P + Qr^\sigma \leq r$ has positive solution r_0. Then Theorem 3.1 implies that BVP(1.4) has at least one solution.

Case 2. $\sigma = 1$. It is easy to see that $P + Qr^\sigma \leq r$ has positive solution r_0 by $Q < 1$. Then Theorem 3.1 implies that BVP(1.4) has at least one solution.

Case 3. $\sigma > 1$.

Choose $r_1 = \left(\frac{P}{Q(\sigma-1)}\right)^{\frac{1}{\sigma}}$. Then r_1 is a positive solution of $P + Qr^\sigma \leq r$ since $P^{\sigma-1}Q \leq \frac{(\sigma-1)^{\sigma-1}}{\sigma^\sigma}$. Then Theorem 3.1 implies that BVP(1.4) has at least one solution.

The proof of Theorem 3.2 is completed. □

Theorem 3.3. Suppose that (a)–(d) hold, and there exist constants $M_f, M_I \geq 0$ such that

$$\left| f\left(t, \frac{x}{(t-t_i)^{n-\alpha}}\right)\right| \leq M_f,\ t \in (t_i, t_{i+1}],\ x \in \mathbf{R}, i \in \mathbf{N}_0^m,$$

$$\left| I_\nu\left(t_i, \frac{x}{(t_i-t_{i-1})^{n-\alpha}}\right)\right| \leq M_I, i \in \mathbf{N}_1^m, \nu \in \mathbf{N}_1^n.$$

Then BVP(1.4) has at least one solution in $PC_{n-\alpha}$.

Proof. In Theorem 3.2, choose $\sigma = 0$, $A = M_f$, $C = M_I$ and $B = D = 0$. It is easy to see that (H3) and (H4) hold. We get Theorem 3.3 from Theorem 3.2. The proof is completed. □

4. Comments on recent published papers

We give the following remarks on Lemma 2.9 in [18] on page 87, on Theorem 3.4 in [21] on page 24 and Theorem 4 on [23] on page 3 in order not to misleading readers.

Remark 4.1. Let $f : [1, e] \times \mathbf{R} \mapsto \mathbf{R}$ and $t \mapsto (\ln t)^{1-\alpha} f(t, u)$ are continuous functions. Then x is a solution of the fractional integral equation

$$x(t) = \begin{cases} \frac{u_0}{\Gamma(\alpha)}(\ln t)^{\alpha-1} + \int_1^t \frac{(\ln t - \ln s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s)) \frac{ds}{s}, t \in (1, t_1], \\ \frac{u_0}{\Gamma(\alpha)}(\ln t)^{\alpha-1} + \sum_{j=1}^{i} \frac{p_j}{\Gamma(\alpha)}(\ln t)^{\alpha-1} \\ + \int_1^t \frac{(\ln t - \ln s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s)) \frac{ds}{s}, t \in (t_i, t_{i+1}], i = 1, 2, \cdots, m \end{cases}$$

if and only if x is a solution of IVP(1.1). We note that Result 1 is wrong. In fact, by Definition 2.2 in Section 2, we have for $t \in (t_i, t_{i+1}]$ that

$$\begin{aligned}
{}^{rh}D_{1^+}^{\alpha} x(t) &= \frac{\left(t\frac{d}{dt}\right)\left[\int_1^t (\ln t - \ln s)^{-\alpha} x(s) \frac{ds}{s}\right]}{\Gamma(1-\alpha)} = \frac{\left(t\frac{d}{dt}\right)\left[\sum_{\chi=0}^{i-1} \int_{t_\chi}^{t_{\chi+1}} (\ln t - \ln s)^{-\alpha} x(s) \frac{ds}{s} + \int_{t_i}^t (\ln t - \ln s)^{1-\alpha} x(s) \frac{ds}{s}\right]}{\Gamma(1-\alpha)} \\
&= \frac{\left(t\frac{d}{dt}\right)\left[\sum_{\chi=0}^{i-1} \int_{t_\chi}^{t_{\chi+1}} (\ln t - \ln s)^{-\alpha} \left(\frac{u_0}{\Gamma(\alpha)}(\ln s)^{\alpha-1} + \sum_{j=1}^{\chi} \frac{p_j}{\Gamma(\alpha)}(\ln s)^{\alpha-1} + \int_1^s \frac{(\ln s - \ln u)^{\alpha-1}}{\Gamma(\alpha)} f(u, x(u)) \frac{du}{u}\right) \frac{ds}{s}\right]}{\Gamma(1-\alpha)} \\
&+ \frac{\left(t\frac{d}{dt}\right)\left[\int_{t_i}^t (\ln t - \ln s)^{-\alpha} \left(\frac{u_0}{\Gamma(\alpha)}(\ln s)^{\alpha-1} + \sum_{j=1}^{i} \frac{p_j}{\Gamma(\alpha)}(\ln s)^{\alpha-1} + \int_1^s \frac{(\ln s - \ln u)^{\alpha-1}}{\Gamma(\alpha)} f(u, x(u)) \frac{du}{u}\right) \frac{ds}{s}\right]}{\Gamma(1-\alpha)} \\
&= \frac{\left(t\frac{d}{dt}\right)\left[\sum_{\chi=0}^{i-1} \int_{t_\chi}^{t_{\chi+1}} (\ln t - \ln s)^{-\alpha} \left(\frac{u_0}{\Gamma(\alpha)}(\ln s)^{\alpha-1} + \sum_{j=1}^{\chi} \frac{p_j}{\Gamma(\alpha)}(\ln s)^{\alpha-1}\right) \frac{ds}{s}\right]}{\Gamma(1-\alpha)} \\
&+ \frac{\left(t\frac{dt}{dt}\right)\left[\int_{t_i}^t (\ln t - \ln s)^{-\alpha} \left(\frac{u_0}{\Gamma(\alpha)}(\ln s)^{\alpha-1} + \sum_{j=1}^{t_i} \frac{p_j}{\Gamma(\alpha)}(\ln s)^{\alpha-1}\right) \frac{ds}{s} + \int_1^t (\ln t - \ln s)^{1-\alpha} \left(\int_1^s \frac{(\ln s - \ln u)^{\alpha-1}}{\Gamma(\alpha)} f(u, x(u)) \frac{du}{u}\right) \frac{ds}{s}\right]}{\Gamma(1-\alpha)} \\
&= \frac{\left(t\frac{d}{dt}\right)\left[\int_1^t (\ln t - \ln s)^{-\alpha} \frac{u_0}{\Gamma(\alpha)}(\ln s)^{\alpha-1} \frac{ds}{s} + \sum_{j=1}^{i-1} \sum_{\chi=j}^{i-1} \frac{p_j}{\Gamma(\alpha)} \int_{t_\chi}^{t_{\chi+1}} (\ln t - \ln s)^{1-\alpha} (\ln s)^{\alpha-1} \frac{ds}{s}\right]}{\Gamma(1-\alpha)} \\
&+ \frac{\left(t\frac{d}{dt}\right)\left[\sum_{j=1}^{i} \frac{p_j}{\Gamma(\alpha)} \int_{t_i}^t (\ln t - \ln s)^{-\alpha} (\ln s)^{\alpha-1} \frac{ds}{s} + \int_1^t \int_u^t (\ln t - \ln s)^{-\alpha} \frac{(\ln s - \ln u)^{\alpha-1}}{\Gamma(\alpha)} \frac{ds}{s} f(u, x(u)) \frac{du}{u}\right]}{\Gamma(1-\alpha)} \\
&= \frac{\left(t\frac{d}{dt}\right)\left[\int_1^t (\ln t - \ln s)^{-\alpha} \frac{u_0}{\Gamma(\alpha)}(\ln s)^{\alpha-1} \frac{ds}{s} + \sum_{j=1}^{i} \frac{p_j}{\Gamma(\alpha)} \int_{t_j}^t (\ln t - \ln s)^{-\alpha} (\ln s)^{\alpha-1} \frac{ds}{s}\right]}{\Gamma(1-\alpha)} \\
&+ \frac{\left(t\frac{d}{dt}\right)\left[\int_1^t \int_u^t (\ln t - \ln s)^{-\alpha} \frac{(\ln s - \ln u)^{\alpha-1}}{\Gamma(\alpha)} \frac{ds}{s} f(u, x(u)) \frac{du}{u}\right]}{\Gamma(1-\alpha)} \text{ by } \frac{\ln s}{\ln t} = w, \frac{\ln s - \ln u}{\ln t - \ln u} = w
\end{aligned}$$

$$= \frac{\left(t\frac{d}{dt}\right)\left[\frac{u_0}{\Gamma(\alpha)}\int_0^1(1-w)^{-\alpha}w^{\alpha-1}dw+\sum\limits_{j=1}^{i}\frac{p_j}{\Gamma(\alpha)}\int_{\frac{\ln t_j}{\ln t}}^1(1-w)^{-\alpha}w^{\alpha-1}dw\right]}{\Gamma(1-\alpha)}$$

$$+\frac{\left(t\frac{d}{dt}\right)\left[\int_1^t\int_0^1(1-w)^{-\alpha}\frac{w^{\alpha-1}}{\Gamma(\alpha)}dwf(u,x(u))\frac{du}{u}\right]}{\Gamma(1-\alpha)}$$

$$= \left(t\frac{d}{dt}\right)\left[u_0+\sum_{j=1}^{i}\frac{p_j}{\Gamma(\alpha)}\int_{\frac{\ln t_j}{\ln t}}^1(1-w)^{-\alpha}w^{\alpha-1}dw\right]+\left(t\frac{d}{dt}\right)\left[\int_1^t f(u,x(u))\frac{du}{u}\right]$$

$$= f(t,x(t))+\left(t\frac{d}{dt}\right)\left[\sum_{j=1}^{i}\frac{p_j}{\Gamma(\alpha)}\int_{\frac{\ln t_j}{\ln t}}^1(1-w)^{-\alpha}w^{\alpha-1}dw\right].$$

It means that ${}^{rh}D_{1^+}^{\alpha}x(t) = f(t,x(t))$ for all $i \in \mathbf{N}_0^m$ if and only if $p_1 = \cdots = p_m = 0$. Hence Lemma 2.9 in [18] is wrong. □

Remark 4.2. In [21], Zhang considered the general solution of the impulsive fractional system

$$\begin{cases} {}^{rh}D_{1^+}^{\frac{3}{2}}x(t) = \ln t, t\in(1,3], t\neq 2,\\ \Delta^h J_{1^+}^{\frac{1}{2}}x(2)=\delta,\\ \Delta^{rh}D_{a^+}^{\frac{1}{2}}x(2)=\overline{\delta},\\ {}^hJ_{1^+}^{\frac{1}{2}}x(1)=x_2,\ {}^{rh}D_{a^+}^{\frac{1}{2}}x(1)=x_1. \end{cases} \tag{4.1}$$

It is claimed that (4.1) has solutions

$$x(t)=\begin{cases} \frac{x_1}{\Gamma(3/2)}\left(\int_1^t\frac{ds}{s}\right)^{\frac{3}{2}-1}+\frac{x_2}{\Gamma(3/2-1)}\left(\int_1^t\frac{ds}{s}\right)^{\frac{3}{2}-2}+\int_1^t\frac{(\ln t-\ln s)^{\frac{3}{2}-1}}{\Gamma(3/2)}\ln s\frac{ds}{s}, t\in(1,2],\\ \frac{x_1}{\Gamma(3/2)}\left(\int_1^t\frac{ds}{s}\right)^{\frac{3}{2}-1}+\frac{x_2}{\Gamma(3/2-1)}\left(\int_1^t\frac{ds}{s}\right)^{\frac{3}{2}-2}+\int_1^t\frac{(\ln t-\ln s)^{\frac{3}{2}-1}}{\Gamma(3/2)}\ln s\frac{ds}{s}\\ +\frac{\overline{\delta}}{\Gamma(3/2)}\left(\int_2^t\frac{ds}{s}\right)^{\frac{3}{2}-1}+\frac{\delta}{\Gamma(3/2-1)}\left(\int_2^t\frac{ds}{s}\right)^{\frac{3}{2}-2}\\ -(\aleph\delta+\hbar\overline{\delta})\left[\frac{x_1}{\Gamma(3/2)}\left(\int_1^t\frac{ds}{s}\right)^{\frac{3}{2}-1}+\frac{x_2}{\Gamma(3/2-1)}\left(\int_1^t\frac{ds}{s}\right)^{\frac{3}{2}-2}\right.\\ +\int_1^t\frac{(\ln t-\ln s)^{\frac{3}{2}-1}}{\Gamma(3/2)}\ln s\frac{ds}{s}-\frac{x_1+\int_1^2\ln s\frac{ds}{s}}{\Gamma(3/2)}\left(\int_2^t\frac{ds}{s}\right)^{\frac{3}{2}-1}\\ \left.-\frac{x_1\ln 2+x_2+\int_1^2\ln\frac{2}{s}\ln s\frac{ds}{s}}{\Gamma(3/2-1)}\left(\int_2^t\frac{ds}{s}\right)^{\frac{3}{2}-2}+\int_2^t\frac{(\ln t-\ln s)^{\frac{3}{2}-1}}{\Gamma(3/2)}\ln s\frac{ds}{s}\right], t\in(2,3]. \end{cases} \tag{4.2}$$

By direct computation, we get

$$x(t)=\begin{cases}\frac{x_1}{\Gamma(3/2)}(\ln t)^{\frac{1}{2}}+\frac{x_2}{\Gamma(1/2)}(\ln t)^{-\frac{1}{2}}+\frac{(\ln t)^{\frac{5}{2}}}{\Gamma(7/2)}, t\in(1,2],\\ \frac{x_1}{\Gamma(3/2)}(\ln t)^{\frac{1}{2}}+\frac{x_2}{\Gamma(1/2)}(\ln t)^{-\frac{1}{2}}+\frac{(\ln t)^{\frac{5}{2}}}{\Gamma(7/2)}\\ +\frac{\overline{\delta}}{\Gamma(3/2)}(\ln t-\ln 2)^{\frac{1}{2}}+\frac{\delta}{\Gamma(1/2)}(\ln t-\ln 2)^{-\frac{1}{2}}\\ -M\Big[\frac{x_1}{\Gamma(3/2)}(\ln t)^{\frac{1}{2}}+\frac{x_2}{\Gamma(1/2)}(\ln t)^{-\frac{1}{2}}\\ +\frac{(\ln t)^{\frac{5}{2}}}{\Gamma(7/2)}-\frac{x_1+\frac{1}{2}(\ln 2)^2}{\Gamma(3/2)}(\ln t-\ln 2)^{\frac{1}{2}}\\ -\frac{x_1\ln 2+x_2+\frac{1}{2}(\ln 2)^3-\frac{1}{3}(\ln 2)^3}{\Gamma(3/2-1)}(\ln t-\ln 2)^{-\frac{1}{2}}+(\ln t)^{\frac{5}{2}}\int_{\frac{\ln 2}{\ln t}}^1\frac{(1-w)^{\frac{1}{2}}}{\Gamma(3/2)}w\,dw\Big], t\in(2,3],\end{cases}$$

where $M=\aleph\delta+\hbar\overline{\delta}$ and $\aleph,\hbar$ are two constants. This result is wrong. In fact, by Definition 2.2([21]), we have for $t\in(2,3]$ that

$$\begin{aligned}{}^{rh}D_{1^+}^{\frac{3}{2}}x(t)&=\left(t\frac{d}{dt}\right)^2\frac{\int_1^2(\ln t-\ln s)^{-\frac{1}{2}}x(s)\frac{ds}{s}+\int_2^t(\ln t-\ln s)^{-\frac{1}{2}}x(s)\frac{ds}{s}}{\Gamma(1/2)}\\
&=\left(t\frac{d}{dt}\right)^2\frac{\int_1^2(\ln t-\ln s)^{-\frac{1}{2}}\left(\frac{x_1}{\Gamma(3/2)}(\ln s)^{\frac{1}{2}}+\frac{x_2}{\Gamma(1/2)}(\ln s)^{-\frac{1}{2}}+\frac{(\ln s)^{\frac{5}{2}}}{\Gamma(7/2)}\right)\frac{ds}{s}}{\Gamma(1/2)}\\
&+\left(t\frac{d}{dt}\right)^2\frac{\int_2^t(\ln t-\ln s)^{-\frac{1}{2}}\left(\frac{x_1}{\Gamma(3/2)}(\ln s)^{\frac{1}{2}}+\frac{x_2}{\Gamma(1/2)}(\ln s)^{-\frac{1}{2}}+\frac{(\ln s)^{\frac{5}{2}}}{\Gamma(7/2)}+\frac{\overline{\delta}}{\Gamma(3/2)}(\ln s-\ln 2)^{\frac{1}{2}}+\frac{\delta}{\Gamma(1/2)}(\ln s-\ln 2)^{-\frac{1}{2}}\right)\frac{ds}{s}}{\Gamma(1/2)}\\
&+\left(t\frac{d}{dt}\right)^2\frac{\int_2^t(\ln t-\ln s)^{-\frac{1}{2}}\left(-M\frac{x_1}{\Gamma(3/2)}(\ln s)^{\frac{1}{2}}+\frac{x_2}{\Gamma(1/2)}(\ln s)^{-\frac{1}{2}}+M\frac{(\ln s)^{\frac{5}{2}}}{\Gamma(7/2)}-\frac{x_1+\frac{1}{2}(\ln 2)^2}{\Gamma(3/2)}(\ln s-\ln 2)^{\frac{1}{2}}\right)\frac{ds}{s}}{\Gamma(1/2)}\\
&+\left(t\frac{d}{dt}\right)^2\frac{\int_2^t(\ln t-\ln s)^{-\frac{1}{2}}\left(-M\frac{x_1\ln 2+x_2+\frac{1}{2}(\ln 2)^3-\frac{1}{3}(\ln 2)^3}{\Gamma(1/2)}(\ln s-\ln 2)^{-\frac{1}{2}}+M(\ln s)^{\frac{5}{2}}\int_{\frac{\ln 2}{\ln s}}^1\frac{(1-w)^{\frac{1}{2}}}{\Gamma(3/2)}w\,dw\right)\frac{ds}{s}}{\Gamma(1/2)}\\
&=\left(t\frac{d}{dt}\right)^2\frac{\int_1^t(\ln t-\ln s)^{-\frac{1}{2}}\left(\frac{x_1}{\Gamma(3/2)}(\ln s)^{\frac{1}{2}}+\frac{x_2}{\Gamma(1/2)}(\ln s)^{-\frac{1}{2}}+\frac{(\ln s)^{\frac{5}{2}}}{\Gamma(7/2)}\right)\frac{ds}{s}}{\Gamma(1/2)}\\
&+\left(t\frac{d}{dt}\right)^2\frac{\int_2^t(\ln t-\ln s)^{-\frac{1}{2}}\left(\frac{\overline{\delta}}{\Gamma(3/2)}(\ln s-\ln 2)^{\frac{1}{2}}+\frac{\delta}{\Gamma(1/2)}(\ln s-\ln 2)^{-\frac{1}{2}}\right)\frac{ds}{s}}{\Gamma(1/2)}\\
&+M\left(t\frac{d}{dt}\right)^2\frac{\int_2^t(\ln t-\ln s)^{-\frac{1}{2}}\left(-\frac{x_1}{\Gamma(3/2)}(\ln s)^{\frac{1}{2}}+\frac{x_2}{\Gamma(1/2)}(\ln s)^{-\frac{1}{2}}+\frac{(\ln s)^{\frac{5}{2}}}{\Gamma(7/2)}\right)\frac{ds}{s}}{\Gamma(1/2)}\\
&+M\left(t\frac{d}{dt}\right)^2\frac{\int_2^t(\ln t-\ln s)^{-\frac{1}{2}}\left(-\frac{x_1\ln 2+x_2+\frac{1}{2}(\ln 2)^3-\frac{1}{3}(\ln 2)^3}{\Gamma(1/2)}(\ln s-\ln 2)^{-\frac{1}{2}}\right)\frac{ds}{s}}{\Gamma(1/2)}\end{aligned}$$

$$+M\left(t\frac{d}{dt}\right)^2\frac{\int_2^t(\ln t-\ln s)^{-\frac{1}{2}}\left(-\frac{x_1+\frac{1}{2}(\ln 2)^2}{\Gamma(3/2)}(\ln s-\ln 2)^{\frac{1}{2}}\right)\frac{ds}{s}}{\Gamma(1/2)}$$

$$+M\left(t\frac{d}{dt}\right)^2\frac{\int_2^t(\ln t-\ln s)^{-\frac{1}{2}}\left((\ln s)^{\frac{5}{2}}\int_{\frac{\ln 2}{\ln s}}^1\frac{(1-w)^{\frac{1}{2}}}{\Gamma(3/2)}wdw\right)\frac{ds}{s}}{\Gamma(1/2)}$$

$$=\left(t\frac{d}{dt}\right)^2\left(x_1\ln t+x_2+\frac{(\ln t)^3}{\Gamma(4)}\right)+\left(t\frac{d}{dt}\right)^2\left(\overline{\delta}(\ln t-\ln 2)+\delta\right)$$

$$+M\left(t\frac{d}{dt}\right)^2\frac{\int_2^t(\ln t-\ln s)^{-\frac{1}{2}}\left(-\frac{x_1}{\Gamma(3/2)}(\ln s)^{\frac{1}{2}}+\frac{x_2}{\Gamma(1/2)}(\ln s)^{-\frac{1}{2}}+\frac{(\ln s)^{\frac{5}{2}}}{\Gamma(7/2)}\right)\frac{ds}{s}}{\Gamma(1/2)}$$

$$-M\left(t\frac{d}{dt}\right)^2\left(\frac{x_1\ln 2+x_2+\frac{1}{2}(\ln 2)^3-\frac{1}{3}(\ln 2)^3}{\Gamma(1/2)}+\frac{x_1+\frac{1}{2}(\ln 2)^2}{\Gamma(3/2)}(\ln t-\ln 2)\right)$$

$$+M\left(t\frac{d}{dt}\right)^2\frac{\int_2^t(\ln t-\ln s)^{-\frac{1}{2}}\left((\ln s)^{\frac{5}{2}}\int_{\frac{\ln 2}{\ln s}}^1\frac{(1-w)^{\frac{1}{2}}}{\Gamma(3/2)}wdw\right)\frac{ds}{s}}{\Gamma(1/2)}$$

$$=\ln t+M\left(t\frac{d}{dt}\right)^2\frac{\int_2^t(\ln t-\ln s)^{-\frac{1}{2}}\left(-\frac{x_1}{\Gamma(3/2)}(\ln s)^{\frac{1}{2}}+\frac{x_2}{\Gamma(1/2)}(\ln s)^{-\frac{1}{2}}+\frac{(\ln s)^{\frac{5}{2}}}{\Gamma(7/2)}\right)\frac{ds}{s}}{\Gamma(1/2)}$$

$$+M\left(t\frac{d}{dt}\right)^2\frac{\int_2^t(\ln t-\ln s)^{-\frac{1}{2}}\left((\ln s)^{\frac{5}{2}}\int_{\frac{\ln 2}{\ln s}}^1\frac{(1-w)^{\frac{1}{2}}}{\Gamma(3/2)}wdw\right)\frac{ds}{s}}{\Gamma(1/2)}\neq\ln t,t\in(2,3].$$

Hence (4.2) in [21] is wrong. We get by using Lemma 2.3 ($\alpha\in(1,2),\lambda=0$) in Section 2 that (4.1) has a unique solution

$$x(t)=\begin{cases}\frac{x_1}{\Gamma(3/2)}(\ln t)^{\frac{1}{2}}+\frac{x_2}{\Gamma(1/2)}(\ln t)^{-\frac{1}{2}}+\frac{(\ln t)^{\frac{5}{2}}}{\Gamma(7/2)},t\in(1,2],\\ \frac{x_1}{\Gamma(3/2)}(\ln t)^{\frac{1}{2}}+\frac{x_2}{\Gamma(1/2)}(\ln t)^{-\frac{1}{2}}\\ +\frac{\overline{\delta}}{\Gamma(3/2)}(\ln t-\ln 2)^{\frac{1}{2}}+\frac{\delta}{\Gamma(1/2)}(\ln t-\ln 2)^{-\frac{1}{2}}+\frac{(\ln t)^{\frac{5}{2}}}{\Gamma(7/2)},t\in(2,3].\end{cases}$$

The following results are directly from Lemma 2.3 ($\alpha\in(1,2),\lambda=0$) and the proofs are omitted:

Result 4.3. x is a solution of IVP(1.2) if and only if

$$x(t)=\begin{cases}x_1(\ln t-\ln)^{\alpha-1}+x_2(\ln t-\ln a)^{\alpha-2}+\int_a^t\frac{(\ln t-\ln s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))\frac{ds}{s},t\in(a,t_1],\\ x_1(\ln t-\ln)^{\alpha-1}+x_2(\ln t-\ln a)^{\alpha-2}+\int_a^t\frac{(\ln t-\ln s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))\frac{ds}{s}\\ +\sum_{t_i<t}\Delta_i(x(t_i))(\ln t-\ln t_j)^{\alpha-1}+\sum_{\bar{t}_l<t}\overline{\Delta}_l(x(\bar{t}_l))(\ln t-\ln\bar{t}_l)^{\alpha-2}\\ +\int_a^t\frac{(\ln t-\ln s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))\frac{ds}{s},t\in(t_1,T].\end{cases}$$

Result 4.4. x is a solution of IVP(1.3) if and only if

$$x(t) = x_1(\ln t - \ln)^{\alpha-1} + x_2(\ln t - \ln a)^{\alpha-2} + \int_a^t \frac{(\ln t-\ln s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))\frac{ds}{s}$$

$$+ \sum_{j=1}^{i} \Delta_j(x(t_j))(\ln t - \ln t_j)^{\alpha-1} + \sum_{j=1}^{i} \overline{\Delta}_j(x(t_j))(\ln t - \ln t_j)^{\alpha-2}$$

$$+ \int_a^t \frac{(\ln t-\ln s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))\frac{ds}{s}, t \in (t_i, t_{i+1}], i \in \mathbf{N}_0^m.$$

Remark 4.5. In [23], the following initial value problem was considered:

$$\begin{cases} {}^hD_{a^+}^q z(t) = f(t, z(t)), t \in (a, T], t \neq t_i, i = 1, 2, \cdots, m, \\ \Delta {}^hI_{a^+}^{1-q} z(t_i) = J_i(z(t_i)), i = 1, 2, \cdots, m, \\ {}^hI_{a^+}^{1-q} z(a) = z_a, \end{cases} \tag{4.3}$$

where $q \in (0, 1), T > a > 0$, $z_a \in \mathbf{R}$, $a = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $f : [a, T] \times \mathbf{R} \to \mathbf{R}$ and $J_i : \mathbf{R} \to \mathbf{R}$ are some appropriate functions. The main result in [23] is as follows:

Theorem 4[23]. Let $\xi \in \mathbf{R}$ be an arbitrary constant. Then IVP(4.3) is equivalent to the following integral equation:

$$z(t) = \frac{z_a}{\Gamma(q)}(\ln t - \ln a)^{q-1} + \int_a^t \frac{(\ln t-\ln s)^{q-1}}{\Gamma(q)} f(s, z(s))\frac{ds}{s}$$

$$+ \sum_{j=1}^{i} \frac{(\ln t-\ln t_j)^{q-1}}{\Gamma(q)} J_j(z(t_j)) - \xi \sum_{j=1}^{i} J_j(z(t_j)) \Big[z_a \frac{(\ln t-\ln a)^{q-1}}{\Gamma(q)} + \int_a^t \frac{(\ln t-\ln s)^{q-1}}{\Gamma(q)} f(s, z(s))\frac{ds}{s}$$

$$-\Big(z_a + \int_a^{t_j} f(s, z(s))\frac{ds}{s}\Big) \frac{(\ln t-\ln t_j)^{q-1}}{\Gamma(q)} - \int_{t_j}^t \frac{(\ln t-\ln s)^{q-1}}{\Gamma(q)} f(s, z(s))\frac{ds}{s}\Big],$$

$$t \in (t_i, t_{i+1}], i = 0, 1, 2, \cdots, m.$$

Example 4.5. In [23], the following example was considered:

$$\begin{cases} {}^hD_{1^+}^{1/2} z(t) = f(t, z(t)), t \in (1, 3], t \neq 2, \\ \Delta {}^hI_{1^+}^{1/2} z(2) = l, \ {}^hI_{1^+}^{1/2} z(1) = z_1. \end{cases} \tag{4.4}$$

By Theorem 4 mentioned, the general solution of (4.4) is given by

$$z(t) = \begin{cases} \frac{z_1}{\Gamma(1/2)}(\ln t)^{-\frac{1}{2}} + \int_1^t \frac{(\ln t-\ln s)^{-\frac{1}{2}}}{\Gamma(1/2)} \ln s\frac{ds}{s}, t \in (1, 2], \\ \frac{z_1}{\Gamma(1/2)}(\ln t)^{-\frac{1}{2}} + \int_1^t \frac{(\ln t-\ln s)^{-\frac{1}{2}}}{\Gamma(1/2)} \ln s\frac{ds}{s} \\ +\frac{(\ln t-\ln 2)^{-\frac{1}{2}}}{\Gamma(1/2)} l - \xi l \Big[z_1 \frac{(\ln t)^{-\frac{1}{2}}}{\Gamma(1/2)} + \int_1^t \frac{(\ln t-\ln s)^{-\frac{1}{2}}}{\Gamma(1/2)} \ln s\frac{ds}{s} \\ -\Big(z_1 + \int_1^2 \ln s\frac{ds}{s}\Big) \frac{(\ln t-\ln 2)^{-\frac{1}{2}}}{\Gamma(1/2)} - \int_2^t \frac{(\ln t-\ln s)^{-\frac{1}{2}}}{\Gamma(1/2)} \ln s\frac{ds}{s}\Big], t \in (2, 3]. \end{cases}$$

One can find by direct computation for $t \in (2, 3]$ that

$$
\begin{aligned}
&{}^hD^{\frac{1}{2}}_{1^+}z(t) = \left(t\frac{d}{dt}\right)^2 \left[\int_1^t \frac{(\ln t - \ln s)^{-\frac{1}{2}}}{\Gamma(1/2)} z(s)\frac{ds}{s}\right] \\
&= \left(t\frac{d}{dt}\right)^2 \left[\int_1^2 \frac{(\ln t - \ln s)^{-\frac{1}{2}}}{\Gamma(1/2)} \left(\frac{z_1}{\Gamma(1/2)}(\ln s)^{-\frac{1}{2}} + \int_1^s \frac{(\ln s - \ln u)^{-\frac{1}{2}}}{\Gamma(1/2)} \ln u \frac{du}{u}\right)\frac{ds}{s}\right] \\
&+ \left(t\frac{d}{dt}\right)^2 \left[\int_1^t \frac{(\ln t - \ln s)^{-\frac{1}{2}}}{\Gamma(1/2)} \left(\frac{z_1}{\Gamma(1/2)}(\ln s)^{-\frac{1}{2}} + \int_1^s \frac{(\ln s - \ln u)^{-\frac{1}{2}}}{\Gamma(1/2)} \ln u \frac{du}{u}\right.\right. \\
&+ \frac{(\ln s - \ln 2)^{-\frac{1}{2}}}{\Gamma(1/2)} l - \xi l \left(z_1 \frac{(\ln s)^{-\frac{1}{2}}}{\Gamma(1/2)} + \int_1^s \frac{(\ln s - \ln u)^{-\frac{1}{2}}}{\Gamma(1/2)} \ln u \frac{du}{u}\right. \\
&\left.\left.\left. - \left(z_1 + \int_1^2 \ln u \frac{du}{u}\right) \frac{(\ln s - \ln 2)^{-\frac{1}{2}}}{\Gamma(1/2)} - \int_2^s \frac{(\ln s - \ln u)^{-\frac{1}{2}}}{\Gamma(1/2)} \ln u \frac{du}{u}\right)\right)\frac{ds}{s}\right] \neq \ln t, t \in (2, 3].
\end{aligned}
$$

So Theorem 4 is wrong.

By Lemma 2.2 and 2.3($\lambda = 0, \alpha = \frac{1}{2}, \beta = 0, h(t) = \ln t, f(t, x) = 1$), we can get the unique solution of (4.4) by

$$
z(t) = \begin{cases} \frac{z_1}{\Gamma(1/2)}(\ln t)^{-\frac{1}{2}} + \int_1^t \frac{(\ln t - \ln s)^{-\frac{1}{2}}}{\Gamma(1/2)} \ln s \frac{ds}{s}, t \in (1, 2], \\ \frac{z_1}{\Gamma(1/2)}(\ln t)^{-\frac{1}{2}} + \frac{l}{\Gamma(1/2)}(\ln t - \ln 2)^{-\frac{1}{2}} + \int_1^t \frac{(\ln t - \ln s)^{-\frac{1}{2}}}{\Gamma(1/2)} \ln s \frac{ds}{s}, t \in (2, 3]. \end{cases}
$$

References

[1] Y. Adjabi, F. Jarad, D. Baleanu, et al. On Cauchy problems with Caputo Hadamard fractional derivatives, J. Comput. Anal. Appl. 21(1)(2016), 661-681.

[2] B. Ahmad, S. K. Ntouyas, J. Tariboon, A study of mixed Hadamard and Riemann-Liouville fractional integro-differential inclusions via endpoint theory, Appl. Math. Letters, 52(2016), 9-14.

[3] A. A. Kilbas, O. I. Marichev, S. G. Samko, Fractional Integral and Derivatives (Theory and Applications), Gordon and Breach, Switzerland, 1993.

[4] S. Karthikeyan, C. Ravichandran, T. Gunasekar, Existence results for Hadamard type fractional functional integro-differential equations with integral boundary conditions, Interna. J. Appl. Engineering Research, 10(3)(2015), 6919-6932.

[5] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations, in North-Holland Mathematics Studies, vol: 204, Elsevier Science B. V. Amsterdam, 2006.

[6] Y. Liu, Survey and new results on boundary value problems of singular fractional differential equations with impulse effects, Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 296, pp. 1-177.

[7] J. Mawhin, Toplogical degree methods in nonlinear boundary value problems, in: NSFCBMS Regional Conference Series in Math., American Math. Soc. Providence, RI, 1979.

[8] S.K. Ntouyas, J. Tariboon, W. Sudsutad, Boundary value problems for Riemann-Liouville fractional differential inclusions with nonlocal Hadamard fractional integral conditions, Meditter. J. Math. http://dx.doi.org/10.1007/s00009-015-0543-1.

[9] Sotiris K. Ntouyas, Jessada Tariboon, Fractional boundary value problems with multiple orders of fractional derivatives and integrals, Electron. J. Differential Equations, Vol. 2017 (2017), No. 100, pp. 1-18.

[10] S.K. Ntouyas, J. Tariboon, Fractional integral problems for Hadamard-Caputo fractional Langevin differential inclusions, J, Appl. Math. Comput. 2015: 1-21. DOI 10.1007/s12190-015-0888-4.

[11] I. Podlubny, Fractional Differential Equations. Mathmatics in Science and Engineering, Vol. 198, Academic Press, San Diego, California, USA, 1999.

[12] J. Tariboon, S.K. Ntouyas, W. Sudsutad, Nonlocal Hadamard fractional integral conditions for nonlinear RiemannLiouville fractional differential equations, Bound. Value Probl. 253(2014), 10-16.

[13] J. Tariboon, S.K. Ntouyas, C. Thaiprayoon, Nonlinear Langevin equation of Hadamard-Caputo type fractional derivatives with nonlocal fractional integral conditions, Adv. Math. Phys. 2014 (2014) 15. Article ID 372749.

[14] J. Tariboon, S.K. Ntouyas, W. Sudsutad, Coupled systems of Riemann-Liouville fractional differential equations with Hadamard fractional integral boundary conditions, J. Nonl. Sci. Appl. 9(1)(2016), 295-308.

[15] J. Tariboon, S. K. Ntouyas, P. Thiramanus, Riemann-Liouville fractional differential equations with Hadamard fractional integral conditions, Inter. J. Appl. Math. Stat. 53(1)(2016), 119-134.

[16] P. Thiramanus, S. K. Ntouyas, J. Tariboon, Existence of solutions for Riemann-Liouville fractional differential equations with nonlocal Erdĺyi-Kober integral boundary conditions on the half-line, Bound. Value Prob. (1)(2015), 1-15.

[17] N. Thongsalee, S. Laoprasittichok, S. K. Ntouyas, et al. System of fractional differential equations with Erdĺyi-Kober fractional integral conditions, Open Mathematics, 13(1)(2015), 1-14.
[18] J. Wang, Y. Zhang, On the concept and existence of solutions for fractional impulsive systems with Hadamard derivatives, Appl. Math. Letters, **39** (2015), 85-90.
[19] W. Yukunthorn, S. Suantai, S.K. Ntouyas, J. Tariboon, Boundary value problems for impulsive multi-order Hadamard fractional differential equations, Bound. Value Probl. (1)(2015), 1-13.
[20] W. Yukunthorna, B. Ahmad, S. K. Ntouyas, J. Tariboon,On Caputo-Hadamard type fractional impulsive hybrid systems with nonlinear fractional integral conditions, Nonl. Anal. Hybrid Systems, 19 (2016), 77-92.
[21] X. Zhang, T. Shu, H. Cao, et al, The general solution for impulsive differential equations with Hadamard fractional derivative of order $q \in (1, 2)$, Adv. Differ. Equ. 2016(14)(2016), 1-36.
[22] Y. Zhang, J. Wang, Existence and finite-time stability results for impulsive fractional differential equations with maximum, J. Appl. Math. Computing, 51(1)(2016) 67-79.
[23] X. Zhang, X. Zhang, Z. Liu, W. Ding, H. Cao, T. Shu, On the general solution of impulsive systems with Hadamard fractional derivatives, Math. Prob. Eng. 2016(2016), Article ID 2814310, 12 pages.
[24] W. Rudin, Real and Complex Analysis, (3rd edition), McGraw-Hill, New York, 1986.
[25] J. A. Goguen, L-fuzzy sets, Journal of Mathematical Analysis and Applications 18 (1967) 145–174.
[26] P. Erdös, S. Shelah, Separability properties of almost-disjoint families of sets, Israel Journal of Mathematics 12 (1972) 207–214.