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Non-polynomial Spline Functions and Quasi-linearization to Approximate Nonlinear Volterra Integral Equation

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Abstract. In this work, we want to use the Non-polynomial spline basis and Quasi-linearization method to solve the nonlinear Volterra integral equation. When the iterations of the Quasilinear technique employed in nonlinear integral equation we obtain a linear integral equation then by using the Non-polynomial spline functions and collocation method the solution of the integral equation can be approximated. Analysis of convergence is investigated. At the end, some numerical examples are presented to show the effectiveness of the method.

1. Introduction

Consider the following nonlinear Volterra integral equation of second kind

$$y(t) = f(t) + \int_0^t k(t, x, y(x))dx \tag{1}$$

When $k(t, x, y)$ is nondecreasing in y and satisfies in lipschitz condition.

There has been a growing interest in the Volterra integral equations in many fields of physics and engineering [11], for example,heat conduction problem [22], concrete problem of mechanics or physics[33], on the unsteady poiseuille flow in a pipe [15], diffusion problems[24], potential theory and Dirichlet problems, electrostatics[16], the particle transport problems of astrophysics and reactor theory and contact problems[14] has arisen. Also, the fractional differential equations of various types, play important roles not only in mathematics but also in physics, control systems, dynamical systems and engineering to create the mathematical modeling of many physical phenomena, can be converted to Volterra integral equation. In [1–7](see also, [20, 21, 23, 32]) A.Akgul et al. solved many important models of fractional differential equations by reproducing kernel method.

Recently, there are many numerical methods for solving Volterra integral equation of the second kinds; for example, Bernstein Polynomial method has been used in [8] for Solving Volterra Integral Equations with Convolution Kernels by Alturk, A. Maleknejad and Aghazadeh proposed Taylor series expansion method for solving this equation[18]. Farshid Mirzaee and Elham Hadadiyan applied hat functions to solve nonlinear Stratonovich Volterra integral equation[13]. In [10] A.Shoja et al. solved the nonlinear

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singular Volterra integral equations of Abel type be using A spectral iterative method. Rashidinia and Zarebnia [30] obtained a numerical solution of the integral equation by Sinc-collection method. In [17] legendre wavelet has been proposed for numerical solution of Volterral integral equation of the second kind by Maleknejad, Tavassoli and Mahmoudi. In [36] Z. Gouyandeha et al. solved the nonlinear Volterra-Fredholm-Hammerstein integral equations via Tau-collocation method.

The structure of this paper is organized as follows. In section 2, We briefly introduce Quasi-linearization method . Section 3, explains the Non-polynomial spline functions. Section 4, shows the collocation method by using Non-polynomials functions to approximate the solution of the integral equations. Section 5, is devoted to convergence analysis. Some numerical results are given to clarify the method in section 6, furthermore in example (3) we employed the nonpolynomial Spline method for solving the linear Volterra integral equation. At the end, we have a conclusion of our study.

2. Linearization

The method of Quasi-linearization pioneered by Bellman and Kalaba [25] and generalized by Lakshmikantham [34, 35] has been applied to a variety of problems. Consider the nonlinear Volterra integral equation (1), to solve this equation we employ the following iterative scheme for $p = 1, 2, ...$

$$y_p(t) = f(t) + \int_0^t \left[k(t, x, y_{p-1}(x)) + k_y(t, x, y_{p-1}(x))(y_p(x) - y_{p-1}(x))\right] dx, \tag{2}$$

which is the linear Volterra integral equation where $y^0(x)$ is the lower solution of (1).
For $T \in \mathbb{R}$ and $T > 0$ let $J = [0, T]$ and $D = \{(t, x) \in J \times J : x \leq t\}$, consider the equation (1) where $f \in C[J, \mathbb{R}]$ and $k \in C[D, \mathbb{R}]$ also, $k(t, x, y)$ is nondecreasing in y for each fixed $(t, x) \in D$ and satisfies Lipschitz condition.

Definition 2.1. *A function $\Im \in C[J, \mathbb{R}]$ is called a lower solution of Eq.(1) on J if*

$$\Im(t) \leq f(t) + \int_0^t k(t, x, \Im(x))dx, \quad t \in J$$

Now, for $\Im_0 \in C[J, \mathbb{R}]$ and $\Im_0 \leq y$ on J, let $\Omega = \{(t, x, y) \in D \times \mathbb{R}; \Im_0(t) \leq y, t \in J\}$

Theorem 2.2. *Assume that*

(a_0) $\Im_0 \in C[J, \mathbb{R}]$ *is lower solution of Eq.(1) on J.*

(a_1) $k \in C^2[\Omega, \mathbb{R}], k_y(t, x, y) \geq 0, k_{yy}(t, x, y) \geq 0$ *for* $(t, x, y) \in \Omega$.

Then the iterative scheme (2) defines a nondecreasing sequence $\{\Im_p(t)\}$ in $C[J, \mathbb{R}]$ such that $\Im \to y$ uniformly on J, and the following quadratic convergent estimate holds:

$$\|y - \Im_p\| \leq \delta \|y - \Im_{p-1}\|^2, \quad \delta > 0$$

The equation (2) may be shown in the form of the following linear integral equation

$$y_p(t) = F_p(t) + \int_0^t k_p(t, x)y_p(x)dx, \quad p = 1, 2, ... \tag{3}$$

where

$$F_p(t) = f(t) + \int_0^t \left[k(t, x, y_{p-1}(x)) - k_y(t, x, y_{p-1}(x))y_{p-1}(x)\right] dx, \quad p = 1, 2, ... \tag{4}$$

and

$$k_p(t, x) = k_y(t, x, y_{p-1}(x)), \quad p = 1, 2, ... \tag{5}$$

In continuation, we want to approximate the solution of Eq(3) by using Non-polynomial spline functions.

3. Non-polynomial Spline Method

We consider a uniform mesh Δ with nodal points x_i on $[a, b]$ such that

$$\Delta : \; a = x_0 < x_1 < ... < x_n = b,$$

where $h = \frac{b-a}{n}$. Let $S_k(x)$ be the interpolating Non-polynomial spline function which interpolate y at x_k, by following our previous study ([26–29, 31]), and others' researches such as [9] for each segment $[x_l, x_{l+1}]$, $l = 0, ..., n-1$ the Non-polynomial spline interpolation,$S_\Delta(x)$, has the form

$$S_\Delta(x, \tau) = a_l + b_l(x - x_l) + c_l sin\tau(x - x_l) + d_l cos\tau(x - x_l), \qquad l = 0, 1, ..., n \tag{6}$$

where a_l, b_l, c_l and d_l are real constant and τ is a arbitrary parameter. we denote the following relations

$$S_\Delta(x_k, \tau) = y_k, S_\Delta(x_{k+1}, \tau) = y_{k+1}, S''_\Delta(x_k, \tau) = M_k, S''_\Delta(x_{k+1}, \tau) = M_{k+1}, \tag{7}$$

using (6) and (7) we have the following expressions

$$a_l = y_l + \frac{M_l}{\tau^2},\; b_l = \frac{y_{l+1} - y_l}{h} + \frac{M_{l+1} - M_l}{\tau},\; c_l = \frac{y_l cos\theta - M_{l+1}}{\tau^2 sin\theta},\; d_l = \frac{-M_l}{\tau^2}, \tag{8}$$

where $\theta = \tau h$ and $l = 0, ..., n$. with the continuity of first derivatives of $s_{l-1}(x)$ and $s_l(x)$ at $x = x_l$,$l = 1, 2, ..., n-1$, we obtain the following consistency relation,

$$\alpha M_{l-1} + 2\beta M_l + \alpha M_{l+1} = \frac{1}{h^2}(y_{l+1} - 2y_l + y_{l-1}), \tag{9}$$

where $\alpha = \frac{1}{\theta^2}(\theta csc\theta - 1)$, $\beta = \frac{1}{\theta^2}(1 - \theta cot\theta)$. If let, $\alpha = \frac{1}{12}$ and $\beta = \frac{5}{12}$ we have the following system which is strictly diagonaly dominant. Obviously system (9) with the natural spline initial condition $M_0 = M_n = 0$ has a unique solution to obtain $M_1, ..., M_{n-1}$. In the matrix notation, the above system has the form:

$$\underbrace{\begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ \vdots \\ \vdots \\ M_{n-2} \\ M_{n-1} \\ M_n \end{bmatrix}}_{M} = \frac{12}{h^2} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 0 \\ 1 & 10 & 1 & 0 & \dots & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 10 & 1 & \dots & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \dots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \dots & 1 & 10 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & 1 \end{bmatrix}^{-1}}_{W^{-1}} \underbrace{\begin{bmatrix} 0 \\ y_0 - 2y_1 + y_2 \\ y_0 - 2y_1 + y_2 \\ \vdots \\ \vdots \\ y_{n-3} - 2y_{n-2} + y_{n-1} \\ y_{n-2} - 2y_{n-1} + y_n \\ 0 \end{bmatrix}}_{JY}, \tag{10}$$

Now, if we suppose $W^{-1} = (u_{i,j})_{1\le i,j\le n+1}$ and $Z = (z_{i,j}) = W^{-1}J$

$$z_{i,j} = \begin{cases} u_{i,j+1} & \text{if } j = 1, \\ u_{i,j+1} - 2u_{i,j} & \text{if } j = 2, \\ u_{i,j-1} - 2u_{i,j} + u_{i,j+1} & \text{if } 1 \le i \le n+1,\;\; 3 \le j \le n-1\,, \\ u_{i,j-1} - 2u_{i,j} & \text{if } j = n, \\ u_{i,j-1} & \text{if } j = n+1\,. \end{cases} \tag{11}$$

4. Non-polynomial Spline Method and Discretization

Considering the nonlinear Volterra integral equation (3), by using equation (6), (8) and employing the collocation method for $i = 0, 1, 2, ..., n$ we have

$$
\begin{aligned}
& y_p(t_i) = F_p(t_i) + \int_0^{t_i} k_p(t_i, x) y_p(x) dx \\
& = F_p(t_i) + \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} k_p(t_i, x) y_p(x) dx \\
& \approx F_p(t_i) + \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} k_p(t_i, x) S_j^p(x) dx + O(h^4) \\
& = F_p(t_i) + \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} k_p(t_i, x) \Big[y_j^p + \frac{M_j^p}{\tau^2} + \left(\frac{y_{j+1}^p}{h} + \frac{M_{j+1}^p}{\tau\theta} \right)(x - x_j) \\
& - \left(\frac{y_j^p}{h} + \frac{M_j^p}{\tau\theta} \right)(x - x_j) + \left(\frac{M_j^p \cos\theta}{\tau^2 \sin\theta} \right) \sin\tau(x - x_j) - \left(\frac{M_{j+1}^p}{\tau^2 \sin\theta} \right) \sin\tau(x - x_j) \\
& - \frac{M_j^p}{\tau^2} \cos\tau(x - x_j) \Big] dx + O(h^4) \\
& = F_p(t_i) + \sum_{j=0}^{i-1} \left(y_j^p + \frac{M_j^p}{\tau^2} \right) \underbrace{\int_{t_j}^{t_{j+1}} k_p(t_i, x) dx}_{a_{ij}} + \sum_{j=0}^{i-1} \left(\frac{y_{j+1}^p}{h} + \frac{M_{j+1}^p}{\tau\theta} \right) \underbrace{\int_{t_j}^{t_{j+1}} k_p(t_i, x)(x - x_j) dx}_{b_{ij}} \\
& - \sum_{j=0}^{i-1} \left(\frac{y_j^p}{h} + \frac{M_j^p}{\tau\theta} \right) \underbrace{\int_{t_j}^{t_{j+1}} k_p(t_i, x)(x - x_j) dx}_{c_{ij}} + \sum_{j=0}^{i-1} \left(\frac{M_j^p \cos\theta}{\tau^2 \sin\theta} \right) \underbrace{\int_{t_j}^{t_{j+1}} k_p(t_i, x) \sin[\tau(x - x_j)] dx}_{d_{ij}} \\
& - \sum_{j=0}^{i-1} \left(\frac{M_{j+1}^p}{\tau^2 \sin\theta} \right) \underbrace{\int_{t_j}^{t_{j+1}} k_p(t_i, x) \sin[\tau(x - x_j)] dx}_{e_{ij}} - \sum_{j=0}^{i-1} \frac{M_j^p}{\tau^2} \underbrace{\int_{t_j}^{t_{j+1}} k_p(t_i, x) \cos[\tau(x - x_j)] dx}_{p_{ij}} + O(h^4) \\
& = F_p(t_i) + \sum_{j=0}^{i} y_j^p a_{ij} + \frac{1}{\tau^2} \sum_{j=0}^{i} M_j^p a_{ij} + \frac{1}{h} \sum_{j=0}^{i} y_j^p b_{ij} + \frac{1}{\tau\theta} \sum_{j=0}^{i} M_j^p b_{ij} - \frac{1}{h} \sum_{j=0}^{i} y_j^p c_{ij} - \frac{1}{\tau\theta} \sum_{j=0}^{i} M_j^p c_{ij} \\
& + \frac{\cos\theta}{\tau^2 \sin\theta} \sum_{j=0}^{i} M_j^p d_{ij} - \frac{1}{\tau^2 \sin\theta} \sum_{j=0}^{i} M_j^p e_{ij} - \frac{1}{\tau^2} \sum_{j=0}^{i} M_j^p p_{ij} + O(h^4), i = 1, 2, ..., n
\end{aligned} \tag{12}
$$

The above integrant can be determined by any quadrature methods such as five-point Gauss Legendre. Also assume, $A^p = (a_{ij}^p), B^p = (b_{ij}^p), C^p = (c_{ij}^p), D^p = (d_{i,j}^p), E^p = (e_{i,j}^p)$ and $P^p = (p_{i,j}^p)$, which are lower triangular matrices.Now, if we suppose $\hat{M}^p \approx M^p = (M_0^p, M_1^p, M_2^p, \ldots, M_{n-1}^p, M_n^p)^T$, $\hat{Y}^p \approx Y^p = (y_0^p, y_1^p, y_2^p, \ldots, y_{n-1}^p, y_n^p)^T$ and $F^p = (F_0^p, F_1^p, F_2^p, \ldots, F_{n-1}^p, F_n^p)^T$, we have

$$
\hat{Y}^p = F^p + \frac{1}{\tau^2} \left(A^p + \frac{1}{h} B^p - \frac{1}{h} C^p + \cot\theta D^p - \csc\theta E^p - P^p \right) \hat{M}^p + (A^p + \frac{1}{h} B^p - \frac{1}{h} C^p) \hat{Y}^p \tag{13}
$$

Using (10) we have

$$\Big[I-\underbrace{(A^p+\frac{1}{h}B^p-\frac{1}{h}C^p)}_{H_1}-\underbrace{\frac{12}{\theta^2}\Big(A^p+\frac{1}{h}B^p-\frac{1}{h}C^p+\cot\theta D^p-\csc\theta E^p-P^p\Big)}_{H_2}Z\Big]\hat{Y}^p=F^p \tag{14}$$

Eventually the collocation Eq(13) is deformed to the following linear algebraic system

$$\Rightarrow\Big[I-(H_1^p+H_2^pZ)\Big]\hat{Y}^p=F^p,\quad p=1,2,... \tag{15}$$

Finally we can approximate the exact solution y by the Non-polynomial spline function $\hat{S}^p$ such that $\hat{S}^p=\hat{S}_i^p$ on $\Omega_i=[x_i,x_{i+1}]$, $i=0,1,...,n-1$, where

$$\begin{aligned}\hat{S}_i^p(x)&=\hat{y}_i^p+\frac{\hat{M}_i^p}{\tau^2}+\left(\frac{\hat{y}_{i+1}^p}{h}+\frac{\hat{M}_{i+1}^p}{\tau\theta}\right)(x-x_i)-\left(\frac{\hat{y}_i^p}{h}+\frac{\hat{M}_i^p}{\tau\theta}\right)(x-x_i)\\&+\left(\frac{\hat{M}_i^p\cos\theta}{\tau^2\sin\theta}\right)\sin\tau(x-x_i)-\left(\frac{\hat{M}_{i+1}^p}{\tau^2\sin\theta}\right)\sin\tau(x-x_i)-\frac{\hat{M}_i^p}{\tau^2}\cos\tau(x-x_i)\end{aligned} \tag{16}$$

5. Analysis of Convergence

Lemma 5.1. *Assume A be a $n\times n$ matrix with $\|A\|_\infty<1$, then, the matrix $(I-A)$ is invertible. in addition to $\|(I-A)^{-1}\|_\infty\le\frac{1}{1-\|A\|_\infty}$.*

Now, If W be a tridiagonal matrix with the inverse $W^{-1}=(u_{i,j})_{1\le i,j\le n+1}$ by using [12] we can proof the following lemmas.

Lemma 5.2. $u_{i,i}=\frac{\alpha_{i-1}\alpha_{n-i+1}}{\alpha_n}$, $\forall i=1,...,n+1$, *where* $\alpha_0=1$ *and* $\alpha_i=\frac{\sqrt{6}}{24}((5+\sqrt{24})^i-(5-\sqrt{24})^i)$
Proof: *for* $i=1,2,n,n+1$ *proof is clear but for* $i=3,...,n-1$ *we have*

$$\begin{aligned}u_{i,i}&=\frac{\alpha_{i-1}\alpha_{n-i+1}}{\alpha_i\alpha_{n-i+1}-\alpha_{n-i}\alpha_{i-1}}\\&=\frac{((5-2\sqrt{6})^{i-1}-(5+2\sqrt{6})^{i-1})(-(5-2\sqrt{6})^{n-i+1}+(5+2\sqrt{6})^{n-i+1})}{-4\sqrt{6}((5+2\sqrt{6})^n-(5-2\sqrt{6})^n)}\\&=\frac{\alpha_{i-1}\alpha_{n-i+1}}{\alpha_n}\end{aligned}$$

Lemma 5.3. $u_{1,n+1}=u_{n+1,1}=u_{1,j}=u_{n+1,j}=0$, *and*

$$u_{i,j}=\begin{cases}(-1)^{j-i}\frac{\alpha_{i-1}}{\alpha_{j-1}}u_{j,j} & \text{if } i<j\,,\\(-1)^{i-j}\frac{\alpha_{n-i+1}}{\alpha_{n-j+1}}u_{j,j} & \text{if } i>j\,.\end{cases}$$

$\forall i=2,...,n$, $\forall j=1,2,...,n+1$

Remark 1. It is clear that $\forall i,j=1,2,...,n+1$, we have
$\alpha_i=10\alpha_{i-1}-\alpha_{i-2}$ and $\alpha_n=\alpha_i\alpha_{n-i+1}-\alpha_{n-i}\alpha_{i-1}$

Lemma 5.4. $u_{i,j}=u_{j,i}$, $\forall i,j=2,3,...,n$

Proof:

$$\frac{u_{i,j}}{u_{j,i}} = \frac{\alpha_{i-1}}{\alpha_{j-1}} \times \frac{\alpha_{n-i+1}}{\alpha_{n-j+1}} \times \frac{\frac{\alpha_i}{\alpha_{i-1}} - \frac{\alpha_{n-i}}{\alpha_{n-i+1}}}{\frac{\alpha_j}{\alpha_{j-1}} - \frac{\alpha_{n-j}}{\alpha_{n-j+1}}}$$

$$= \frac{\alpha_i\alpha_{n-i+1} - \alpha_{i-1}\alpha_{n-i}}{\alpha_j\alpha_{n-j+1} - \alpha_{j-1}\alpha_{n-j}}$$

$$= \frac{(5+\sqrt{24})^{n+1} + (5-\sqrt{24})^{n+1} - (5+\sqrt{24})^{n-1} - (5-\sqrt{24})^{n-1}}{(5+\sqrt{24})^{n+1} + (5-\sqrt{24})^{n+1} - (5+\sqrt{24})^{n-1} - (5-\sqrt{24})^{n-1}} = 1 \quad \square.$$

Remark 2. In addition above lemmas it is easy to show that the following properties hold
$u_{i,1} = -u_{i,2}, \quad u_{i,n} = -u_{i,n+1}, \forall i, j = 2, 3, ..., n$
$u_{n-i+2,n-i+2} = u_{i,i} > 0$ and $u_{n-i+2,n-j+2} = u_{i,j}, \forall i, j = 1, 2, 3, ..., n+1$

Lemma 5.5. $\|Z\|_\infty \leq \frac{241(5+2\sqrt{6})}{20\sqrt{6}(48+2\sqrt{6})}$
Proof: *We suppose* $\zeta_i = \sum_{j=1}^{n+1} |z_{i,j}|$ *for* $2 \leq i \leq n-1$, *then*

$$\zeta_i \leq 4\left(\frac{u_{i,i}}{\alpha_{i-1}}\sum_{j=2}^{i}\alpha_{j-1} + \frac{u_{i,i}}{\alpha_{n-i+1}}\sum_{j=i+1}^{n}\alpha_{n-j+1}\right) + 2\frac{u_{i,i}}{\alpha_{n-i+1}}$$

$$= 4u_{i,i}\left(\frac{\frac{1}{2}\left[(5+2\sqrt{6})\frac{(5+2\sqrt{6})^{i-1}-1}{2+\sqrt{6}} - (5-2\sqrt{6})\frac{(5-2\sqrt{6})^{i-1}-1}{2-\sqrt{6}}\right]}{(5+2\sqrt{6})^{i-1} - (5-2\sqrt{6})^{i-1}}\right)$$

$$+ 2u_{i,i}\left(\frac{(5+2\sqrt{6})\frac{(5+2\sqrt{6})^{n-i}-1}{2+\sqrt{6}} - (5-2\sqrt{6})\frac{(5-2\sqrt{6})^{n-i}-1}{2-\sqrt{6}} + 1}{(5+2\sqrt{6})^{n-i+1} - (5-2\sqrt{6})^{n-i+1}}\right)$$

$$\leq \frac{241}{20\sqrt{6}}u_{i,i}$$

$$\leq \frac{241(5+2\sqrt{6})}{20\sqrt{6}(48+2\sqrt{6})}, \quad \square.$$

also, it can be shown that $\zeta_1 = \zeta_{n+1} = 0$, *and*

$$\zeta_n = \sum_{j=1}^{n+1}|z_{n,j}| = \sum_{j=1}^{n-1}|z_{n,j}| + |u_{n,n-1} - 2u_{n,n}| + |u_{n,n}|$$

$$\leq 4\sum_{j=2}^{n}|u_{n,j}| + |u_{n,n-1} - 2u_{n,n}| + |u_{n,n}|$$

$$= 4\frac{u_{n,n}}{\alpha_{n-1}}\sum_{j=1}^{n-2}\alpha_j + \frac{7\alpha_{n-1}}{\alpha_n} + \frac{\alpha_{n-2}}{\alpha_n}$$

$$= 2u_{n,n}\left(\frac{(5+2\sqrt{6})\frac{(5+2\sqrt{6})^{n-2}-1}{2+\sqrt{6}} - (5-2\sqrt{6})\frac{(5-2\sqrt{6})^{n-2}-1}{2-\sqrt{6}}}{(5+2\sqrt{6})^{n-1} - (5-2\sqrt{6})^{n-1}}\right) + \frac{7\alpha_{n-1}}{\alpha_n} + \frac{\alpha_{n-2}}{\alpha_n}$$

$$\leq \frac{166+68\sqrt{6}}{218+89\sqrt{6}} \quad \square.$$

Theorem 5.6. *Let $f \in C^4(I)$ and $k \in C^4(I \times I)$ such that*

$$||K||(b-a)\left[\frac{2892(5+2\sqrt{6})}{20\sqrt{6}(48+2\sqrt{6})\theta^2}(1-\tan\frac{\theta}{2})+2\right] < 1,$$

then (16) *define a unique approximation and the resulting error $\hat{e} := y - \hat{s}$ satisfies*

$$||\hat{e}||_{\infty\psi} \leq \alpha h^4, \forall\psi \subset I,$$

where α is a constant .
Proof:*It is easy to verify that $||A||_\infty, ||D||_\infty, ||E||_\infty$ and $||P||_\infty \leq ||k||_\infty(b-a)$ and also $||B||_\infty, ||C||_\infty \leq \frac{||k||_\infty(b-a)h}{2}$, hence, $||H_1^p||_\infty \leq 2||k||_\infty(b-a)$ and $||H_2^p||_\infty \leq \frac{12}{\theta^2}||k||_\infty(b-a)[1-\tan\frac{\theta}{2}]$, then we have*

$$||H_1^p + H_2^p Z||_\infty \leq ||K||_\infty(b-a)\left[\frac{2892(5+2\sqrt{6})}{20\sqrt{6}(48+2\sqrt{6})\theta^2}(1-\tan\frac{\theta}{2})+2\right] < 1$$

*Now by **lemma** 5.1 the system (15) has a unique solution $\hat{y}$. It follows that the Eq(16) define a unique solution $\hat{S}$. Now, let $\hat{e} = y - \hat{y} = (y_0 - \hat{y}_0, ..., y_n - \hat{y}_n)^T$. then from (12) we get $(I - (H_1^p + H_2^p Z))\hat{e} = O(h^4)$. Therefor,$\hat{e} = (I - (H_1^p + H_2^p Z))O(h^4)$, which implies by **lemma** 5.1, that there exist $\alpha_1 > 0$ such that*

$$||\hat{e}||_\infty \leq \underbrace{\frac{\alpha_1}{1-||K||_\infty(b-a)\left[\frac{2892(5+2\sqrt{6})}{20\sqrt{6}(48+2\sqrt{6})\theta^2}(1-\tan\frac{\theta}{2})+2\right]}}_{\alpha_2} h^4$$

. On the other hand, from (10), we have $(M - \hat{M}) = \frac{12}{h^2}Z\hat{e}$.Therefor, $||Z - \hat{Z}||_\infty \leq 12\alpha_2 h^4$.
In consequence, for all $i = 0, 1..., n-1$ and $x \in \Omega_i$, we have

$$|S_i(x) - \hat{S}_i(x)| \leq 12\alpha_2 h^4$$

It follows that

$$||Y - \hat{S}||_\infty \leq ||Y - S||_\infty + ||S - \hat{S}||_\infty \leq \alpha_1 h^4 + 12\alpha_2 h^4$$

Thus, the proof is completed by taking $\alpha = \alpha_1 + 12\alpha_2$. □

6. Computational Illustrations

In this section, we have implemented our method (NPS) for solving examples of the nonlinear Volterra integral equation, to show the efficiency of the proposed numerical method. The absolute error in the solution are compared with the similar method in [19]. All the computations are performed by **MATLAB R2013a**.
Example 1: Consider the following Volterra integral equation

$y(t) = 2 - e^t + \int_0^t e^{t-x}y^2(x)dx, \quad t \in [0, 1],$

where, one of the lower solutions is $y^0(t) = 1 - e^t$ and the iterative scheme is

$$y_p(t) = \left(2 - e^t - \int_0^t e^{t-x}(y_{p-1}(x))^2 dx\right) + 2\int_0^t e^{t-x}(y_{p-1}(x))y_p(x)dx.$$

The absolute errors in the solution presented in **Table 1**. The exact solution is given by the relation $y(t) = 1$.

Table 1: Absolute errors for Example 1.

p	3	3	6	**5**
t	Best in [19]	NPS method(τ=1.0×10^{+6})	Best in [19]	NPS method(τ=1.0×10^{+9})
0	2.2×10^{-3}	0.0	1.4×10^{-15}	0.0
0.1	2.5×10^{-3}	9.9×10^{-9}	1.7×10^{-15}	2.2×10^{-16}
0.2	7.9×10^{-4}	1.0×10^{-6}	6.6×10^{-16}	2.2×10^{-16}
0.3	1.4×10^{-3}	2.1×10^{-5}	1.5×10^{-15}	2.2×10^{-16}
0.4	1.6×10^{-3}	2.1×10^{-4}	2.1×10^{-15}	3.3×10^{-16}
0.5	8.9×10^{-4}	1.3×10^{-3}	1.2×10^{-15}	2.2×10^{-16}
0.6	3.6×10^{-3}	6.0×10^{-3}	1.1×10^{-15}	1.1×10^{-16}
0.7	1.7×10^{-2}	2.1×10^{-2}	2.4×10^{-15}	8.8×10^{-16}
0.8	5.5×10^{-2}	5.7×10^{-2}	1.3×10^{-15}	1.1×10^{-16}
0.9	1.3×10^{-1}	1.3×10^{-1}	1.5×10^{-14}	1.3×10^{-15}
1	2.6×10^{-1}	2.6×10^{-1}	4.5×10^{-14}	1.9×10^{-15}

Example 2: Consider the following Volterra integral equation

$$y(t) = \sin(t) - e^{\sin(t)} + 1 + \int_0^t \cos(x) e^{y(x)} dx, \quad t \in [0, 1],$$

where, one of the lower solutions is $y^0(t) = 0$.

The absolute errors in the solution presented in **Table 2**. The exact solution is given by the relation $y(t) = \sin(t)$.

Table 2: Absolute errors for Example 2.

p	2	2	5	5
t	Best in [19]	NPS method(τ=1.0×10^{+3})	Best in [19]	NPS method(τ=1.0×10^{+9})
0	9.7×10^{-5}	0.0	7.1×10^{-6}	0.0
0.1	8.2×10^{-5}	4.7×10^{-9}	6.2×10^{-6}	7.6×10^{-12}
0.2	5.7×10^{-5}	5.4×10^{-8}	4.6×10^{-6}	2.9×10^{-11}
0.3	8.3×10^{-5}	7.3×10^{-7}	6.2×10^{-6}	7.1×10^{-11}
0.4	1.3×10^{-4}	5.7×10^{-6}	1.4×10^{-6}	1.4×10^{-10}
0.5	6.7×10^{-4}	2.9×10^{-5}	7.0×10^{-6}	2.4×10^{-10}
0.6	2.1×10^{-3}	1.0×10^{-4}	3.6×10^{-6}	3.8×10^{-10}
0.7	4.9×10^{-3}	3.1×10^{-4}	4.2×10^{-6}	5.7×10^{-10}
0.8	8.6×10^{-3}	7.8×10^{-4}	4.7×10^{-6}	7.9×10^{-10}
0.9	1.2×10^{-2}	1.6×10^{-3}	5.6×10^{-6}	1.0×10^{-9}
1	1.3×10^{-2}	3.0×10^{-3}	6.7×10^{-6}	2.9×10^{-8}

Example 3: Consider the following Volterra integral equation

$$y(t) = \sin(t) + \cos(t) + \int_0^t 2\sin(t-x) y(x) dx, \quad t \in [0, 1],$$

The absolute errors in the solution presented in **Table 3**. The exact solution is given by the relation $y(t) = e^t$

Table 3: Absolute errors for Example 3.

t	Best in [8]	NPS method(τ=1.0×10^{+6})	NPS method(τ=1.0×10^{+9})
0	2.8×10^{-6}	0.0	0.0
0.1	1.1×10^{-6}	9.9×10^{-9}	6.5×10^{-16}
0.2	2.7×10^{-7}	1.0×10^{-6}	2.3×10^{-16}
0.3	1.2×10^{-6}	2.1×10^{-5}	1.6×10^{-16}
0.4	1.2×10^{-5}	2.1×10^{-5}	2.7×10^{-16}
0.5	4.7×10^{-5}	1.3×10^{-5}	8.6×10^{-17}
0.6	1.4×10^{-4}	6.0×10^{-6}	2.7×10^{-16}
0.7	3.6×10^{-4}	2.1×10^{-5}	1.1×10^{-16}
0.8	8.2×10^{-4}	5.7×10^{-5}	2.2×10^{-16}
0.9	1.7×10^{-3}	1.3×10^{-5}	1.7×10^{-15}
1	3.2×10^{-3}	2.6×10^{-5}	2.5×10^{-15}

7. Conclusion

The present work is an effort to obtaining the numerical solution of Volterra integral equation of the second kind. Analysis of convergence is investigated. Three test examples are considered from previous work in [19]. The computational solutions are compared with the exact solution. The absolute errors in the solutions by our *NPS* method are accurate in comparison with [19] and [8].

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