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Non-Archimedean Closed Graph Theorems

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Abstract. We consider linear maps $T\colon X \to Y$, where X and Y are polar local convex spaces over a complete non-archimedean field K. Recall that X is called polarly barrelled, if each weakly* bounded subset in the dual X' is equicontinuous. If in this definition *weakly* bounded subset* is replaced by *weakly* bounded sequence* or *sequence weakly* converging to zero*, then X is said to be ℓ^∞-barrelled or c_0-barrelled, respectively. For each of these classes of locally convex spaces (as well as the class of spaces with weakly* sequentially complete dual) as domain class, the maximum class of range spaces for a closed graph theorem to hold is characterized. As consequences of these results, we obtain non-archimedean versions of some classical closed graph theorems.

The final section deals with the necessity of the above-named barrelledness-like properties in closed graph theorems. Among others, counterparts of the classical theorems of Mahowald and Kalton are proved.

1. Introduction

The Banach closed graph theorem is valid under very general circumstances. If X and Y are complete metrizable topological vector spaces over an arbitrary complete valued field K, then each linear map $T\colon X \to Y$ with closed graph is continuous. In the classical case, where $K = \mathbb{R}$ or $K = \mathbb{C}$, a well-known result of V. Pták [15, Theorem 4.9] states that the theorem remains true if X is barrelled and Y is a B_r-complete locally convex space. (Recall that a locally convex space Y is said to be B_r-complete, if every weakly* dense subspace S of the dual Y' whose intersection with each equicontinuous subset $H \subset Y'$ is weakly* closed in H, is necessarily weakly* closed, that is, $S = Y'$.) Since Fréchet spaces are B_r-complete, Pták's theorem holds for any Fréchet space Y. Moreover, by a famous theorem of M. Mahowald [9], the assumption of barrelledness of X is necessary for this result to hold. N. J. Kalton [6] has considered the case where the range space Y is an arbitrary separable B_r-complete locally convex space. He proved that in this situation each linear map $T\colon X \to Y$ with closed graph is weakly continuous if and only if the dual X' is weakly* sequentially complete. If, in addition, X is a Mackey space, then (according to the Hellinger-Toeplitz theorem) *weakly continuous* can be replace by *continuous.* The maximal class of range spaces (L_r-spaces) for Kalton's closed graph theorem was identified by J. H. Qiu [16].

Another line of development in closed graph theorms (also starting from the classic Banach theorem) is concerned with webbed spaces introduced by M. De Wilde. The class of webbed spaces contains all Fréchet spaces and has good stability properties. One of De Wilde's closed graph theorems states (see [4]): If X is an

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ultrabornological space (i.e. an inductive limit of Banach spaces) and Y is a webbed space, then any linear map $T\colon X \to Y$ with sequentially closed graph is continuous. For a comparison of De Wilde's approach to closed graph theorems with the one due to Pták, we refer to the book of G. Köthe [7, the end of §35.5].

The non-archimedean version of Pták's theorem in which the range space Y is an arbitrary Fréchet space (originally proved by T. E. Gilsdorf and J. Kakol [5]) one finds in the book by C. Perez-Garcia and W. H. Schikhof [13, Theorem 11.1.10]. The proof is based on usual technique of metrizable topological vector spaces. In this paper, using methods from duality theory, we establish some closed graph theorems of this type, where barrelled spaces X are replaced by some other barrelled-like locally convex spaces. We study (weak) continuity of linear maps $T\colon X \to Y$ with closed graph, where X is assumed to be either polarly barrelled, ℓ^∞-barrelled, c_0-barrelled, or to have the property, that the dual space X' is weakly* sequentially complete. By use of a general technique (based on duality arguments) which is adapted from a paper by J. Boos and the author [2], we prove in Section 4 Theorem 4.1, from which we deduce maximal classes of range space Y in each of four cases. Also we obtain non-archimedean versions of some related classical theorems, most importantly among them theorems of Pták type and of Kalton type. Here the concept of B_r-completeness plays a crucial role. This concept is justified by the Krein-Šmulian theorem proved by Perez-Garcia and Schikhof [11, Corollary 3.6 and Proposition 3.7] for Fréchet spaces over non-archimedean fields.

Section 5 is concerned with the necessity of the above-named barrelledness-like properties in closed graph theorems. For example, if Y is a Fréchet space of countable type, then a linear map $T\colon X \to Y$ with closed graph is weakly continuous if and only if X' is weakly* sequentially complete (Theorem 5.2).

Section 3 presents (besides of a brief discussion on B_r-completeness of Fréchet spaces) a theorem of Hellinger-Toeplitz type in the non-archimedean situation. As in the classical case, weak continuity is an intermediate step in the proof of closed graph theorems; the rest (that is, the continuity with respect to the Mackey topologies (provided that they exist)) follows from the Hellinger-Toeplitz theorem.

It turns out that if K is spherically complete, then the classical closed graph theorems considered here hold word for word for the non-archimedean setting. If K is not spherically complete, the trouble is that there exist Banach spaces which are not B_r-complete (Proposition 3.4). In addition to that, we do not know to what extent the Hellinger-Toeplitz theorem holds true. Furthermore, it is still open whether the (polar) Mackey topology in any polar locally convex spaces exists if K is not spherically complete.

Note that non-archimedean webbed spaces were introduced by C. T. M. Vinagre [18]. She investigated stability properties of them and proved the non-archimedean version of the above-quoted De Wilde's closed graph theorem. See also F. Bambozzi [3].

2. Preliminaries

Throughout this paper, $K = (K, |\,|)$ is a complete non-archimedean non-trivially valued field. By B_K we denote the closed unit ball of K. For basic notations and properties concerning locally convex spaces over K, we refer to the book [13]. Here we recall some definitions and results needed in this paper.

Let X be a vector space over K, let X^* denote its algebraic dual. A subset $A \subset X$ is called *absolutely convex* if it is a B_K-submodule of X. For a nonempty subset $D \subset X$, the *absolutely convex hull* is defined by $\mathrm{aco}D := \{\sum_{k=1}^{n} a_k x_k \mid n \in \mathbb{N},\ a_k \in B_K,\ x_k \in D\}$. For an absolutely convex subset A of X, we set $A^e := A$ if the valuation of K is discrete and $A^e := \bigcap \{aA \mid a \in K \setminus B_K\}$ if the valuation is dense. A is called *edged*, if $A^e = A$.

A *seminorm* on X is a function $p\colon X \to \overline{\{|a| \mid a \in K\}}$ (the closure in $\mathbb{R}$) such that $p(ax) = |a|\,p(x)$ and $p(x+y) \le \max\{p(x), p(y)\}$ for all $x, y \in X$ and $a \in K$. We denote by B_p the closed unit ball of p. If p is a seminorm on X, then the function $\overline{p}\colon X/\ker p \to \mathbb{R}$, $x + \ker p \mapsto p(x)$ is a norm on the quotient space $X/\ker p$.

A seminorm p is said to be

• *of countable type* if the normed space $X/\ker p$ is of countable type, that is, it has a countable subset C whose linear hull, denoted by span C, is dense in $X/\ker p$;

• *polar* if $p(x) = \sup\left\{\left|f(x)\right| \mid f \in \mathcal{F}_p\right\}$ for all $x \in X$, where

$$\mathcal{F}_p := \left\{f \in X^* \mid \left|f(x)\right| \leq p(x) \ \ (x \in X)\right\}.$$

If K is spherically complete, then each seminorm is polar. Every seminorm of countable type is polar.

Let $X = (X, \tau)$ be a Hausdorff locally convex space (abbreviated LCS) over K. Its topological dual will be denoted by X'. X is said to be *of countable type* if any continuous seminorm on X is of countable type. If the topology τ is defined by a family of polar seminorms, then X (as well as the topology τ) is called *polar*. For any Hausdorff polar LCS X, the dual X' separates the points of X. The weak topology $\sigma(X, X')$ on X with respect to the dual pair $\langle X, X'\rangle$ is defined as usual, as well as the weak* topology $\sigma(X', X)$ on X'.

A locally convex topology ξ on X is called *compatible* (with τ) if $(X, \xi)' = X'$. If there exists a finest compatible polar topology, then it is called the *Mackey topology* and denoted by $\tau(X, X')$. If K is spherically complete, then $\tau(X, X')$ always exists. For a polar LCS over a non-spherically complete field, the problem of existence of the Mackey topology is open. In any case, the Mackey topology on an LCS X (whenever it exists) is determined by the collection of all polar seminorms $p\colon X \to \mathbb{R}$ satisfying $\mathcal{F}_p \subset X'$ ([13, Theorems 5.7.8 and 5.8.8]).

In a polar LCS X, a subset is weakly bounded if and only if it is bounded. Every weakly convergent sequence in a LCS X is convergent when either X is of countable type or the field K is spherically complete. In a LCS over a spherically complete K, every weakly closed absolutely convex subset is closed. If X is of countable type, then every weakly closed edged subset is closed.

Let $A \subset X$ and $B \subset X'$. The *polar sets* A^0 and B^0 (relative to $\langle X, X'\rangle$) are defined by

$$A^0 := \left\{f \in X' \mid \left|f(x)\right| \leq 1 \ \ (x \in A)\right\} \ \text{ and } \ B^0 := \left\{x \in X \mid \left|f(x)\right| \leq 1 \ \ (f \in B)\right\},$$

respectively. The non-archimedean counterpart of the classical Bipolar Theorem states that $A^{00} = \left(\overline{A}^{\sigma(X,X')}\right)^e$ for any absolutely convex subset A of X (cf. [13, Theorem 5.2.7]). A subset $D \subset X$ with $D^{00} = D$ is called *polar*. An LCS X is polar if and only if the polar neighbbourhoods of zero form a neighbourhood basis of zero.

The polar of $A \subset X$ with respect to the duality $\langle X, X^*\rangle$ we denote by $A^{\oplus}$. Note that $\mathcal{F}_p = \left(B_p\right)^{\oplus}$ for a polar seminorm $p\colon X \to \mathbb{R}$.

A Hausdorff LCS X is called
• *barrelled* if each pointwise bounded family of continuous seminorms on X is equicontinuous,
• *polarly barrelled* if each pointwise bounded family of polar continuous seminorms on X is equicontinuous (or, equivalently, each $\sigma(X', X)$-bounded subset of X' is equicontinuous),
• *ℓ^∞-barrelled*, if each $\sigma(X', X)$-bounded sequence in X' is equicontinuous,
• *c_0-barrelled* if each sequence converging weakly* to zero in X' is equicontinuous.
Any polar polarly barrelled LCS admits the Mackey topology. If K is spherically complete or X is of countable type, then X is polarly barrelled if and only if it is barrelled.

A complete metrizable LCS is called a *Fréchet space*; a normable Fréchet space is called a *Banach space*. For a set I, the vector space $\ell^\infty(I)$ of all bounded functions $\phi\colon I \to K$ is a polar Banach space with the sup-norm $\left\|\phi\right\|_\infty := \sup\left\{\left|\phi(\iota)\right| \mid \iota \in I\right\}$. Note that $(\ell^\infty(I), \|\cdot\|_\infty)$ is of countable type if and only if I is finite. The subspace

$$c_0(I) := \left\{\phi \in \ell^\infty(I) \mid \text{for each } \varepsilon > 0, \text{ the set } \left\{\iota \in I \mid \left|\phi(\iota)\right| > \varepsilon\right\} \text{ is finite}\right\}$$

is of countable type if and only if I is countable. Thus the sequence spaces

$$\ell^{\infty} := \ell^{\infty}(\mathbb{N}) = \left\{ \mathbf{x} = (x_k) \mid \|\mathbf{x}\|_{\infty} = \sup_{k\in\mathbb{N}} |x_k| < \infty \right\},$$
$$c_0 := c_0(\mathbb{N}) = \left\{ \mathbf{x} = (x_k) \mid \lim_{k\to\infty} x_k = 0 \text{ in } K \right\},$$
$$c := \left\{ \mathbf{x} = (x_k) \mid \lim \mathbf{x} := \lim_{k\to\infty} x_k \text{ exists} \right\},$$

equipped with the sup-norm, are polar Banach spaces. The sequences $\mathbf{e}_1 := (1,0,0,\dots), \mathbf{e}_2 := (0,1,0,\dots), \dots$ form a Schauder basis of c_0. Since $c = c_0 \oplus \operatorname{span}\{\mathbf{e}\}$, where $\mathbf{e} := (1,1,1,\dots)$, then c is of countable type. Obviously, $\lim \in c'$. It is well known, that $(c_0)' = \ell^{\infty}$. If K is not spherically complete, then $(\ell^{\infty})' = c_0$.

For a topological space H, the function space

$$BC(H) := \left\{ \phi \in \ell^{\infty}(H) \mid \phi\colon H \to K \text{ is continuous} \right\},$$

equipped with the sup-norm, is a Banach space.

All LCS's X and Y considered in this paper are assumed to be Hausdorff. The vector space of all continuous linear maps $T\colon X \to Y$ is denoted by $L(X,Y)$. Finally, X'_{σ} and X^*_{σ} stand for $(X', \sigma(X',X))$ and $(X^*, \sigma(X^*,X))$, respectively. If X is a normed space, then B_X stands for the closed unit ball of X.

3. Non-archimedean Theorems of Hellinger-Toeplitz and Krein-Šmulian Type

Let $X = (X,\tau)$ and $Y = (Y,\tau')$ be LCSs. For a linear map $T\colon X \to Y$ we write $\Delta_T := \{g \in Y' \mid g \circ T \in X'\}$. Then
- $\Delta_T = Y'$ if and only if T is weakly continuous, that is, $T\colon (X, \sigma(X,X')) \to (Y, \sigma(Y,Y'))$ is continuous,
- $\overline{\Delta_T}^{\sigma(Y',Y)} = Y'$ if and only if the graph $\operatorname{Gr}T := \{(x, T(x)) \mid x \in X\}$ of T is closed in $X \times Y$.

Clearly, every continuous linear map is weakly continuous. If T is weakly continuous, then the dual map $T'\colon Y'_{\sigma} \to X'_{\sigma}$ defined by $T'(g) := g \circ T$ is continuous.

In classical functional analysis, a well-known extension of the Hellinger-Toeplitz theorem asserts that every linear map which is weakly continuous, is continuous with respect to the Mackey topologies. The following theorem provides a non-archimedean counterpart of this result.

Theorem 3.1 (*non-archimedean Hellinger-Toeplitz Theorem*). *Let T be a weakly continuous linear map from a polar LCS (X,τ) into a polar LCS (Y,τ'). Suppose that τ is the Mackey topology $\tau(X,X')$. If either*
(a) *K is spherically complete or*
(b) *(X,τ) is polarly barrelled or*
(c) *X'_{σ} is sequentially complete and Y is of countable type,*
then $T \in L(X,Y)$.

Proof. Under the condition (a), the assertion was proved by D. P. Pombo, Jr. [14]. Assume that K is not spherically complete. To prove the assertion for both cases (b) and (c), let q be a continuous polar seminorm on Y. We have to verify that the inclusion $\mathcal{F}_p \subset X'$ holds for the seminorm $p := q \circ T$.

As T is weakly continuous, $T'\colon Y'_{\sigma} \to X'_{\sigma}$ is continuous and $H := T'\left(\mathcal{F}_q\right) \subset X'$.Thus

$$H^0 = \left\{ x \in X \mid \left| f(x) \right| \le 1 \ (f \in H) \right\} = \{ x \in X \mid p(x) \le 1 \} = B_p,$$

from which it follows that $\mathcal{F}_p = H^{0\oplus} = \left(\overline{H}^{\sigma(X^*,X)}\right)^e$ (here $H^{0\oplus}$ is the bipolar of H in X^*). To finish the proof, we show that $\overline{H}^{\sigma(X^*,X)} \subset X'$.

(b) Let X be polarly barrelled, then X'_σ is quasicomplete (cf. [13, Corollary 7.3.7]). For a fixed $f \in \overline{H}^{\sigma(X^*,X)}$, we choose in H a net $\left(f_\gamma\right)$ such that $f_\gamma(x) \to f(x)$ for all $x \in X$. Then $\left(f_\gamma\right)$ (as a bounded Cauchy net) is convergent in X'_σ, which implies $f \in X'$.

(c) Assume that X'_σ is sequentially complete. If Y is of countable type, then $\mathcal{F}_q$ is metrizable in Y'_σ ([13, Theorem 7.6.10]), hence H is a metrizable compactoid in X'_σ ([13, Corollary 3.8.27]). By [13, Theorem 3.8.25], there exists a sequence (h_k) in X'_σ such that $h_k \to 0$ and $H \subset \overline{\mathrm{aco}\{h_k \mid k \in \mathbb{N}\}}^{\sigma(X',X)} =: A$. By [13, Theorem 3.8.24], $(A, \sigma(X',X)|_A)$ is metrizable, then it is complete by assumption. Thus $\overline{A}^{\sigma(X^*,X)} = A$, so we have $\overline{H}^{\sigma(X^*,X)} \subset A \subset X'$. □

As noted above, a linear map $T\colon X \to Y$ has closed graph if and only if the subspace Δ_T is dense in Y'_σ. Since T is weakly continuous if and only if $\Delta_T = Y'$, it is natural to look for properties of Y that guarantee that Δ_T is closed in Y'_σ. Following the approach of Ptàk [15], we attempt to reduce the problem of closedness of Δ_T to that of $\Delta_T \cap H$, where $H \subset Y'$ is equicontinuous.

Definition. Let X be a polar LCS. A subset $A \subset X'$ is said to be *aw^*-closed* if $A \cap \mathcal{F}_p$ is $\sigma(X',X)$-closed in $\mathcal{F}_p$ for every continuous polar seminorm p on X. X is called *B_r-complete* if each aw^*-closed dense subspace S of X'_σ is closed (that is, $S = X'$).

Note that any closed subset of X'_σ is aw^*-closed. A subspace S of the dual X' of a normed space X is aw^*-closed if and only if $S \cap B_{X'}$ is closed in X'_σ.

The following non-archimedean Krein-Šmulian Theorem due to Perez-Garcia and Schikhof ([12, Corollary 3.6 and Proposition 3.7]), characterizes aw^*-closed subsets of the dual of a Fréchet space.

Theorem 3.2. *Let X be a Fréchet space over K.*
(a) *If K is spherically complete, then each absolutely convex aw^*-closed subset $A \subset X'_\sigma$ is closed.*
(b) *If X is of countable type, then each edged aw^*-closed subset $A \subset X'_\sigma$ is closed.*

Corollary 3.3. *Let X be a Fréchet space over K. If either*
(a) *K is spherically complete or*
(b) *X is of countable type,*
then X is B_r-complete.

In [17], a normed space $(X, \|\cdot\|)$ is defined to be a *Krein-Šmulian space* if every edge absolutely convex subset A is closed in X'_σ whenever $\left\{f \in A \mid \left\|f\right\| \leq n\right\}$ is closed for all $n \in \mathbb{N}$. Unlike the classical situation, a Banach space over a non-spherically complete field K need not be a Krein-Šmulian space. Schikhof [17, Corollaries 3.6 an 3.7] proved, that X is not a Krein-Šmulian space if

- $X = c_0(I)$, where $\#I \geq \#K$ (here $\#I$ denotes the cardinality of I), or
- $X = \ell^\infty(I)$, when I is a small set with $\#I \geq \#K^K$ (for the definition of a small set we refer to [13, Appendix 2]).

Moreover, in the first case, there exists a subspace S of $c_0(I)' = \ell^\infty(I)$ that is not $\sigma(\ell^\infty(I), c_0(I))$-closed while $S \cap B_{\ell^\infty(I)}$ is (cf. [17, p. 187]).

Proposition 3.4. *Let K be not spherically complete.*
(a) *There exist Banach spaces, which are not B_r-complete.*
(b) *There exist B_r-complete Banach spaces, which are not Krein-Šmulian spaces and not of countable type.*

Proof. (a) By the preceding remark, we can find a Banach space X such that there is an aw^*-closed subspace S of X'_σ with $\overline{S}^{\sigma(X',X)} \neq S$. We verify that the Banach space X/D is not B_r-complete, whenever $D := S^\perp = \cap \{\ker f \mid f \in S\}$. The adjoint π' of the quotient map $\pi \colon X \to X/D$ is an isometry of $(X/D)'$ to $D^\perp = \overline{S}^{\sigma(X',X)}$, so it is a linear homeomorphism with respect to the weak* topologies. Then $L := (\pi')^{-1}(S)$ is a dense proper subspace of $(X/D)'_\sigma$. Since $S \cap B_{X'}$ is closed in $\overline{S}^{\sigma(X',X)}$, then $L \cap B_{(X/D)'} = (\pi')^{-1}(S \cap B_{X'})$ is closed in $(X/D)'_\sigma$. Hence X/D is not B_r-complete.

(b) Given a small set I, verify that $X := (\ell^\infty(I), \|\cdot\|_\infty)$ is B_r-complete. It is known (cf. [13, Theorem 7.4.3]) that $X' = c_0(I)$. Let S be a dense proper subspace of $X'_\sigma = (c_0(I), \sigma(c_0(I), \ell^\infty(I)))$, that is, $S \subsetneq c_0(I) \subset \overline{S}^{\sigma(c_0(I),\ell^\infty(I))}$; then there exists a $\phi \in B_{c_0(I)} \setminus S$. Evidently, $S \supset \operatorname{span}\{\mathbf{e}_i \mid i \in I\}$, where $(\mathbf{e}_i)_{i\in I}$ is the natural orthonormal basis of the Banach space $c_0(I)$. Then $\phi \in \overline{B_{c_0(I)} \cap \operatorname{span}\{\mathbf{e}_i \mid i \in I\}}^{\|\cdot\|_\infty} \subset \overline{B_{c_0(I)} \cap S}^{\sigma(c_0(I),\ell^\infty(I))}$. We see that $B_{c_0(I)} \cap S$ is not closed in X'_σ; it follows that X is B_r-complete.

According to the preceding remark, we can choose I such that X is not a Krein-Šmulian space. Clearly, X is not of countable type. □

4. Closed Graph Theorems

We first prove a general theorem, from which we obtain the nonarchimedean counterparts of some classical closed graph theorems. The approach proposed here is taken in [2].

Let θ be a condition, which defines in any LCS $H = (H, \zeta)$ a class of bounded nets. If H is fixed, we denote this class by $\theta[H]$ or, more formally, $\theta[(H,\zeta)]$. Assume furthermore that the following requirements are satisfied:

1^0 for each $h \in H$ there exists an $(h_\alpha) \in \theta[H]$ such that $h_\alpha \to h$ in H,

2^0 $(ah_\alpha) \in \theta[H]$ for all $a \in K$ and $(h_\alpha) \in \theta[H]$,

3^0 if $(f_\alpha)_{\alpha\in A}, \left(g_\beta\right)_{\beta\in B} \in \theta[H]$ with $f_\alpha \to f$ and $g_\beta \to g$ in H, then there exists an $\left(h_\gamma\right) \in \theta[H]$ such that $h_\gamma \in \{f_\alpha \mid \alpha \in A\} + \left\{g_\beta \mid \beta \in B\right\}$ and $h_\gamma \to f + g$,

4^0 if X ja Y are vector spaces over K and $T \colon X \to Y$ is a linear map, then $\left(g_\gamma\right) \in \theta[Y^*_\sigma] \Rightarrow \left(g_\gamma \circ T\right) \in \theta[X^*_\sigma]$.

Referring to 4^0, note that, for any bounded subset B of Y^*_σ, the subset $\{g \circ T \mid g \in B\}$ is bounded in X^*_σ. Thus, for each $(g_\alpha) \in \theta[Y^*_\sigma]$, the net $(g_\alpha \circ T)$ is $\sigma(X^*, X)$-bounded.

Definition. A LCS H is *θ-complete* if each Cauchy net $(h_\alpha) \in \theta[H]$ is convergent in H.

Let Y be a vector space over K. For a subset $D \subset Y^*$ we define

$$\overset{\ulcorner\urcorner\theta}{D} := \left\{ g \in Y^* \mid \exists \left(g_\gamma\right)_{\gamma\in\Gamma} \in \theta[Y^*_\sigma] : \left\{g_\gamma \mid \gamma \in \Gamma\right\} \subset D \text{ and } g_\gamma \to g \text{ in } Y^*_\sigma \right\}.$$

The subset D is said to be *θw^*-closed* whenever $D = \overset{\ulcorner\urcorner\theta}{D}$. If $S \subset Y^*$ is a subspace, the set

$$\overset{\llcorner\lrcorner\theta}{S} := \bigcap \left\{ M \subset Y^* \mid S \subset M,\ M \text{ is a subspace and } M = \overset{\ulcorner\urcorner\theta}{M} \right\}$$

is called the *θw^*-closure* of S.

Note that $\overset{\llcorner\lrcorner\theta}{S}$ is the smallest θw^*-closed subspace of Y^* including S. In general, $\overset{\ulcorner\urcorner\theta}{S} \neq \overset{\llcorner\lrcorner\theta}{S}$.

Definition. A polar LCS Y is called an *L_θ-space* if $\overset{\llcorner\lrcorner\theta}{S} \supset Y'$ for any dense subspace $S \subset Y'_\sigma$.

We are now ready to present a ***general nonarchimedean closed graph theorem***.

Theorem 4.1. *For a polar LCS* (Y, τ') *the following statements are equivalent:*
(a) *Y is an L_θ-space,*
(b) *if (X, τ) is a polar LCS such that X'_σ is θ-complete, then each linear map $T\colon (X, \tau) \to (Y, \tau')$ with closed graph is weakly continuous.*
If K is spherically complete and τ is the Mackey topology $\tau(X, X')$, then in (b) weakly continuous *can be replaced by* continuous*.*

Proof. (a) ⇒ (b): Let X be a LCS such that X'_σ is θ-complete, and let $T\colon X \to Y$ be a linear map with closed graph. Since the subspace Δ_T is dense in Y'_σ, then $Y' \subset \overset{\llcorner\lrcorner\theta}{\Delta_T}$ by the assumption (a). In order to prove that T is weakly continuous, it suffices to show that Δ_T is θw^*-closed. For that, let (g_α) be a net in Δ_T such that $(g_\alpha) \in \theta\left[Y^*_\sigma\right]$ and $g_\alpha \to g$ in Y^*_σ. Then $g_\alpha \circ T \in X'$ for each α and $(g_\alpha \circ T) \in \theta\left[X^*_\sigma\right]$. Moreover, $(g_\alpha \circ T)$ is a Cauchy net in X'_σ. So $g \circ T \in X'$ by the assumption that X'_σ is θ-complete. That is, $g \in \Delta_T$, which means that Δ_T is θw^*-closed.

(b) ⇒ (a): Let S be a dense subspace of Y'_σ. Observe that $G := \overset{\llcorner\lrcorner\theta}{S}$ is θ-complete in Y^*_σ (if a net $(g_\alpha) \in \theta\left[Y^*_\sigma\right]$ is $\sigma(Y^*, Y)$-Cauchy in G, then the functional $g \in Y^*$, defined by $g(y) := \lim_\alpha g_\alpha(y)\ \ (y \in Y)$, belongs to G, because G is θw^*-closed). Since the identity map $i\colon (Y, \sigma(Y, G)) \to (Y, \tau')$ clearly has closed graph, then, by (b), it is continuous and we have $G \supset Y'$. Therefore Y is an L_θ-space.

The conclusion of the second part of the theorem follows directly from Theorem 3.1(a). □

Next we apply Theorem 4.1 to obtain closed graph theorems for the following four choices of θ:
- $\theta = \theta_{bn}$, where $\theta_{bn}\left[Y^*_\sigma\right]$ is the class of all bounded nets in Y^*_σ,
- $\theta = \theta_{bs}$, where $\theta_{bs}\left[Y^*_\sigma\right]$ is the class of all bounded sequences in Y^*_σ,
- $\theta = \theta_{\ell^\infty}$, which is defined as follows: $\left(g_\gamma\right)_\gamma \in \theta_{\ell^\infty}\left[Y^*_\sigma\right]$ if and only if there exists a bounded sequence (h_k) in Y^*_σ such that $g_\gamma \in \overline{\operatorname{aco}\{h_k \mid k \in \mathbb{N}\}}$ for each γ,
- $\theta = \theta_{c_0}$, which is defined as follows: $\left(g_\gamma\right)_{\gamma\in\Gamma} \in \theta_{c0}\left[Y^*_\sigma\right]$ if and only if there exists a sequence (h_k) converging to zero in Y^*_σ with $g_\gamma \in \overline{\operatorname{aco}\{h_k \mid k \in \mathbb{N}\}}$ for each γ.

Below, like in the classical case (cf. Qiu [16]), the $L_{\theta_{bs}}$-spaces are called *L_r-spaces;* $\overset{\ulcorner\urcorner}{D}$ and $\overset{\llcorner\lrcorner}{S}$ stand for $\overset{\ulcorner\urcorner\theta_{bs}}{D}$ and $\overset{\llcorner\lrcorner\theta_{bs}}{S}$, respectively.

The case $\theta = \theta_{bn}$. If X is a polarly barrelled LCS, then X'_σ is quasicomplete (cf. [13, Corollary 7.3.7]), that is, θ_{bn}-complete. Then from Theorems 4.1 and 3.1(b), it follows that *a polar LCS Y is an $L_{\theta_{bn}}$-space if and only if any linear map $T\colon X \to Y$ with closed graph is continuous for each polar polarly barrelled space X.*

Proposition 4.2. *Any B_r-complete LCS Y is an $L_{\theta_{bn}}$-space.*

Proof. Suppose that Y is a B_r-complete LCS, let S be a dense subspace of Y'_σ. We show that $M := \overset{\llcorner\lrcorner\theta_{bn}}{S} \cap Y'$ is aw^*-closed (then $\overset{\llcorner\lrcorner\theta_{bn}}{S} \supset Y'$ by assumption). For that, let q be a polar continuous seminorm on Y and let $\left(g_\gamma\right)$ be a net in $M \cap \mathcal{F}_q$ converging to some g in Y'_σ. Then $g \in \mathcal{F}_q$ because $\mathcal{F}_q$ is closed in Y^*_σ. Moreover, since $\left(g_\gamma\right) \in \theta_{bn}\left[Y^*_\sigma\right]$, then $g \in M$, and thus $g \in M \cap \mathcal{F}_q$. So we have that M is aw^*-closed. □

The following theorem, which is the non-archimedean version of the Pták closed graph theorem [15, Theorem 4.9], is an immediate consequence of the preceding observations.

Theorem 4.3 (*non-archimedean Pták's closed graph theorem*). *Let X be a polar, polarly barrelled LCS. If Y is a B_r-complete LCS, then each linear map $T\colon X \to Y$ with closed graph is continuous. If K is spherically complete, then the assertion holds for each Fréchet space Y over K.*

The case $\theta = \theta_{bs}$. Clearly, a subset $A \subset Y^*$ is $\theta_{bs}w^*$-closed if and only if it is sequentially closed, and θ_{bs}-completeness means the sequential completeness.

It follows from Theorem 4.1 that *a polar LCS Y is an L_r-space if and only if any linear map $T\colon X \to Y$ with closed graph is weakly continuous whenever X is an arbitrary LCS such that X'_σ is sequentially complete.* If K is spherically complete and $\tau = \tau(X, X')$, then, by Theorem 3.1(a), *weakly continuous* may be replaced by *continuous.*

Proposition 4.4. *Any B_r-complete LCS of countable type is an L_r-space.*

Proof. Assume Y is a B_r-complete LCS of countable type, take a dense subspace S of Y'_σ. We show that $M := \overset{\sqcup\sqcup}{S} \cap Y'$ is aw^*-closed (so $\overset{\sqcup\sqcup}{S} \supset Y'$ by assumption). For that we verify that $M \cap \mathcal{F}_q$ is complete in Y'_σ for every continuous seminorm q on Y.

Since Y is of countable type, then the equicontinuous subset $M \cap \mathcal{F}_q$ of Y' is metrizable in Y'_σ (see [13, Theorem 7.6.10]). So, it sufficies to prove that any $\sigma(Y', Y)$-Cauchy sequence (g_k) of points of $M \cap \mathcal{F}_q$ is $\sigma(Y', Y)$-convergent in $M \cap \mathcal{F}_q$. Indeed, because $\mathcal{F}_q$ is $\sigma(Y', Y)$-complete, $g_k \to g$ for some $g \in \mathcal{F}_q$. On the other hand, $g \in \overset{\sqcup\sqcup}{S}$ because $\overset{\sqcup\sqcup}{S}$ is $\sigma(Y^*, Y)$-sequentially closed. Thus $g \in \overset{\sqcup\sqcup}{S} \cap \mathcal{F}_q = M \cap \mathcal{F}_q$ and we are done. □

From the preceding remark and Proposition 4.4 (in view of Theorem 3.1(c)), we get the counterpart of a classical result by Kalton [6].

Theorem 4.5 (*non-archimedean Kalton's closed graph theorem*). *Let (X, τ) be a polar LCS such that τ is the Mackey topology $\tau(X, X')$ and X'_σ is sequentially complete. If Y is a B_r-complete LCS of countable type, then any linear map $T\colon X \to Y$ with closed graph is continuous.*

If $K = \mathbb{R}$ or $K = \mathbb{C}$, then the Banach space ℓ^∞ is not an L_r-space (cf. [1, Example 3.13(a)]). The next example says that in the nonarchimedean case the situation is essentially different.

Example 4.6. *The Banach space ℓ^∞ is an L_r-space if and only if K is not spherically complete.*

Proof. First suppose K is shperically complete. Then every $\sigma(c_0, \ell^\infty)$-convergent sequence is norm convergent in c_0. Therefore, $(c_0, \sigma(c_0, \ell^\infty))$ is sequentially complete (see [13, Theorem 5.5.4(ii)]). If ℓ^∞ were an L_r-space, then the identity map $i : (\ell^\infty, \tau(\ell^\infty, c_0)) \to (\ell^\infty, \|\cdot\|_\infty)$ (which clearly has closed graph) is continuous. Thus, $(\ell^\infty, \|\cdot\|_\infty)' = c_0$, a contradiction.

Conversely, suppose K is not spherically complete. Then $(\ell^\infty)'_\sigma = (c_0, \sigma(c_0, \ell^\infty))$. In order to prove that ℓ^∞ is an L_r-space, we need to show that $c_0 \subset \overset{\sqcup\sqcup}{S}$ for arbitrary $\sigma(c_0, \ell^\infty)$-dense subspace $S \subset c_0$. Obviously, $\varphi :=\text{span}\{\mathbf{e}_k \mid k \in \mathbb{N}\} \subset S$. Since for every $\mathbf{z} = (z_k) \in c_0$ we have

$$\langle \mathbf{y}, \mathbf{z} \rangle = \lim_{n\to\infty} \left\langle \mathbf{y}, \sum_{k=1}^{n} z_k \mathbf{e}_k \right\rangle = \lim_{n\to\infty} \sum_{k=1}^{n} z_k y_k = \sum_{k=1}^{\infty} z_k y_k \quad (\mathbf{y} \in \ell^\infty),$$

then $c_0 \subset \overset{\sqcup\sqcup}{\varphi} \subset \overset{\sqcup\sqcup}{S}$. □

The case $\theta = \theta_{\ell^\infty}$. Since $\theta_{bs}[X^*_\sigma] \subset \theta_{\ell^\infty}[X^*_\sigma]$, we have that *any L_r-space is an $L_{\theta_{\ell^\infty}}$-space.* The next proposition presents the main connection between ℓ^∞-barrelledness and weak* θ_{ℓ^∞}-completeness of the dual for a LCS.

Proposition 4.7. *If a LCS (X, τ) is ℓ^∞-barrelled, then X'_σ is θ_{ℓ^∞}-complete. The converse holds when τ is the Mackey topology $\tau(X, X')$.*

Proof. Suppose X is ℓ^∞-barrelled. Let $\left(f_\gamma\right) \in \theta_{\ell^\infty}\left[X^*_\sigma\right]$ be a Cauchy net in X'_σ and let (h_k) be a $\sigma(X', X)$-bounded sequence such that $f_\gamma \in H := \overline{\operatorname{aco}\{h_k \mid k \in \mathbb{N}\}}^{\sigma(X',X)}$ for each γ. Then H is equicontinuous, and hence complete in X'_σ. Thus, $\left(f_\gamma\right)$ is convergent in X'_σ, which shows that X'_σ is θ_{ℓ^∞}-complete.

For the converse, assume that $\tau = \tau(X, X')$, and let X'_σ be θ_{ℓ^∞}-complete. To prove that each bounded sequence (h_k) in X'_σ is equicontinuous, we first verify that $H := \overline{\operatorname{aco}\{h_k \mid k \in \mathbb{N}\}}^{\sigma(X',X)}$ is $\sigma(X', X)$-complete. Indeed, if $\left(f_\gamma\right)$ is a $\sigma(X', X)$-Cauchy net in H, then $\left(f_\gamma\right) \in \theta_{\ell^\infty}\left[X^*_\sigma\right]$ and, by assumption, the linear functional f defined by $f(x) := \lim_\gamma f_\gamma(x)$ $(x \in X)$, belongs to X'. Thus $H = \overline{H}^{\sigma(X^*,X)}$, and so we have that $H^{0\oplus} = H^e \subset X'$. Hence the seminorm $p(x) = \sup_{f\in H}\left|f(x)\right|$ $(x \in X)$ is $\tau(X, X')$-continuous. So, (h_k) is equicontinuous. □

From Proposition 4.7 and Theorem 4.1 we obtain the following ***closed graph theorem for*** ℓ^∞***-barrelled spaces.*** For a classical counterpart see [10].

Theorem 4.8. *For a polar LCS Y the following statements are equivalent:*
(i) *Y is an $L_{\theta_{\ell^\infty}}$-space,*
(ii) *if (X, τ) is ℓ^∞-barrelled, then any linear map $T\colon X \to Y$ with closed graph is weakly continuous.*
If K is spherically complete and τ is the Mackey topology $\tau(X, X')$, then in (ii) weakly continuous *may be replaced by* continuous.

The case $\theta = \theta_{c0}$. The proof of the next result is almost identical to that of Proposition 4.7.

Proposition 4.9. *If a LCS (X, τ) is c_0-barrelled, then X'_σ is θ_{c0}-complete. The converse holds when τ is the Mackey topology $\tau(X, X')$.*

The following ***closed graph theorem for*** c_0***-barrelled spaces*** follows directly from Theorem 4.1 and Proposition 4.9.

Theorem 4.10. *For a polar LCS Y the following statements are equivalent:*
(i) *Y is an $L_{\theta_{c0}}$-space,*
(ii) *if (X, τ) is c_0-barrelled, then any linear map $T\colon (X, \tau) \to (Y, \tau')$ with closed graph is weakly continuous.*
If K is spherically complete and $\tau = \tau(X, X')$, then in (ii) weakly continuous *may be replaced by* continuous.

Example 4.11. *The Banach space $Y := c_0$ is an $L_{\theta_{c0}}$-space.*

Proof. Let S be a dense subspace of $Y'_\sigma = (\ell^\infty, \sigma(\ell^\infty, c_0))$. Since c_0 is B_r-complete (cf. Corollary 3.3(b)), it suffices to show that the subset $A := S^{\llcorner\lrcorner\theta_{c0}} \cap B_{\ell^\infty}$ (which obviously is $\theta_{c0}w^*$-closed) is $\sigma(\ell^\infty, c_0)$-closed. For that we prove that $A^{\ulcorner\urcorner\theta_{c0}} \supset \overline{A}^{\sigma(\ell^\infty,c_0)}$, or equivalently, that any $\sigma(\ell^\infty, c_0)$-convergent net $\left(\mathbf{z}_\gamma\right)$ in A belongs to $\theta_{c0}\left[Y^*_\sigma\right]$.

For all γ and $\mathbf{u} \in c_0$ there exists

$$\lim_{n\to\infty}\left\langle \mathbf{u}, \sum_{k=1}^{n} z_k^{(\gamma)}\mathbf{e}_k\right\rangle = \sum_{k=1}^{\infty} z_k^{(\gamma)} u_k$$

(here $\mathbf{z}_\gamma = \left(z_k^{(\gamma)}\right)_{k\in\mathbb{N}}$), so $\left(\sum_{k=1}^{n} z_k^{(\gamma)}\mathbf{e}_k\right)_{n\in\mathbb{N}}$ is a $\sigma(\ell^\infty, c_0)$-Cauchy sequence in B_{ℓ^∞}. Then it converges to $\mathbf{z}_\gamma$ (since B_{ℓ^∞} is weakly* complete). Thus $\mathbf{z}_\gamma \in \overline{\operatorname{aco}\{\mathbf{e}_k \mid k \in \mathbb{N}\}}^{\sigma(\ell^\infty,c_0)}$, whereas $\left|z_k^{(\gamma)}\right| \leq 1$ for all γ and k. Then $\left(\mathbf{z}_\gamma\right) \in \theta_{c0}\left[Y^*_\sigma\right]$ follows, since $\mathbf{e}_k \to \mathbf{0}$ in Y'_σ. □

5. Converse Theorems

In the preceding section we have characterized the maximum class of range spaces for closed graph theorems to hold. This is the main idea of Theorem 4.1. For a given class $\mathcal{D}$ of LCS X, we have looked for (possibly maximum) class $\mathcal{R}$ of LCS Y such that each linear map $T\colon X \to Y$ with closed graph is (weakly) continuous. For example, if $\mathcal{D}$ is the class of all LCS X with the Mackey topology for which X'_σ is sequentially complete, then the L_r-spaces form the maximum class $\mathcal{R}$. Likewise, polarly barrelled, ℓ^∞-barrelled and c_0-barrelled spaces as domain spaces are considered.

This section studies the necessity of this barrelledness-like properties in closed graph theorems. In the classical situation, results of this kind (known as converse closed graph theorems) are studied in Wilansky's book [19, Chapter 12.6].

Theorem of the Mahowald type. In the classical case, the famous Mahowald [9] theorem asserts that a LCS X is barrelled if and only if for each Banach space Y each linear map $T\colon X \to Y$ with closed graph is continuous. By the next theorem we see that, in the non-archimedean setting, this result holds when K is spherically complete.

Theorem 5.1 (*non-archimedean Mahowald's theorem*). *For a polar LCS X, consider the following statements:*
(i) *X is barrelled, i.e., each absolutely convex, closed, absorbing subset is a neighbourhood of zero,*
(ii) *for each Fréchet space Y, any linear map $T\colon X \to Y$ with closed graph is continuous,*
(iii) *for each polar Fréchet space Y, any linear map $T\colon X \to Y$ with closed graph is continuous,*
(iv) *each linear map $T\colon X \to (BC(H), \|\cdot\|_\infty)$ with closed graph is continuous for every Hausdorff topological space H,*
(v) *X is polarly barrelled.*
Then we have that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v).
If K is spherically complete, then statements (i) – (v) *are equivalent. In this case, the following statement is equivalent to* (i) – (v):
(iii*) *for each B_r-complete LCS Y, any linear map $T\colon X \to Y$ with closed graph is continuous.*

Proof. The implication (i) $\Rightarrow$ (ii) is [13, Theorem 11.1.10]; (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious.
(iv) $\Rightarrow$ (v): Let H be a bounded subset in X'_σ endowed with $\sigma(X', X)|_H$. To prove that H is equicontinuous, we define the map $T\colon X \to BC(H)$ by

$$(T(x))(f) := f(x) \quad (x \in X,\ f \in H).$$

Then T is obviously linear and has closed graph (if $x_\gamma \to 0$ in X and $T\left(x_\gamma\right) \to d$ in $BC(H)$, then $d = \lim_\gamma f\left(x_\gamma\right) = 0$). By assumption (iv), $T \in L(X, BC(H))$. Thus, $H^0 = T^{-1}\left(B_{BC(H)}\right)$ is a zero neighbourhood in X, so H is equicontinuous.

If K is spherically complete, then (i) $\Leftrightarrow$ (v), that is, statements (i) – (v) are equivalent. Moreover, (iii*) is equivalent to these statements. Indeed, then (v) $\Rightarrow$ (iii*) by Theorem 4.3, and (iii*) $\Rightarrow$ (iii) because each Fréchet space is B_r-complete (cf. Corollary 3.3(a)). □

Converse theorems of Kalton's type. The classical counterpart of the following results can be found in [6].

Theorem 5.2. *If X is a polar LCS, then the following statements are equivalent:*
(i) *X'_σ is sequentially complete,*
(ii) *for each L_r-space Y, each linear map $T\colon X \to Y$ with closed graph is weakly continuous,*
(iii) *for each B_r-complete LCS Y of countable type, each linear map $T\colon X \to Y$ with closed graph is weakly continuous,*
(iv) *for each Fréchet space Y of countable type, each linear map $T\colon X \to Y$ with closed graph is weakly continuous,*
(v) *each linear map $T\colon X \to c$ with closed graph is weakly continuous,*

Proof. The implication (i) ⇒ (ii) follows from Theorem 4.1 (see the remark before Proposition 4.4). Further, (ii) ⇒ (iii) (cf. Proposition 4.4), (iii) ⇒ (iv) (cf. Corollary 3.3(b)) and (iv) ⇒ (v).

(v) ⇒ (i): Let (f_k) be a Cauchy sequence in X'_σ. We have to verify that the linear functional $f\colon X \to K$ defined by $f(x) := \lim_{k\to\infty} f_k(x)$ is continuous on X. To this end, consider the linear map

$$T\colon X \to c,\ \ x \mapsto (f_k(x)).$$

Since T has closed graph (if $x_\gamma \to 0$ in X and $T\left(x_\gamma\right) \to \mathbf{d} = (d_k)$ in c, then $d_k = \lim_\gamma f_k\left(x_\gamma\right) = 0$ for all k), then, by (v), T is weakly continuous. Therefore, $f = \lim \circ T \in X'$. □

Theorem 5.3. *Let (X,τ) be a LCS, where τ is the Mackey topology $\tau(X,X')$. The following statements are equivalent:*
(i) *X'_σ is sequentially complete,*
(ii) *for each B_r-complete LCS Y of countable type, each linear map $T\colon X \to Y$ with closed graph is continuous,*
(iii) *for each Fréchet space (Y,τ') of countable type, each linear map $T\colon X \to Y$ with closed graph is continuous,*
(iv) *each linear map $T\colon X \to c$ with closed graph is continuous,*
(v) *each linear map $T\colon X \to c_0$ with closed graph is continuous.*

Proof. Since $c = c_0 \oplus \operatorname{span}\{\mathbf{e}\}$, then it is easy to check that (iv) and (v) are equivalent. According to the preceding theorem, we only have to prove (i) ⇒ (ii). This follows from Theorems 5.2 and 3.1(c). □

Converse theorems for ℓ^∞- and c_0-barrelled spaces.

Theorem 5.4. *For a polar LCS (X,τ), consider the following statements:*
(i) *X is ℓ^∞-barrelled,*
(ii) *each linear map $T\colon X \to \ell^\infty$ with closed graph is continuous.*
Then (ii) ⇒ (i). *If, in addition, either K is not spherically complete or $\tau = \tau(X,X')$, then* (i) ⇔ (ii).

Proof. Let (f_k) be a bounded sequence in X'_σ. If (ii) holds, then the linear map $T\colon X \to \ell^\infty,\ x \mapsto (f_k(x))$ is continuous or, equivalently, $\{f_k \mid k \in \mathbb{N}\}$ is equicontinuous. So, (ii) ⇒ (i).

Suppose K is not spherically complete. Then ℓ^∞ is an L_r-space (cf. Example 4.6), hence an $L_{\theta_{\ell^\infty}}$-space. Let X be ℓ^∞-barrelled and let $T\colon X \to \ell^\infty$ be a linear map with closed graph. To prove (i) ⇒ (ii), we verify that the seminorm $p := \|\cdot\|_\infty \circ T$ is continuous on X.

By Theorem 4.8, T is weakly continuous, so the dual map $T'\colon (c_0, \sigma(c_0,\ell^\infty)) \to X'_\sigma$ is continuous. Let $\phi_k := T'(\mathbf{e}_k)$, i.e., $\phi_k(x) = T(x)_k$ for each $x \in X$ and each k. From $\sigma(c_0,\ell^\infty)$-boundedness of $(\mathbf{e}_k)$ we have that $\left\{\phi_k \mid k \in \mathbb{N}\right\}$ is $\sigma(X',X)$-bounded, hence equicontinuous by (i). Since each $f \in T'(B_{c_0})$ has the form

$$f(x) = \sum_{k=1}^{\infty} u_k T(x)_k = \sum_{k=1}^{\infty} u_k \phi_k(x) \quad (x \in X),$$

where $|u_k| \le 1$ for each k, then $f \in \overline{\operatorname{aco}\left\{\phi_k \mid k \in \mathbb{N}\right\}}^{\sigma(X',X)}$. We see that $T'(B_{c_0})$ is equicontinuous, therefore its polar

$$T'(B_{c_0})^0 = \left\{x \in X \mid \sup_{\mathbf{u}\in B_{c_0}} \left|\sum_{k=1}^{\infty} u_k T(x)_k\right| \le 1\right\} = \{x \in X \mid \|T(x)\|_\infty \le 1\} = B_p$$

is a neighbourhood of zero in X. So we have that p is continuous.

If $\tau = \tau(X,X')$, then (i) ⇒ (ii) follows from Theorem 4.8. □

Theorem 5.5. *For a polar LCS (X,τ), consider the following statements:*
(i) *X is c_0-barrelled,*
(ii) *each linear map $T\colon X \to c_0$ with closed graph is continuous.*
Then (i) ⇒ (ii). *If, in addition, τ is the Mackey topology $\tau(X,X')$, then* (i) ⇔ (ii).

Proof. (i) ⇒ (ii): Let $T\colon X \to c_0$ be a linear map with closed graph. Since c_0 is an $L_{\theta_{c_0}}$-space, then Theorem 4.10 ensures the weak continuity of T. Hence, the dual map $T'\colon (\ell^\infty, \sigma(\ell^\infty, c_0)) \to X'_\sigma$ is continuous. To verify that the seminorm $p := \|\cdot\|_\infty \circ T$ is continuous on X, let $\phi_k := T'(\mathbf{e}_k)$ for each k. Obviously, $\mathbf{e}_k \to \mathbf{0}$ in $(\ell^\infty, \sigma(\ell^\infty, c_0))$, which implies that $\phi_k \to 0$ in X'_σ. By (i), $\overline{\operatorname{aco}\left\{\phi_k \mid k \in \mathbb{N}\right\}}^{\sigma(X',X)}$ is equicontinuous. Since $T'(B_{\ell^\infty}) \subset \overline{\operatorname{aco}\left\{\phi_k \mid k \in \mathbb{N}\right\}}^{\sigma(X',X)}$, then $T'(B_{\ell^\infty})^0 = B_p$ is a neighbourhood of zero in X. Hence p is continuous.

Assume now that (ii) holds with $\tau = \tau(X, X')$. Then X'_σ is sequentially complete by Theorem 5.3. Let (f_k) be a sequence converging to 0 in X'_σ. To prove that it is equicontinuous, we put $A := \overline{\operatorname{aco}\{f_k \mid k \in \mathbb{N}\}}^{\sigma(X',X)}$ and show that the seminorm p on X defined by $p(x) := \sup\left\{\left|f(x)\right| \mid f \in A\right\}$ $(x \in X)$ is $\tau(X, X')$-continuous. To see this, we first notice that A is a metrizable compactoid (see [13, Theorem 3.8.24]). Then, by the fact that X'_σ is sequentially complete, A is a complete subset in X'_σ. Therefore, $\mathcal{F}_p = A^{0\oplus} = A^e \subset X'$. Hence B_p is a $\tau(X, X')$-neighbourhood of zero and so p is continuous on (X, τ). □

Finally, we close this section with some examples that say that the barrelledness conditions considered above are different in general. For a polar LCS (X, τ), we consider the following conditions:
(a) X is polarly barrelled,
(b) X is ℓ^∞-barrelled,
(c) X is c_0-barrelled,
(d) X'_σ is sequentially complete.
Clearly, (a) ⇒ (b) ⇒ (c). The implication (c) ⇒ (d) follows from Theorems 5.2 and 5.5. If τ is the Mackey topology, then (d) ⇒ (c) by Theorems 5.3 and 5.5.

Note that (c) ⇒ (d) may be false in the classical case (for example, $(\ell, \tau(\ell, c_0))$; see [8, p. 102]).

The following examples show that all the implications (a) ⇒ (b) ⇒ (c) ⇒ (d) are irreversible in general.

Example 5.6. *There exist ℓ^∞-barrelled LCS's which are not polarly barrelled.*

Proof. See [11, Example 2]. □

Example 5.7. *There exist c_0-barrelled LCS's which are not ℓ^∞-barrelled.*

Proof. Let ζ be the polar topology on ℓ^∞ defined by the seminorms $p_{(\mathbf{u}_n)}(\mathbf{z}) := \sup_{n\in\mathbb{N}} \left|\sum_{k=1}^\infty u_k^{(n)} z_k\right|$ $(\mathbf{z} \in \ell^\infty)$, where $\mathbf{u}_n = \left(u_k^{(n)}\right)_{k\in\mathbb{N}} \in c_0$ and $\mathbf{u}_n \to \mathbf{0}$ in $(c_0, \sigma(c_0, \ell^\infty))$. Clearly, $(\ell^\infty, \zeta)' \supset c_0$. Since

$$p_{(\mathbf{u}_n)}\left(\mathbf{z} - \sum_{k=1}^m z_k \mathbf{e}_k\right) = \sup_{n\in\mathbb{N}} \left|\sum_{k=m+1}^\infty u_k^{(n)} z_k\right| \le \|\mathbf{z}\|_\infty \sup_{k>m} \sup_{n\in\mathbb{N}} \left|u_k^{(n)}\right| \to 0 \quad (m \to \infty,\ \mathbf{z} \in \ell^\infty),$$

then, for each $f \in (\ell^\infty, \zeta)'$, we have $f(\mathbf{z}) = \sum_{k=1}^\infty v_k z_k$ $(\mathbf{z} \in \ell^\infty)$ with $\mathbf{v} = (f(\mathbf{e}_k)) \in c_0$. Thus, $(\ell^\infty, \zeta)' = c_0$.

Obviously, (ℓ^∞, ζ) is c_0-barrelled. To verify that it is not ℓ^∞-barrelled, we show that the sequence $(\sum_{k=1}^m \mathbf{e}_k)_{m\in\mathbb{N}}$ (which clearly is $\sigma(c_0, \ell^\infty)$-bounded in c_0) is not ζ-equicontinuous. Assume, therefore, that this is not the case. Then the polar seminorm q defined by $q(\mathbf{z}) := \sup_{m\in\mathbb{N}} \left|\sum_{k=1}^m z_k\right|$ $(\mathbf{z} \in \ell^\infty)$ is continuous on (ℓ^∞, ζ). Hence we can find a sequence $(\mathbf{u}_n)$ such that $\mathbf{u}_n \to \mathbf{0}$ in $(c_0, \sigma(c_0, \ell^\infty))$ and $q(\mathbf{z}) \le \sup_{n\in\mathbb{N}} \left|\sum_{k=1}^\infty u_k^{(n)} z_k\right|$ $(\mathbf{z} \in \ell^\infty)$. Since $\|\mathbf{u}_n\|_\infty \to 0$, there exists an $N \in \mathbb{N}$ with $\sup_{n+k>N} \left|u_k^{(n)}\right| < \frac{1}{2}$. So, the sequence $\mathbf{t} := (0, \dots, 0, 1, 1, \dots)$, where $t_1 = \dots = t_N = 0$, yields the contradiction:

$$1 = \sup_{m>N} \left|\sum_{k=N+1}^m t_k\right| = q(\mathbf{t}) \le \sup_{n\in\mathbb{N}} \left|\sum_{k=N+1}^\infty u_k^{(n)}\right| \le \sup_{n\in\mathbb{N}} \sup_{k>N} \left|u_k^{(n)}\right| \le \frac{1}{2}.$$

□

Example 5.8. *There exist a LCS X which is not c_0-barrelled but X'_σ is sequentially complete.*

Proof. Let $X := (c_0, \sigma(c_0, \ell^\infty))$, then $X'_\sigma = (\ell^\infty, \sigma(\ell^\infty, c_0)) = (c_0, \|\cdot\|_\infty)'_\sigma$ is sequentially complete. It is not difficult to see that the sequence $(\mathbf{e}_n)$ (which obviously converges to $\mathbf{0}$ in X'_σ) is not $\sigma(c_0, \ell^\infty)$-equicontinuous. Hence X is not c_0-barrelled. □

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