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Some Formulas for Bell Numbers

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Abstract. We give elementary proofs of three formulas involving Bell numbers, including a generalization of the Gould-Quaintance formula and a generalization of Spivey's formula. We find variants for two of our formulas which involve some well-known sequences, among them the Fibonacci, Bernoulli and Euler numbers.

1. Introduction

The n-th Bell number is given by

$$B(n) = \sum_{j=0}^{n} S(n, j), \tag{1}$$

where $S(n, j)$ denotes the Stirling number of the second kind which counts the partitions of an n-element set having exactly j blocks. Then $B(n)$ counts the total number of partitions of an n-element set (see, e.g., [14, p. 33]). The Bell numbers $B(n)$ may be defined by Dobiński's formula

$$B(n) = e^{-1} \sum_{m=0}^{\infty} \frac{n^m}{m!}$$

(see [3]), or by the generating function

$$e^{e^x - 1} = \sum_{n=0}^{\infty} B(n) \frac{x^n}{n!}$$

(see [4]). A recurrence for the sequence $B(n)$ is given by

$$B(n+1) = \sum_{j=0}^{n} \binom{n}{j} B(j). \tag{2}$$

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The inverse relation of (2) is

$$B(n)=\sum_{j=0}^{n}\binom{n}{j}(-1)^{n+j}B(j+1). \tag{3}$$

In 2008, Spivey [13] proved the following generalized recurrence for Bell numbers

$$B(n+k)=\sum_{m=0}^{k}\sum_{j=0}^{n}\binom{n}{j}m^{n-j}S(k,m)\,B(j)\,, \tag{4}$$

which is called "Spivey's formula" nowadays. When $n=0$ in (4), this is reduced to the definition (1). When $k=1$ in (4), this is reduced to the recurrence (2). Besides the generalizations of Bell numbers and their recurrences (see, e.g., [7] and [2, 8, 9]), we also have several generalizations of Spivey's formula (4) (see, e.g., [1, 5, 10, 12, 15]).

In this work, we give elementary proofs of the following three formulas involving Bell numbers: for arbitrary non-negative integers n and k, we have

$$\sum_{j=0}^{n}\binom{n}{j}B(k+j)=\sum_{j=0}^{k}\binom{k}{j}(-1)^{k+j}B(n+j+1), \tag{5}$$

$$\sum_{j=0}^{n}\binom{n}{j}a^{j}b^{n-j}B(j)=\sum_{j=0}^{k}(-1)^{k+j}s(k,j)\sum_{l=0}^{n}\binom{n}{l}a^{l}(b-ak)^{n-l}B(l+j), \tag{6}$$

$$\sum_{j=0}^{n+k}\binom{n+k}{j}a^{n+k-j}b^{j}B(n+k-j)=\sum_{m=0}^{k}\sum_{j=0}^{n}\sum_{l=0}^{k}\sum_{i=0}^{j}\binom{n}{j}\binom{k}{l}\binom{j}{i}a^{n+l+i-j}b^{k+j-l-i}m^{n-j}S(l,m)\,B(i)\,. \tag{7}$$

In (6), $s(k,j)$ denotes the unsigned Stirling numbers of the first kind whose recurrence is given by $s(k+1,j)=s(k,j-1)+ks(k,j)$. In (6) and (7), a and b are arbitrary complex numbers.

Observe that the case $a=1, b=0$ of (7) is Spivey's formula (4). More particular cases in (5), (6) and (7) are discussed in the next section.

2. The Main Results

In this section, we prove formulas (5), (6) and (7).

Proposition 1. *Formula (5) holds.*

Proof. We proceed by induction on k. By (2), the result is true for $k=0$. Suppose that the result is true for a given $k\in\mathbb{N}$. Then

$$\begin{aligned}&\sum_{j=0}^{k+1}\binom{k+1}{j}(-1)^{k+1+j}B(n+j+1)\\&=\sum_{j=0}^{k}\binom{k}{j}(-1)^{k+1+j}B(n+j+1)+\sum_{j=1}^{k+1}\binom{k}{j-1}(-1)^{k+1+j}B(n+j+1)\\&=\sum_{j=0}^{k}\binom{k}{j}(-1)^{k+1+j}B(n+j+1)+\sum_{j=0}^{k}\binom{k}{j}(-1)^{k+j}B(n+1+j+1)\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{j=0}^{n}\binom{n}{j}B(k+j)+\sum_{j=0}^{n+1}\binom{n+1}{j}B(k+j)\\
&= \sum_{j=1}^{n+1}\binom{n}{j-1}B(k+j)\\
&= \sum_{j=0}^{n}\binom{n}{j}B(k+1+j),
\end{aligned}
$$

as desired. □

When $n = 0$ in (5), we obtain the inverse relation (3). Formula (5) can be written as the following recurrence for Bell numbers:

$$
B(k+n+1)=\sum_{j=0}^{k-1}\binom{k}{j}(-1)^{k+1+j}B(n+j+1)+\sum_{j=0}^{n}\binom{n}{j}B(k+j). \tag{8}
$$

For example, the Bell number $B(8)$ can be expressed in terms of $B(1),\dots,B(7)$, by taking k,n such that $k+n=7$:

n	k	$B(8)=$
6	1	$B(1)+6B(2)+15B(3)+20B(4)+15B(5)+6B(6)+2B(7)$
5	2	$B(2)+5B(3)+10B(4)+10B(5)+4B(6)+3B(7)$
4	3	$B(3)+4B(4)+7B(5)+B(6)+4B(7)$
3	4	$7B(5)-3B(6)+5B(7)$
2	5	$B(3)-5B(4)+11B(5)-8B(6)+6B(7)$
1	6	$-B(2)+6B(3)-15B(4)+20B(5)-14B(6)+7B(7)$

Proposition 2. *Formula (6) holds.*

Proof. The proof is done by induction on k. If $k = 0$, formula (6) is trivial. If $k = 1$, then the result holds as

$$
\begin{aligned}
&\sum_{j=0}^{n}\binom{n}{j}a^{j}B(j+1)\sum_{i=0}^{n-j}\binom{n-j}{i}(-a)^{n-j-i}b^{i}\\
&= \sum_{j=0}^{n}\sum_{i=0}^{n-j}\binom{n}{i}\binom{n-i}{n-i-j}(-1)^{n-i-j}a^{n-i}b^{i}B(j+1)\\
&= \sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^{i}\sum_{j=0}^{n-i}\binom{n-i}{j}(-1)^{n-i-j}B(j+1)\\
&= \sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^{i}B(n-i),
\end{aligned}
$$

where we have used relation (3) in the last equality. Suppose formula (6) is true in the k-case for some $k \geq 1$. Then

$$
\sum_{j=0}^{k+1}(-1)^{k+1+j}s(k+1,j)\sum_{l=0}^{n}\binom{n}{l}a^{l}\Big(b-a(k+1)\Big)^{n-l}B(l+j)
$$

$$= \sum_{j=0}^{k+1}(-1)^{k+1+j}\Big(s(k,j-1)+ks(k,j)\Big)\sum_{l=0}^{n}\binom{n}{l}a^l(b-a-ak)^{n-l}B(l+j)$$
$$= \sum_{j=0}^{k}(-1)^{k+j}s(k,j)\sum_{l=0}^{n}\binom{n}{l}a^l(b-a-ak)^{n-l}B(l+j+1)$$
$$- k\sum_{j=0}^{n}\binom{n}{j}a^j(b-a)^{n-j}B(j). \tag{9}$$

Now observe that

$$\sum_{j=0}^{k}(-1)^{k+j}s(k,j)\sum_{l=0}^{n}\binom{n}{l}a^l(b-a-ak)^{n-l}B(l+j+1)$$
$$= \sum_{j=0}^{k}(-1)^{k+j}s(k,j)\sum_{l=0}^{n+1}(\binom{n+1}{l}-\binom{n}{l})a^{l-1}(b-a-ak)^{n+1-l}B(l+j)$$
$$= a^{-1}\sum_{j=0}^{n+1}\binom{n+1}{j}a^j(b-a)^{n+1-j}B(j) - a^{-1}(b-a-ak)\sum_{j=0}^{n}\binom{n}{j}a^j(b-a)^{n-j}B(j)$$
$$= a^{-1}\sum_{j=0}^{n+1}\binom{n+1}{j}a^j(b-a)^{n+1-j}B(j) - a^{-1}\sum_{j=0}^{n}\binom{n}{j}a^j(b-a)^{n+1-j}B(j)$$
$$+ k\sum_{j=0}^{n}\binom{n}{j}a^j(b-a)^{n-j}B(j).$$

Thus, from (9), we have

$$\sum_{j=0}^{k+1}(-1)^{k+1+j}s(k+1,j)\sum_{l=0}^{n}\binom{n}{l}a^l\big(b-a(k+1)\big)^{n-l}B(l+j)$$
$$= a^{-1}\sum_{j=0}^{n+1}\binom{n+1}{j}a^j(b-a)^{n+1-j}B(j) - a^{-1}\sum_{j=0}^{n}\binom{n}{j}a^j(b-a)^{n+1-j}B(j)$$
$$= a^{-1}\sum_{j=1}^{n+1}\binom{n}{j-1}a^j(b-a)^{n+1-j}B(j)$$
$$= \sum_{j=0}^{n}\binom{n}{j}a^j(b-a)^{n-j}B(j+1)$$
$$= \sum_{j=0}^{n}\binom{n}{j}a^jb^{n-j}B(j).$$

as desired. Note that in the last step, we have used the $k=1$ case of (6). □

When $a \neq 0$ and $b = 0$, (6) is

$$B(n) = \sum_{j=0}^{k}(-1)^{k+j}s(k,j)\sum_{l=0}^{n}\binom{n}{l}(-k)^{n-l}B(l+j). \tag{10}$$

This is the so-called "Gould-Quaintance Formula" in [5, (7)]. When $a = 0$ and $b \neq 0$ or $n = 0$ in (6), we have the identity

$$\sum_{j=0}^{k} (-1)^{k+j} s(k, j) B(j) = 1,$$

which is also derived from (1) by using the orthogonality of the two type of Stirling numbers. When $a = 1$ and $b = k$ in (6), we have the formula

$$\sum_{j=0}^{n} \binom{n}{j} k^{n-j} B(j) = \sum_{j=0}^{k} (-1)^{k+j} s(k, j) B(n + j), \tag{11}$$

contained also in [5, (6)]. When $k = 1$ in (6), we also have

$$\sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j} B(j) = \sum_{j=0}^{n} \binom{n}{j} a^j (b - a)^{n-j} B(j + 1), \tag{12}$$

which contains the recurrence (2) with $a = b = 1$ and the inverse relation (3) with $a = 1$ and $b = 0$ as particular cases.

If $a \neq 0$, from (6), we obtain the recurrence

$$B(n + k + 1) = \sum_{j=0}^{n+k} \left(\binom{n+1}{j} z^{n+1-j} - \sum_{t=0}^{k} (-1)^{k+t} s(k, t) \binom{n+1}{j-t} (z - k)^{n+1-j+t} \right) B(j), \tag{13}$$

where $z = a^{-1}b$ is an arbitrary complex number and n and k are arbitrary integers with $n \geq 0$ and $k > 0$. The case $z = k$ of (13) is essentially the same as (11).

Formula (13) gives us the Bell number $B(n + k + 1)$ written as a linear combination (with polynomial coefficients) of the previous Bell numbers $B(0), B(1), \ldots, B(n + k)$: each coefficient is a polynomial in z of degree $\leq n + 1$. For example, $B(6)$ can be written as a linear combination of $B(0), B(1), \ldots, B(5)$, in 5 different ways, corresponding to $(n, k) = (0, 5), (1, 4), (2, 3), (3, 2), (4, 1)$. In the simplest case where $n = 0$ and $k = 5$, we have the following family of recurrence formulas for $B(6)$:

$$\begin{aligned} B(6) &= zB(0) + (-24z + 121)B(1) + (50z - 274)B(2) \\ &\quad + (-35z + 225)B(3) + (10z - 85)B(4) + (-z + 15)B(5). \end{aligned}$$

Proposition 3. *Formula (7) holds.*

Proof. Both sides of (7) are $(n + k)$-th degree polynomials in the variable b. We shall prove that the values of these polynomials are equal when b is a non-negative integer, implying that the polynomials are equal. We proceed by induction on b. When $b = 0$, formula (7) is exactly the same as Spivey's formula (4). Suppose that the result is valid for $b \in \mathbb{N}$. Then we have

$$\begin{aligned} &\sum_{j=0}^{n+k} \binom{n+k}{j} a^j (b+1)^{n+k-j} B(j) \\ &= \left(\frac{b+1}{b} \right)^{n+k} \sum_{j=0}^{n+k} \binom{n+k}{j} \left(\frac{ab}{b+1} \right)^j b^{n+k-j} B(j) \\ &= \left(\frac{b+1}{b} \right)^{n+k} \sum_{m=0}^{k} \sum_{j=0}^{n} \sum_{l=0}^{k} \sum_{i=0}^{j} \binom{n}{j} \binom{k}{l} \binom{j}{i} \left(\frac{ab}{b+1} \right)^{n+l+i-j} b^{k+j-l-i} m^{n-j} S(l, m) B(i) \end{aligned}$$

$$= \sum_{m=0}^{k}\sum_{j=0}^{n}\sum_{l=0}^{k}\sum_{i=0}^{j}\binom{n}{j}\binom{k}{l}\binom{j}{i}a^{n+l+i-j}\left(b+1\right)^{k+j-l-i}m^{n-j}S\left(l,m\right)B\left(i\right),$$

as desired. □

When $n = 0$, (7) is

$$\sum_{j=0}^{k}\binom{k}{j}a^{j}b^{k-j}B\left(j\right) = \sum_{m=0}^{k}\sum_{l=0}^{k}\binom{k}{l}a^{l}b^{k-l}S\left(l,m\right). \tag{14}$$

When $a = 1$ and $b = 0$, (14) is reduced to the definition (1). In addition, by using the relation $\sum_{l=0}^{k}\binom{k}{l}S\left(l,m\right) = S\left(k+1,m+1\right)$, we see that (14) becomes $B\left(k+1\right) = \sum_{m=0}^{k}S\left(k+1,m+1\right)$ when $a = b = 1$, which is essentially the definition (1). When $k = 1$, (7) becomes

$$\sum_{j=0}^{n+1}\binom{n+1}{j}a^{j}b^{n+1-j}B\left(j\right) = \sum_{i=0}^{n}\binom{n}{i}a^{i}b^{n+1-i}B\left(i\right) + \sum_{j=0}^{n}\sum_{i=0}^{j}\binom{n}{j}\binom{j}{i}a^{n+1+i-j}b^{j-i}B\left(i\right). \tag{15}$$

When $a = 1$ and $b = 0$, (15) is the recurrence (2), and when $a = b = 1$, it becomes the recurrence

$$B\left(n+2\right) = B\left(n+1\right) + \sum_{j=0}^{n}\binom{n}{j}B\left(j+1\right). \tag{16}$$

Formula (7) can be written as the following generalized recurrence for Bell numbers:

$$B\left(n+k\right) = \sum_{m=0}^{k}\sum_{l=m}^{k}\sum_{i=0}^{n}\sum_{j=i}^{n}\binom{n}{j}\binom{k}{l}\binom{j}{i}z^{k+j-l-i}m^{n-j}S\left(l,m\right)B\left(i\right) - \sum_{j=0}^{n+k-1}\binom{n+k}{j}z^{n+k-j}B\left(j\right), \tag{17}$$

where $z = a^{-1}b$ is an arbitrary complex number. Moreover, some elementary algebra gives us the following more interesting form for (17):

$$B\left(n+k\right) = \sum_{j=0}^{n}\binom{n}{j}P_{k,j}\left(z\right)B\left(j\right) + z^{k}\sum_{j=0}^{n+k-1}\left(\binom{n}{j} - \binom{n+k}{j}\right)z^{n-j}B\left(j\right), \quad k \geq 1, \tag{18}$$

where the polynomial $P_{k,j}\left(z\right)$ is given by

$$P_{k,j}\left(z\right) = \sum_{m=1}^{k}\sum_{l=m}^{k}\binom{k}{l}S\left(l,m\right)z^{k-l}\left(z+m\right)^{n-j}. \tag{19}$$

When $z = 0$, (18) is reduced to Spivey's formula (4). Furthermore, if $k = 1$, we have $P_{1,j}\left(z\right) = \left(z+1\right)^{n-j}$, so (18) becomes

$$B\left(n+1\right) = \sum_{j=0}^{n}\binom{n}{j}\left(z+1\right)^{n-j}B\left(j\right) - \sum_{j=0}^{n-1}\binom{n}{j}z^{n-j}B\left(j+1\right), \tag{20}$$

which contains the recurrence (2) if $z = 0$, and the inverse relation (3) if $z = -1$. When $k = 2$, formula (18) looks as follows:

$$B\left(n+2\right) = \sum_{j=0}^{n}\binom{n}{j}\left(\left(2z+1\right)\left(z+1\right)^{n-j} + \left(z+2\right)^{n-j}\right)B\left(j\right) + z^{2}\sum_{j=0}^{n+1}\left(\binom{n}{j} - \binom{n+2}{j}\right)z^{n-j}B\left(j\right). \tag{21}$$

If we set $z = -1$ in (21), the recurrence can be written as follows:

$$B\left(n+2\right) = \left(n+3\right)B(n+1) - \binom{n+2}{2}B\left(n\right) + \sum_{j=0}^{n-1}\left(\binom{n}{j} - \binom{n+2}{j}\right)\left(-1\right)^{n-j}B\left(j\right).$$

3. Some Additional Identities

In this section, we shall consider the Fibonacci sequence F_n, defined recursively by $F_{n+2} = F_{n+1} + F_n$ ($n \geq 0$) with $F_0 = 0$ and $F_1 = 1$, and the Lucas sequence L_n, defined by $L_{n+2} = L_{n+1} + L_n$ ($n \geq 0$) with $L_0 = 2$ and $L_1 = 1$. Recall that $F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$ and $L_n = \alpha^n + \beta^n$, where $\alpha = \frac{1}{2}(1+\sqrt{5})$, $\beta = \frac{1}{2}(1-\sqrt{5})$.

Formula (6) can be seen as an identity of two n-th degree polynomials in the variable b, namely,

$$\sum_{j=0}^{n} c_j(n,a)b^{n-j} = \sum_{l=0}^{n}\sum_{i=0}^{l} d_{l,i}(n,k,a)b^i, \tag{22}$$

where $c_j(n,a) = \binom{n}{j}a^jB(j)$ and $d_{l,i}(n,k,a) = \binom{n}{l}\binom{l}{i}(-k)^{l-i}a^{n-i}\sum_{j=0}^{k}(-1)^{k+j}s(k,j)B(n-l+j)$. If we set $b = \alpha$ in (22), by using the known identity $\alpha^n = \alpha F_n + F_{n-1}$ (see [6, Lemma 5.1, p. 78]), we get

$$\sum_{j=0}^{n} c_j(n,a)(\alpha F_{n-j} + F_{n-j-1}) = \sum_{l=0}^{n}\sum_{i=0}^{l} d_{l,i}(n,k,a)(\alpha F_i + F_{i-1}). \tag{23}$$

Both sides of (23) are of the form $A\sqrt{5} + B$, where A and B are integers. By equating the coefficients of $\sqrt{5}$ in both sides of (23), we get

$$\sum_{j=0}^{n} c_j(n,a)F_{n-j} = \sum_{l=0}^{n}\sum_{i=0}^{l} d_{l,i}(n,k,a)F_i, \tag{24}$$

and by equating the other (independent) coefficients of (23), we get

$$\sum_{j=0}^{n} c_j(n,a)(F_{n-j} + 2F_{n-j-1}) = \sum_{l=0}^{n}\sum_{i=0}^{l} d_{l,i}(n,k,a)(F_i + 2F_{i-1}). \tag{25}$$

Since $F_t + 2F_{t-1} = F_{t+1} + F_{t-1} = L_t$ (see, e.g., [6, Formula 5.14, p. 80]), we can write (25) as

$$\sum_{j=0}^{n} c_j(n,a)L_{n-j} = \sum_{l=0}^{n}\sum_{i=0}^{l} d_{l,i}(n,k,a)L_i. \tag{26}$$

Thus, by substituting the powers b^{n-j} on the left-hand side and b^i on the right-hand side in formula (22), by F_{n-j} (or L_{n-j}) and F_i (or L_i), respectively, we can obtain the identity (24) (or the identity (26)), respectively.

Formula (22) is still useful, by using [11, Theorem 1], we obtain

$$\sum_{j=0}^{n} c_j(n,a)\mathbb{B}_{n-j}(b) = \sum_{l=0}^{n}\sum_{i=0}^{l} d_{l,i}(n,k,a)\mathbb{B}_i(b), \tag{27}$$

where $\mathbb{B}_n(b)$ is the n-Bernoulli b-polynomial. By setting $b = 0$ in (27), we obtain the following identity involving Bernoulli numbers $\mathfrak{B}_*$:

$$\sum_{j=0}^{n} c_j(n,a)\mathfrak{B}_{n-j} = \sum_{l=0}^{n}\sum_{i=0}^{l} d_{l,i}(n,k,a)\mathfrak{B}_i. \tag{28}$$

By using some known relations between Bernoulli polynomials $\mathbb{B}_n(x)$ and Euler polynomials $\mathbb{E}_n(x)$, including $\mathbb{E}_n(x) = \frac{2^{n+1}}{n+1}\left(\mathbb{B}_{n+1}\left(\frac{x+1}{2}\right) - \mathbb{B}_{n+1}\left(\frac{x}{2}\right)\right)$, together with [11, Theorem 1], we can see that the identity (27) is also valid for Euler polynomials:

$$\sum_{j=0}^{n} c_j(n,a)\mathbb{E}_{n-j}(b) = \sum_{l=0}^{n}\sum_{i=0}^{l} d_{l,i}(n,k,a)\mathbb{E}_i(b). \tag{29}$$

By setting $b = \frac{1}{2}$ in (29), as $E_n = 2^n \mathbb{E}_n\left(\frac{1}{2}\right)$, we obtain the following identity involving Euler numbers E_*.

$$\sum_{j=0}^{n} c_j(n,a) 2^{-(n-j)} E_{n-j} = \sum_{l=0}^{n} \sum_{i=0}^{l} d_{l,i}(n,k,a) 2^{-i} E_i. \tag{30}$$

In fact, (30) can be simplified. Set $b = 2b_1$ in (22). Then, similarly to (29), using the argument above, we obtain an identity involving Euler polynomials,

$$\sum_{j=0}^{n} c_j(n,a) 2^{n-j} \mathbb{E}_{n-j}(b_1) = \sum_{l=0}^{n} \sum_{i=0}^{l} d_{l,i}(n,k,a) 2^i \mathbb{E}_i(b_1),$$

and setting $b_1 = \frac{1}{2}$, we obtain

$$\sum_{j=0}^{n} c_j(n,a) E_{n-j} = \sum_{l=0}^{n} \sum_{i=0}^{l} d_{l,i}(n,k,a) E_i. \tag{31}$$

4. More Variations

In the previous section, by replacing the powers b^r by F_r, L_r, $\mathfrak{B}_r$ or E_r in the polynomial identity (22), we obtain new identities. Moreover, the corresponding identities (24), (26), (28) and (31) are, in turn, identities between two a-polynomials, so we can use the same ideas to obtain identities replacing the powers a^t by F_t, L_t, $\mathfrak{B}_t$ or E_t.

Including the constant sequence $1, 1, \ldots$, we can have 25 identities of the form

$$\sum_{j=0}^{n} \binom{n}{j} X_j Y_{n-j} B(j) = \sum_{j=0}^{k} \sum_{l=0}^{n} \sum_{i=0}^{l} \binom{n}{l} \binom{l}{i} (-1)^{k+j} s(k,j) (-k)^{l-i} X_{n-i} Y_i B(n-l+j).$$

That is, both X_* and Y_* can be any of the 5 sequences of numbers: constant 1, Fibonacci F_*, Lucas L_*, Bernoulli $\mathfrak{B}_*$ and Euler E_*. All of these identities involve the Stirling numbers of the first kind and the Bell numbers, and all of them have some "Gould-Quaintance formula flavor".

The previous discussion can be also applied to formula (7). We have 25 identities of the form

$$\sum_{j=0}^{n+k} \binom{n+k}{j} X_j Y_{n+k-j} B(j) = \sum_{m=0}^{k} \sum_{j=0}^{n} \sum_{l=0}^{k} \sum_{i=0}^{j} \binom{n}{j} \binom{k}{l} \binom{j}{i} m^{n-j} X_{n+l+i-j} Y_{k+j-l-i} S(l,m) B(i),$$

where both X_* and Y_* can be any of the 5 sequences of numbers: constant 1, Fibonacci F_*, Lucas L_*, Bernoulli $\mathfrak{B}_*$, Euler E_*. All these identities involve the Stirling numbers of the second kind and the Bell numbers, and all of them have some "Spivey's formula flavor".

Finally, we mention that if we set simultaneously $a = \alpha$ and $b = \beta$ in (6), it is possible to obtain two new identities:

$$\sum_{j=0}^{n} \binom{n}{j} (-1)^j F_{2j} B(j) = \sum_{j=0}^{k} \sum_{l=0}^{n} \sum_{i=0}^{l} \binom{n}{l} \binom{l}{i} (-1)^{n+l+k+j} s(k,j) k^{l-i} F_{2(n-i)} B(n-l+j),$$

and

$$\sum_{j=0}^{n} \binom{n}{j} (-1)^j L_{2j} B(j) = \sum_{j=0}^{k} \sum_{l=0}^{n} \sum_{i=0}^{l} \binom{n}{l} \binom{l}{i} (-1)^{n+l+k+j} s(k,j) k^{l-i} L_{2(n-i)} B(n-l+j).$$

Similarly, from (7) one can obtain the following identities:

$$\sum_{j=0}^{n+k}\binom{n+k}{j}(-1)^{j}F_{2j}B(j)=\sum_{m=0}^{k}\sum_{j=0}^{n}\sum_{l=0}^{k}\sum_{i=0}^{j}\binom{n}{j}\binom{k}{l}\binom{j}{i}(-1)^{n+l+i+j}m^{n-j}F_{2(n+l+i-j)}S(l,m)B(i)$$

and

$$\sum_{j=0}^{n+k}\binom{n+k}{j}(-1)^{j}L_{2j}B(j)=\sum_{m=0}^{k}\sum_{j=0}^{n}\sum_{l=0}^{k}\sum_{i=0}^{j}\binom{n}{j}\binom{k}{l}\binom{j}{i}(-1)^{n+l+i+j}m^{n-j}L_{2(n+l+i-j)}S(l,m)B(i).$$

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