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Some Smoothness Properties of the Lupaş-Kantorovich Type Operators Based on Pólya Distribution

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Abstract. In the present article, we study some smoothness properties of new Lupaş-Kantorovich type operators based on Pólya distribution, as uniform convergence and asymptotic behavior. In order to get the degree of approximation, some quantitative type theorems will be established. The bivariate extension of these operators, with some indispensable results will be also presented.

1. Introduction

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In 1930, Kantorovich [24] has considered the following integral form of the well-known Bernstein operators, in order to approximate functions in space $L_1[0,1]$ (the class of Lebesgue integrable functions on $[0,1]$, which means $\int_0^1 |f(x)|dx < \infty$), given by

$$K_n(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt = (n+1)\sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k}\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt, \tag{1}$$

for $x \in [0,1]$ and $n \in \mathbb{N}$, where $p_{n,k}$ are the Bernstein fundamental polynomials. Looking for papers which contain certain researches and studies of the Kantorovich operators (1) or generalizations of them, we find thousand of published articles until nowadays. Some representative examples in this sense could be the following papers [1, 2, 4, 5, 8, 11, 12, 14, 19–21, 26, 29, 34, 36, 37]. Among highly investigated operators in the last period, which have the preservation property of the linear functions are the Lupaş operators [27], defined for any function $f \in C[0,1]$ by

$$P_n^{(1/n)}(f;x) = \sum_{k=0}^{n} p_{n,k}^{(1/n)}(x) f\left(\frac{k}{n}\right) = \frac{2(n!)}{(2n)!}\sum_{k=0}^{n}\binom{n}{k}\prod_{\nu=0}^{k-1}(nx+\nu)\prod_{\mu=0}^{n-k-1}(n-nx+\mu) f\left(\frac{k}{n}\right). \tag{2}$$

Having as fundamental basis $p_{n,k}^{(1/n)}$, the Lupaş operators (2) are based on Pólya distribution and they could be obtained as a special case from the well-known class of operators introduced by Stancu [35]. Some

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approximation properties, as well as the bivariate extension of Lupaş operators (2) are investigated in two recent papers [28], [30]. In 2014, Gupta and Rassias [22] have introduced the Durrmeyer type integral modification of the operators (2) and studied asymptotic behavior, local, respectively global results. Further modifications in sense of Lupaş operators having as start point the paper of Gupta and Rassias are given in [6, 7, 15]. A new interesting sequence of positive linear operators is presented by Özarslan and Duman [33] as follows

$$K_{n,\alpha}(f;x)=\sum_{k=0}^{n}p_{n,k}(x)\int_0^1 f\left(\frac{k+t^\alpha}{n+1}\right)dt,$$

for a given $\alpha>0$ and $n\in\mathbb{N}$. The main results investigated by the authors for these modified operators $K_{n,\alpha}$ are geometric properties concerning convexity and some approximation properties involving uniform convergence, the order of approximation and simultaneous approximation.

Inspired by [33], for any integrable function $f:[0,1]\to\mathbb{R}$, $\rho>0$ and $n\in\mathbb{N}$ we introduce the Lupaş-Kantorovich type operators based on Pólya distribution, given by

$$\begin{aligned}\mathcal{K}_{n,\rho}(f;x)&=\sum_{k=0}^{n}p_{n,k}^{(1/n)}(x)\int_0^1 f\left(\frac{k+t^\rho}{n+1}\right)dt \\ &=\frac{2(n!)}{(2n)!}\sum_{k=0}^{n}\binom{n}{k}\prod_{\nu=0}^{k-1}(nx+\nu)\prod_{\mu=0}^{n-k-1}(n-nx+\mu)\int_0^1 f\left(\frac{k+t^\rho}{n+1}\right)dt.\end{aligned} \tag{3}$$

The purpose of this paper is to present these new Lupaş-Kantorovich type operators (3) based on Pólya distribution, studying the uniform convergence and asymptotic behavior. In order to get the degree of approximation, some quantitative type theorems will be established. The bivariate extension of these operators, with some approximation results will be also presented.

2. Auxiliary Results

For Lupaş-Kantorovich type operators (3) we establish some necessary results. The monomials $e_k(x)=x^k$, for $k\in\mathbb{N}_0$ called also test functions play an important role in uniform approximation by linear positive operators. In order to determine the images of the monomials by operators (3) we present a useful form of these operators.

Lemma 2.1. *For any $n\in\mathbb{N}$, $\rho>0$ and $x\in[0,1]$, it follows*

$$\mathcal{K}_{n,\rho}(e_m;x)=\frac{1}{(n+1)^m}\sum_{i=0}^{m}\binom{m}{i}\frac{n^i}{\rho(m-i)+1}P_n^{(1/n)}(e_i;x), \tag{4}$$

where $P_n^{(1/n)}$ is given at (2) and $m\in\mathbb{N}_0$.

Proof. Using the definition of the $\mathcal{K}_{n,\rho}$, we get

$$\begin{aligned}\mathcal{K}_{n,\rho}(e_m;x)&=\sum_{k=0}^{n}p_{n,k}^{(1/n)}(x)\int_0^1\left(\frac{k+t^\rho}{n+1}\right)^m dt=\frac{1}{(n+1)^m}\sum_{i=0}^{m}\binom{m}{i}\sum_{k=0}^{n}p_{n,k}^{(1/n)}(x)k^i\int_0^1 t^{\rho(m-i)}dt \\ &=\frac{1}{(n+1)^m}\sum_{i=0}^{m}\binom{m}{i}\frac{n^i}{\rho(m-i)+1}\sum_{k=0}^{n}p_{n,k}^{(1/n)}(x)\left(\frac{k}{n}\right)^i=\frac{1}{(n+1)^m}\sum_{i=0}^{m}\binom{m}{i}\frac{n^i}{\rho(m-i)+1}P_n^{(1/n)}(e_i;x).\end{aligned}$$

□

The images of the monomials $e_i(x)=x^i$, for $i=0,1,2,3,4$ by Lupaş operators (2) are given in [27] or [28].

Lemma 2.2. *For any $n \in \mathbb{N}$ hold*
$P_n^{(1/n)}(e_0;x) = 1$; $P_n^{(1/n)}(e_1;x) = x$; $P_n^{(1/n)}(e_2;x) = x^2 + \frac{2x(1-x)}{n+1}$; $P_n^{(1/n)}(e_3;x) = x^3 + \frac{6nx^2(1-x)}{(n+1)(n+2)} + \frac{6x(1-x)}{(n+1)(n+2)}$;
$P_n^{(1/n)}(e_4;x) = x^4 + \frac{12(n^2+1)x^3(1-x)}{(n+1)(n+2)(n+3)} + \frac{12(3n-1)x^2(1-x)}{(n+1)(n+2)(n+3)} + \frac{2(13n-1)x(1-x)}{n(n+1)(n+2)(n+3)}$.

Corollary 2.3. *For the Lupaş-Kantorovich type operators* (3) *hold*
$\mathcal{K}_{n,\rho}(e_0;x) = 1$; $\mathcal{K}_{n,\rho}(e_1;x) = x + \frac{1-(1+\rho)x}{(n+1)(1+\rho)}$;
$\mathcal{K}_{n,\rho}(e_2;x) = x^2 - \frac{\left(4n^2+3n+1\right)x^2}{(n+1)^3} + \frac{2n(1+n(2+\rho))x}{(n+1)^3(1+\rho)} + \frac{1}{(n+1)^2(1+2\rho)}$.

Proof. Using the relation (4) and Lemma 2.2, it follows the desired results. □

Corollary 2.4. *The computation of the central moments up to the second order for Lupaş-Kantorovich type operators* (3)*, is given by*
$\mathcal{K}_{n,\rho}(e_1 - x;x) = \frac{1-(1+\rho)x}{(n+1)(1+\rho)}$;
$\mathcal{K}_{n,\rho}\left((e_1 - x)^2;x\right) = \frac{\left(1+n-2n^2\right)x^2}{(n+1)^3} + \frac{2(n(n\rho+n-1)-1)x}{(n+1)^3(1+\rho)} + \frac{1}{(n+1)^2(1+2\rho)}$.

Proof. The above results follow from Corollary 2.3 and simple computations. □

Proposition 2.5. *Let f be a real-valued function continuous on* $[0,1]$*, with* $\|f\| = \sup_{x\in[0,1]} |f(x)|$*, then*

$$|\mathcal{K}_{n,\rho}(f;x)| \le \|f\|.$$

Proof. Taking the definition of Lupaş-Kantorovich type operators (3) and Corollary 2.3 into account, it follows

$$|\mathcal{K}_{n,\rho}(f;x)| \le \sum_{k=0}^{n} p_{n,k}^{(1/n)}(x) \int_0^1 \left| f\left(\frac{k+t^\rho}{n+1}\right)\right| dt \le \|f\|\mathcal{K}_{n,\rho}(e_0;x) = \|f\|.$$

□

Lemma 2.6. *For any $n \in \mathbb{N}$, we can write*

$$\mathcal{K}_{n,\rho}\left((e_1 - x)^2;x\right) \le \frac{C}{n+1} \cdot \delta_n^2(x), \tag{5}$$

where $C > 1$, $\delta_n^2(x) = \phi^2(x) + \frac{1}{(n+1)(1+\rho)}$ and $\phi^2(x) = x(1-x)$.

3. Direct Results for Lupaş-Kantorovich Type Operators

Our further studies focus on the qualitative properties of Lupaş-Kantorovich type operators (3), involving the uniform convergence and asymptotic behavior.

Theorem 3.1. *For every $f \in C[0,1]$ and $\rho > 0$, yields $\lim_{n\to\infty} \mathcal{K}_{n,\rho}(f;x) = f(x)$ uniformly on* $[0,1]$.

Proof. Since $\mathcal{K}_{n,\rho}(e_0;x) = 1$, $\mathcal{K}_{n,\rho}(e_1;x) = x + \frac{1-(1+\rho)x}{(n+1)(1+\rho)}$, $\mathcal{K}_{n,\rho}(e_2;x) = x^2 - \frac{\left(4n^2+3n+1\right)x^2}{(n+1)^3} + \frac{2n(1+n(2+\rho))x}{(n+1)^3(1+\rho)} + \frac{1}{(n+1)^2(1+2\rho)}$, it follows

$$\lim_{n\to\infty} \mathcal{K}_{n,\rho}(e_i;x) = e_i(x), \text{ for } i = 0,1,2.$$

Applying the well-known Korovkin's theorem, we get

$$\lim_{n\to\infty} \mathcal{K}_{n,\rho}(f;x) = f(x) \text{ uniformly on } [0,1].$$

□

The next result provides the asymptotic behavior of the Lupaş-Kantorovich type operators.

Theorem 3.2. *If $f \in C^2[0,1]$, then*

$$\lim_{n\to\infty} n\left(\mathcal{K}_{n,\rho}(f;x)-f(x)\right)=\left(\frac{1}{1+\rho}-x\right)f'(x)+x(1-x)f''(x) \tag{6}$$

uniformly on $[0,1]$.

Proof. Using Taylor's expansion formula of the function f, it follows

$$f(t)=f(x)+(t-x)f'(x)+\tfrac{1}{2}(t-x)^2f''(x)+r(t,x)(t-x)^2, \tag{7}$$

where $r(t,x):=r(t-x)$ is a bounded function and $\lim\limits_{t\to x} r(t,x)=0$. Taking the linearity of operators (3) into account and applying the operators $\mathcal{K}_{n,\rho}$ on both side of the above equation (7), we get

$$\mathcal{K}_{n,\rho}(f;x)-f(x)=\mathcal{K}_{n,\rho}(e_1-x;x)f'(x)+\tfrac{1}{2}\mathcal{K}_{n,\rho}\left((e_1-x)^2;x\right)f''(x)+\mathcal{K}_{n,\rho}\left(r(t,x)\cdot(e_1-x)^2;x\right).$$

Therefore, using Corollary 2.4 it follows

$$\lim_{n\to\infty} n\left(\mathcal{K}_{n,\rho}(f;x)-f(x)\right)=\left(\frac{1}{1+\rho}-x\right)f'(x)+x(1-x)f''(x)+\lim_{n\to\infty} n\left(\mathcal{K}_{n,\rho}\left(r(t,x)\cdot(e_1-x)^2;x\right)\right). \tag{8}$$

We estimate the last term on the right-hand side of the above equality, applying the Cauchy-Schwarz inequality, so we obtain

$$\mathcal{K}_{n,\rho}\left(r(t,x)\cdot(e_1-x)^2;x\right)\le\sqrt{\mathcal{K}_{n,\rho}\left((r^2(t,x);x\right)}\sqrt{\mathcal{K}_{n,\rho}\left((e_1-x)^4;x\right)}. \tag{9}$$

Because $r^2(x,x)=0$ and $r^2(\cdot,x)\in C[0,1]$, using the convergence from Theorem 3.1, we get

$$\lim_{n\to\infty}\mathcal{K}_{n,\rho}\left(r^2(t;x);x\right)=r^2(x,x)=0. \tag{10}$$

Therefore, from (9), respectively (10) yields

$$\lim_{n\to\infty} n\left(\mathcal{K}_{n,\rho}\left(r(t,x)\cdot(e_1-x)^2;x\right)\right)=0$$

and using (8) we obtain the asymptotic behavior of the Lupaş-Kantorovich type operators (3). □

The main tools to measure the degree of approximation of linear positive operators towards the identity operators are moduli of smoothness. For $f\in C[0,1]$ and $\delta\ge 0$ we consider the moduli of smoothness of first, respectively second order, given by

$$\omega_1(f,\delta):=\sup\{|f(x+h)-f(x)|\,:\,x,x+h\in[0,1],\ 0\le h\le\delta\},$$

respectively

$$\omega_2(f,\delta):=\sup\{|f(x+h)-2f(x)+f(x-h)|\,:\,x,x\pm h\in[0,1],\ 0\le h\le\delta\}.$$

Moreover, let us consider Peetre's K-functional

$$K_2(f,\delta)=\inf\{\|f-g\|+\delta\|g''\|\,:\,g\in C^2[0,1]\},\ \text{for }\delta>0. \tag{11}$$

There exists an absolute constant $M>0$, such that

$$K_2(f,\delta)\le M\cdot\omega_2\left(f,\sqrt{\delta}\right), \tag{12}$$

according to Theorem 2.4, p. 177 in [16].

Below, we derive some quantitative estimates in terms of moduli of smoothness and Peetre's K-functional for the Lupaş-Kantorovich type operators (3).

Theorem 3.3. *Let $f \in C[0,1]$, $\rho > 0$ and $n \geq 1$, then for any $x \in [0,1]$ yields*

$$|\mathcal{K}_{n,\rho}(f;x) - f(x)| \leq M_1 \cdot \omega_2\left(f, \tfrac{1}{2}\delta_{n,x}\right) + \omega_1\left(f, \delta_\omega\right),$$

where M_1 is an absolute constant and $\delta_{n,x} = \left(\mathcal{K}_{n,\rho}\left((e_1-x)^2;x\right) + \left(\mathcal{K}_{n,\rho}(e_1-x;x)\right)^2\right)^{1/2}$, $\delta_\omega = |\mathcal{K}_{n,\rho}(e_1-x;x)|$.

Proof. For any $n \in \mathbb{N}$, $\rho > 0$ and $x \in [0,1]$, we construct the auxiliary operators as

$$\mathcal{T}_{n,\rho}(f;x) = \mathcal{K}_{n,\rho}(f;x) + f(x) - f\left(x + \tfrac{1-(1+\rho)x}{(n+1)(1+\rho)}\right). \tag{13}$$

By Corollary 2.3, we remark that $\mathcal{T}_{n,\rho}(e_0;x) = 1$ and $\mathcal{T}_{n,\rho}(e_1;x) = x$, i.e. the defined operators (13) preserve constants as well as linear functions. Therefore

$$\mathcal{T}_{n,\rho}(e_1 - x;x) = 0. \tag{14}$$

For any $g \in C^2[0,1]$ and $t, x \in [0,1]$, by using the Taylor's expansion formula, we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du.$$

Applying the operators $\mathcal{T}_{n,\rho}$ on both sides of the above equation, we get

$$\begin{aligned}\mathcal{T}_{n,\rho}(g;x) - g(x) &= g'(x)\mathcal{T}_{n,\rho}(e_1 - x;x) + \mathcal{T}_{n,\rho}\Big(\int_x^t (t-u)g''(u)du;x\Big)\\ &= \mathcal{K}_{n,\rho}\Big(\int_x^t (t-u)g''(u)du;x\Big) - \int_x^{x+\frac{1-(1+\rho)x}{(n+1)(1+\rho)}} \left(x + \tfrac{1-(1+\rho)x}{(n+1)(1+\rho)} - u\right)g''(u)du.\end{aligned}$$

On the other hand

$$\left|\int_x^t (t-u)g''(u)du\right| \leq (t-x)^2 \cdot \|g''\|,$$

then

$$|\mathcal{T}_{n,\rho}(g;x) - g(x)| \leq \left(\mathcal{K}_{n,\rho}\left((e_1-x)^2;x\right) + \left(\mathcal{K}_{n,\rho}(e_1-x;x)\right)^2\right) \cdot \|g''\| = \delta_{n,x}^2 \cdot \|g''\|. \tag{15}$$

Using the definition (13) of the operators $\mathcal{T}_{n,\rho}$ and Proposition 2.5, it follows

$$\begin{aligned}|\mathcal{K}_{n,\rho}(f;x) - f(x)| &= \left|\mathcal{T}_{n,\rho}(f;x) - f(x) + f\left(x + \tfrac{1-(1+\rho)x}{(n+1)(1+\rho)}\right) - f(x)\right|\\ &\leq \left|\mathcal{T}_{n,\rho}(f-g;x)\right| + |\mathcal{T}_{n,\rho}(g;x) - g(x)| + |g(x) - f(x)| + \left|f\left(x + \tfrac{1-(1+\rho)x}{(n+1)(1+\rho)}\right) - f(x)\right|\\ &\leq 4 \cdot \|f-g\| + \delta_{n,x}^2 \cdot \|g''\| + \omega_1\left(f, \delta_\omega\right),\end{aligned}$$

with $\delta_{n,x}^2 = \mathcal{K}_{n,\rho}\left((e_1-x)^2;x\right) + \left(\mathcal{K}_{n,\rho}(e_1-x;x)\right)^2$ and $\delta_\omega = |\mathcal{K}_{n,\rho}(e_1-x;x)|$.
Now, taking the infimum on the right-hand side over all $g \in C^2[0,1]$ and using the relation (11), we get

$$|\mathcal{K}_{n,\rho}(f;x) - f(x)| \leq 4 \cdot K_2\left(f, \tfrac{1}{4}\delta_{n,x}^2\right) + \omega_1\left(f, \delta_\omega\right) \leq M_1 \cdot \omega_2\left(f, \tfrac{1}{2}\delta_{n,x}\right) + \omega_1\left(f, \delta_\omega\right),$$

where M_1 is an absolute constant. □

In order to prove a global approximation theorem for the Lupaş-Kantorovich type operators involving the Ditzian-Totik modulus of smoothness, we recall some results from [17]. For any $f \in C_B[0,1]$ and $\delta \geq 0$, the Ditzian-Totik modulus of smoothness of second order and appropriate K-functional are given by

$$\omega_2^{\phi}\left(f, \sqrt{\delta}\right) = \sup_{0<h\leq \sqrt{\delta}} \sup_{x\pm h\phi(x)\in[0,1]} |f(x+h\phi(x)) - 2f(x) + f(x-h\phi(x))|,$$

and

$$K_2^{\phi}(f,\delta) = \inf\left\{\|f-g\| + \delta\|\phi^2 g''\| + \delta^2\|g''\| : g \in W^2(\phi)\right\},\ (\delta > 0),$$

with $\phi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$, where $W^2(\phi) = \{g \in C[0,1] : g' \in AC_{\rm loc}[0,1],\ \phi^2 g'' \in C[0,1]\}$ and $g' \in AC_{\rm loc}[0,1]$ means that g is differentiable and g' is absolutely continuous on every closed interval $[a,b] \subset (0,1)$. Conformable with [17], there exists a positive constant $N > 0$, such that

$$K_2^{\phi}(f,\delta) \leq N \cdot \omega_2^{\phi}\left(f, \sqrt{\delta}\right). \tag{16}$$

Further, the Ditzian-Totik modulus of smoothness of first order is given by

$$\omega_{\psi}(f,\delta) = \sup_{0<h\leq\delta} \sup_{x\pm(h/2)\psi(x)\in[0,1]} \left|f\left(x+\tfrac{1}{2}h\psi(x)\right) - f\left(x-\tfrac{1}{2}h\psi(x)\right)\right|,$$

where ψ is an admissible step-weight function on $[0,1]$.

Theorem 3.4. *Let be $f \in C[0,1]$, $\rho > 0$ and $n \geq 1$, then for any $x \in [0,1]$ we get*

$$|\mathcal{K}_{n,\rho}(f;x) - f(x)| \leq N_1 \cdot \omega_2^{\phi}\left(f, \tfrac{1}{\sqrt{n+1}}\right) + \omega_{\psi}\left(f, \tfrac{1}{n+1}\right),$$

where N_1 is a positive constant and $\psi(x) = |1-(1+\rho)x|$.

Proof. We consider again the auxiliary operators defined at (13), given by the following relation

$$\mathcal{T}_{n,\rho}(f;x) = \mathcal{K}_{n,\rho}(f;x) + f(x) - f\left(x + \tfrac{1-(1+\rho)x}{(n+1)(1+\rho)}\right).$$

Taking the definition of the Lupaş-Kantorovich operators (3) into account, proceeding as in above theorem, for $g \in W^2(\phi)$, we get

$$|\mathcal{T}_{n,\rho}(g;x) - g(x)| \leq \mathcal{K}_{n,\rho}\Big(\int_x^t |t-u| \cdot |g''(u)|du; x\Big) + \int_x^{x+\frac{1-(1+\rho)x}{(n+1)(1+\rho)}} \left|x + \tfrac{1-(1+\rho)x}{(n+1)(1+\rho)} - u\right| \cdot |g''(u)|du. \tag{17}$$

Since $\delta_n^2(x) = \phi^2(x) + \frac{1}{(n+1)(1+\rho)}$ is a concave function on $[0,1]$, for $u = \lambda x + (1-\lambda)t$, with $t < u < x$ and $\lambda \in [0,1]$, it follows

$$\frac{|t-u|}{\delta_n^2(u)} = \frac{\lambda|t-x|}{\delta_n^2(\lambda x + (1-\lambda)t)} \leq \frac{\lambda|t-x|}{\lambda\delta_n^2(x) + (1-\lambda)\delta_n^2(t)} \leq \frac{|t-x|}{\delta_n^2(x)}. \tag{18}$$

Thus, using inequality (18), the relation (17) leads us to

$$\begin{aligned}|\mathcal{T}_{n,\rho}(g;x) - g(x)| &\leq \mathcal{K}_{n,\rho}\left(\int_x^t \frac{|t-u|}{\delta_n^2(u)}du; x\right) \cdot \|\delta_n^2 g''\| + \left(\int_x^{x+\frac{1-(1+\rho)x}{(n+1)(1+\rho)}} \frac{\left|x + \frac{1-(1+\rho)x}{(n+1)(1+\rho)} - u\right|}{\delta_n^2(u)}du\right) \cdot \|\delta_n^2 g''\| \\ &\leq \frac{1}{\delta_n^2(x)}\|\delta_n^2 g''\|\left(\mathcal{K}_{n,\rho}\left((e_1-x)^2; x\right) + \left(\frac{1-(1+\rho)x}{(n+1)(1+\rho)}\right)^2\right).\end{aligned}$$

Now, applying the inequality (5), we find

$$|\mathcal{T}_{n,\rho}(g;x)-g(x)|\leq \frac{C}{n+1}\|\delta_n^2 g''\|\leq \frac{C}{n+1}\Big(\|\phi^2 g''\|+\frac{1}{n+1}\|g''\|\Big).$$

For any $f\in C[0,1]$, using the above inequalities and Proposition 2.5, it follows

$$\begin{aligned}|\mathcal{K}_{n,\rho}(f;x)-f(x)| &\leq |\mathcal{T}_{n,\rho}(f-g,x)|+|\mathcal{T}_{n,\rho}(g;x)-g(x)|+|g(x)-f(x)|+\left|f\left(x+\tfrac{1-(1+\rho)x}{(n+1)(1+\rho)}\right)-f(x)\right|\\ &\leq 4\cdot\|f-g\|+\frac{C}{n+1}\|\phi^2 g''\|+\frac{C}{(n+1)^2}\|g''\|+\left|f\left(x+\tfrac{1-(1+\rho)x}{(n+1)(1+\rho)}\right)-f(x)\right|.\end{aligned}$$

Taking the infimum on the right hand side over all $g\in W^2(\phi)$, we obtain

$$|\mathcal{K}_{n,\rho}(f;x)-f(x)|\leq C_1\cdot K_2^{\phi}\left(f,\tfrac{1}{n+1}\right)+\left|f\left(x+\tfrac{1-(1+\rho)x}{(n+1)(1+\rho)}\right)-f(x)\right|. \tag{19}$$

On the other hand

$$\begin{aligned}\left|f\left(x+\frac{1-(1+\rho)x}{(n+1)(1+\rho)}\right)-f(x)\right| &= \left|f\left(x+\psi(x)\frac{1-(1+\rho)x}{(n+1)(1+\rho)\psi(x)}\right)-f(x)\right|\\ &\leq \sup_{t\in[0,1]}\left|f\left(t+\psi(x)\frac{\frac{1}{(1+\rho)}-x}{(n+1)\psi(x)}\right)-f(t)\right|\\ &\leq \omega_\psi\left(f,\frac{\left|\frac{1}{1+\rho}-x\right|}{(n+1)\psi(x)}\right)\leq \omega_\psi\left(f,\frac{1}{n+1}\right).\end{aligned} \tag{20}$$

Hence, combining (16), (19) and (20), we get the desired result. □

4. Construction of Bivariate Operators

Gurdek et al. [23] established some direct results for the bivariate extension of the Baskakov and Baskakov-Kantorovich operators. Doğru and Gupta [18] presented a bivariate generalization of the Meyer-König and Zeller operators based on q-integers and obtained the rate of convergence of these operators with the help of Korovkin theorem for the bivariate functions. Wafi and Khatoon [38] established some approximation properties for the bivariate extension of generalized Baskakov operators defined by Miheşan [31]. Mishra et al. [32] introduced a bivariate generalization of the discrete q-Beta operators and obtained their statistical approximation properties. In this section we present the bivariate form of the operators defined at (3), respectively we establish their rate of approximation. For $J^2:=J\times J$, with $J=[0,1]$, let $C(J^2)$ be the space of all real-valued continuous functions on J^2, endowed with the norm given by $\|f\|_{C(J^2)}=\sup_{(x,y)\in J^2}|f(x,y)|$. Let $C^2(J^2)$ be the space of all functions $f\in C(J^2)$, such that $\frac{\partial^i f}{\partial x^i},\frac{\partial^i f}{\partial y^i}$, for $i=1,2$ belong to the space $C(J^2)$. The appropriate norm on the space $C^2(J^2)$ is defined as

$$\|f\|_{C^2(J^2)}=\|f\|_{C(J^2)}+\sum_{i=1}^{2}\left(\left\|\frac{\partial^i f}{\partial x^i}\right\|_{C(J^2)}+\left\|\frac{\partial^i f}{\partial y^i}\right\|_{C(J^2)}\right).$$

The Peetre's K-functional of $f\in C(J^2)$ is defined by

$$K(f;\delta)=\inf_{g\in C^2(J^2)}\left\{\|f-g\|_{C(J^2)}+\delta\|g\|_{C^2(J^2)},\ \delta>0\right\}.$$

It is also known that the following inequality

$$K(f;\delta) \le M^* \cdot \left\{ \overline{\omega_2}(f;\sqrt{\delta}) + \min(1,\delta)\|f\|_{C(J^2)} \right\} \tag{21}$$

holds, for all $\delta > 0$ conformable with [13]. The constant M^* is independent of δ and f and $\overline{\omega_2}(f;\sqrt{\delta})$ is the second order modulus of continuity defined in a similar manner as in the above section.
For any integrable function $f : J^2 \to \mathbb{R}$, $\rho_1,\rho_2 > 0$ and $n_1,n_2 \in \mathbb{N}$ we introduce the bivariate Lupaş-Kantorovich type operators based on Pólya distribution, given by

$$\mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}(f;x,y) = \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2} p_{n_1,k_1}^{(1/n_1)}(x) p_{n_2,k_2}^{(1/n_2)}(y) \int_0^1\int_0^1 f\left(\frac{k_1+t^{\rho_1}}{n_1+1},\frac{k_2+s^{\rho_2}}{n_2+1}\right) dt ds, \tag{22}$$

with $p_{n_1,k_1}^{(1/n_1)}(x) = \frac{2(n_1!)}{(2n_1)!}\binom{n_1}{k_1}\prod_{\nu=0}^{k_1-1}(n_1x+\nu)\prod_{\mu=0}^{n_1-k_1-1}(n_1-n_1x+\mu)$ and $p_{n_2,k_2}^{(1/n_2)}(y) = \frac{2(n_2!)}{(2n_2)!}\binom{n_2}{k_2}\prod_{\nu=0}^{k_2-1}(n_2y+\nu)\prod_{\mu=0}^{n_2-k_2-1}(n_2-n_2y+\mu)$. The test functions in the bivariate case are given by $e_{i,j}(x,y) = x^iy^j$, for $(i,j) \in \mathbb{N}_0 \times \mathbb{N}_0$. In the following, we establish some auxiliary results.

Lemma 4.1. *For the bivariate Lupaş-Kantorovich type operators (22), the following formulae hold*
$\mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}(e_{0,0};x,y) = 1$; $\mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}(e_{1,0};x,y) = x + \frac{1-(1+\rho_1)x}{(n_1+1)(1+\rho_1)}$; $\mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}(e_{0,1};x,y) = y + \frac{1-(1+\rho_2)y}{(n_2+1)(1+\rho_2)}$;
$\mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}(e_{2,0};x,y) = x^2 - \frac{(4n_1^2+3n_1+1)x^2}{(n_1+1)^3} + \frac{2n_1(1+n_1(2+\rho_1))x}{(n_1+1)^3(1+\rho_1)} + \frac{1}{(n_1+1)^2(1+2\rho_1)}$;
$\mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}(e_{0,2};x,y) = y^2 - \frac{(4n_2^2+3n_2+1)y^2}{(n_2+1)^3} + \frac{2n_2(1+n_2(2+\rho_2))y}{(n_2+1)^3(1+\rho_2)} + \frac{1}{(n_2+1)^2(1+2\rho_2)}$.

Proof. The proof of this lemma easily follows taking relation (22) and Corollary 2.3 into account. □

Corollary 4.2. *The computation of the central moments up to the second order for the bivariate Lupaş-Kantorovich type operators, is given by*
$\mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}((e_{1,0}-x);x,y) = \frac{1-(1+\rho_1)x}{(n_1+1)(1+\rho_1)}$; $\mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}((e_{0,1}-y);x,y) = \frac{1-(1+\rho_2)y}{(n_2+1)(1+\rho_2)}$;
$\mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}\left((e_{2,0}-x)^2;x,y\right) = \frac{(1+n_1-2n_1^2)x^2}{(n_1+1)^3} + \frac{2(n_1(n_1\rho_1+n_1-1)-1)x}{(n_1+1)^3(1+\rho_1)} + \frac{1}{(n_1+1)^2(1+2\rho_1)}$;
$\mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}\left((e_{0,2}-y)^2;x,y\right) = \frac{(1+n_2-2n_2^2)y^2}{(n_2+1)^3} + \frac{2(n_2(n_2\rho_2+n_2-1)-1)y}{(n_2+1)^3(1+\rho_2)} + \frac{1}{(n_2+1)^2(1+2\rho_2)}$.

Theorem 4.3. *The sequence of bivariate Lupaş-Kantorovich type operators $\left(\mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}f\right)$ converges uniformly to f, for any $f \in C(J^2)$.*

Proof. Since

$$\lim_{n_1,n_2\to\infty} \mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}(e_{i,j};x,y) = e_{i,j}(x,y), \quad (i,j) \in \{(0,0),(0,1),(1,0)\}$$

and

$$\lim_{n_1,n_2\to\infty} \mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}(e_{2,0}+e_{0,2};x,y) = e_{2,0}(x,y) + e_{0,2}(x,y)$$

uniformly on J^2, then

$$\lim_{n_1,n_2\to\infty} \mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}(f;x,y) = f(x,y) \text{ uniformly on } J^2.$$

□

Remark 4.4. *A detailed proof of the above theorem in the case of any bivariate operators is given in [10].*

For any function $f \in C(J^2)$, the complete modulus of continuity for bivariate case is defined by

$$\omega^{(c)}(f;\delta) = \sup\left\{ |f(t,s) - f(x,y)| : (t,s),(x,y) \in J^2 \text{ and } \sqrt{(t-x)^2+(s-y)^2} \le \delta \right\}.$$

Further, the partial moduli of continuity with respect to x, respectively y are defined by

$$\omega^{(1)}(f;\delta) = \sup\{|f(x_1,y) - f(x_2,y)| : y \in J \text{ and } |x_1 - x_2| \le \delta\}$$

and

$$\omega^{(2)}(f;\delta) = \sup\{|f(x,y_1) - f(x,y_2)| : x \in J \text{ and } |y_1 - y_2| \le \delta\}.$$

It is clear that they satisfy the properties of the usual modulus of continuity. The details of the modulus of continuity for the bivariate case can be found in [9]. Below, we get the degree of approximation for the bivariate Lupaş-Kantorovich type operators (22).

Theorem 4.5. *Let the function $f \in C(J^2)$. The following inequality*

$$|\mathcal{K}^{\rho_1,\rho_2}_{n_1,n_2}(f;x,y) - f(x,y)| \le 2\cdot\left(\omega^{(1)}\left(f;(n_1+1)^{-1/2}\right) + \omega^{(2)}\left(f;(n_2+1)^{-1/2}\right)\right)$$

holds.

Proof. Taking the definition of the partial moduli of continuity into account and applying Cauchy-Schwarz inequality, we may write

$$\begin{aligned}
|\mathcal{K}^{\rho_1,\rho_2}_{n_1,n_2}(f;x,y) - f(x,y)| &\le \mathcal{K}^{\rho_1,\rho_2}_{n_1,n_2}(|f(t,s) - f(x,y)|;x,y)\\
&\le \mathcal{K}^{\rho_1,\rho_2}_{n_1,n_2}(|f(t,s) - f(t,y)|;x,y) + \mathcal{K}^{\rho_1,\rho_2}_{n_1,n_2}(|f(t,y) - f(x,y)|;x,y)\\
&\le \mathcal{K}^{\rho_1,\rho_2}_{n_1,n_2}\left(\omega^{(2)}(f;|s-y|);x,y\right) + \mathcal{K}^{\rho_1,\rho_2}_{n_1,n_2}\left(\omega^{(1)}(f;|t-x|);x,y\right)\\
&\le \omega^{(2)}(f;\delta_{n_2})\left(1 + \frac{1}{\delta_{n_2}}\mathcal{K}^{\rho_2}_{n_2}(|s-y|;y)\right) + \omega^{(1)}(f;\delta_{n_1})\left(1 + \frac{1}{\delta_{n_1}}\mathcal{K}^{\rho_1}_{n_1}(|t-x|;x)\right)\\
&\le \omega^{(2)}(f;\delta_{n_2})\left(1 + \frac{1}{\delta_{n_2}}\left(\mathcal{K}^{\rho_2}_{n_2}((s-y)^2;y)\right)^{1/2}\right) + \omega^{(1)}(f;\delta_{n_1})\left(1 + \frac{1}{\delta_{n_1}}\left(\mathcal{K}^{\rho_1}_{n_1}((t-x)^2;x)\right)^{1/2}\right) = A_1 + A_2.
\end{aligned}$$

Taking $\delta_{n_2} = (n_2+1)^{-1/2}$, we obtain $A_1 \le 2\cdot\omega^{(2)}\left(f;(n_2+1)^{-1/2}\right)$. Analogously, taking $\delta_{n_1} = (n_1+1)^{-1/2}$, we get $A_2 \le 2\cdot\omega^{(1)}\left(f;(n_1+1)^{-1/2}\right)$. Hence, we get the desired result. □

As in [25], for $0 < \eta_1 \le 1$ and $0 < \eta_2 \le 1$ we define the Lipschitz class $Lip_{M^L}(\eta_1,\eta_2)$ for the bivariate case as follows

$$|f(t,s) - f(x,y)| \le M^L\cdot|t-x|^{\eta_1}|s-y|^{\eta_2}.$$

The degree of approximation for the bivariate Lupaş-Kantorovich type operators (22) on $Lip_M^{L(\eta_1,\eta_2)}$could be established by the following result.

Theorem 4.6. *Let the function $f \in Lip_{M^L}(\eta_1,\eta_2)$, then we have*

$$|\mathcal{K}^{\rho_1,\rho_2}_{n_1,n_2}(f;x,y) - f(x;y)| \le M^L_1\cdot\delta^{\eta_1/2}_{n_1}(x)\delta^{\eta_2/2}_{n_2}(y), \tag{23}$$

where $\delta_{n_1}(x) = \mathcal{K}^{\rho_1}_{n_1}\left((t-x)^2;x\right)$ and $\delta_{n_2}(y) = \mathcal{K}^{\rho_2}_{n_2}\left((s-y)^2;y\right)$.

Proof. Taking the definition (22) of the Lupaş-Kantorovich type operators into account, we may write

$$\begin{aligned}
|\mathcal{K}^{\rho_1,\rho_2}_{n_1,n_2}(f;x,y) - f(x,y)| &\le \mathcal{K}^{\rho_1,\rho_2}_{n_1,n_2}(|f(t,s) - f(x,y)|;x,y)\\
&\le M^L_1\cdot\mathcal{K}^{\rho_1,\rho_2}_{n_1,n_2}(|t-x|^{\eta_1}|s-y|^{\eta_2};x,y)\\
&= M^L_1\cdot\mathcal{K}^{\rho_1}_{n_1}(|t-x|^{\eta_1};x)\mathcal{K}^{\rho_2}_{n_2}(|s-y|^{\eta_2};y).
\end{aligned}$$

Using the Hölder's inequality with $w_1 = \frac{2}{\eta_1}$, $u_1 = \frac{2}{2-\eta_1}$ and $w_2 = \frac{2}{\eta_2}$, $u_2 = \frac{2}{2-\eta_2}$, we get the desired result

$$|\mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}(f;x,y) - f(x,y)| \le M^L \cdot \left(\mathcal{K}_{n_1}^{\rho_1}((t-x)^2;x)\right)^{\frac{\eta_1}{2}} \left(\mathcal{K}_{n_1}^{\rho_1}(e_0;x)\right)^{\frac{2-\eta_1}{2}} \cdot \left(\mathcal{K}_{n_2}^{\rho_2}((s-y)^2;y)\right)^{\frac{\eta_2}{2}} \left(\mathcal{K}_{n_2}^{\rho_2}(e_0;y)\right)^{\frac{2-\eta_2}{2}}$$
$$= M_1^L \cdot \delta_{n_1}^{\eta_1/2}(x)\delta_{n_2}^{\eta_2/2}(y).$$

□

Theorem 4.7. *Let the function $f \in C^1(J^2)$ (which means that partial derivatives $f'_x, f'_y \in C(J^2)$), then*

$$|\mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}(f;x,y) - f(x,y)| \le \|f'_x\| \sqrt{\delta_{n_1}(x)} + \|f'_y\| \sqrt{\delta_{n_2}(y)},$$

with $\delta_{n_1}(x)$ and $\delta_{n_2}(y)$ defined in the above theorem.

Proof. For $(x,y) \in J^2$ fixed, we may write

$$f(t,s) - f(x,y) = \int_x^t f'_w(w,s)dw + \int_y^s f'_u(x,u)du.$$

Applying the Lupaş-Kantorovich type operators $\mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}$ on both sides of the above equality, we get

$$\left|\mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}(f;x,y) - f(x,y)\right| \le \mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}\left(\int_x^t f'_w(w,s)dw;x,y\right) + \mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}\left(\int_y^s f'_u(x,u)du;x,y\right).$$

Since

$$\left|\int_x^t f'_w(w,s)dw\right| \le \|f'_x\| \cdot |t-x| \text{ and } \left|\int_y^s f'_u(x,u)du\right| \le \|f'_y\| \cdot |s-y|,$$

we have

$$\left|\mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}(f;x,y) - f(x,y)\right| \le \|f'_x\| \cdot \mathcal{K}_{n_1}^{\rho_1}(|t-x|;x) + \|f_y\| \cdot \mathcal{K}_{n_2}^{\rho_2}(|s-y|;y).$$

Using the Cauchy-Schwarz inequality, we obtain

$$\left|\mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}(f;x,y) - f(x,y)\right| \le \|f'_x\|\left(\mathcal{K}_{n_1}^{\rho_1}((t-x)^2;x)\right)^{\frac{1}{2}} \left(\mathcal{K}_{n_1}^{\rho_1}(e_0;x)\right)^{\frac{1}{2}} + \|f'_y\|\left(\mathcal{K}_{n_2}^{\rho_2}((s-y)^2;y)\right)^{\frac{1}{2}} \left(\mathcal{K}_{n_2}^{\rho_2}(e_0;y)\right)^{\frac{1}{2}}$$
$$\le \|f'_x\| \sqrt{\delta_{n_1}(x)} + \|f'_y\| \sqrt{\delta_{n_2}(y)}.$$

□

The next result provides the asymptotic behavior of the bivariate Lupaş-Kantorovich type operators (22).

Theorem 4.8. *Let the function $f : J^2 \to \mathbb{R}$. If $f \in C^2(J^2)$, then*

$$\lim_{n\to\infty} n\left(\mathcal{K}_{n,n}^{\rho,\rho}(f;x,y) - f(x,y)\right) = \left(\frac{1}{1+\rho} - x\right) f'_x(x,y) + \left(\frac{1}{1+\rho} - y\right) f'_y(x,y)$$
$$+ x(1-x)f''_{x^2}(x,y) + y(1-y)f''_{y^2}(x,y).$$

Proof. Let $(x,y) \in J^2$ be arbitrary fixed. By the Taylor's expansion bivariate formula with the Peano's form of remainder term, for any $(t,s) \in J^2$ we get

$$f(t,s) = f(x,y) + f'_x(x,y)(t-x) + f'_y(x,y)(s-y) + \frac{1}{2}\left(f''_{x^2}(x,y)(t-x)^2\right. \tag{24}$$
$$\left.+2f''_{xy}(x,y)(t-x)(s-y) + f''_{y^2}(x,y)(s-y)^2\right) + r(t,s;x,y)\sqrt{(t-x)^4 + (s-y)^4},$$

where $r(t,s;x,y) \in C(J^2)$ and $r(t,s;x,y) \to 0$ as $(t,s) \to (x,y)$. Applying $\mathcal{K}_{n,n}^{\rho,\rho}$ on the equation (24), we get

$$\begin{aligned}\mathcal{K}_{n,n}^{\rho,\rho}(f;x,y) &= f(x,y) + f_x'(x,y)\mathcal{K}_n^{\rho}((t-x);x) + f_y'(x,y)\mathcal{K}_n^{\rho}((s-y);x) \\ &+ \tfrac{1}{2}\left(f_{x^2}''(x,y)\mathcal{K}_n^{\rho}((t-x)^2;x) + f_{y^2}''(x,y)\mathcal{K}_n^{\rho}((s-y)^2;y) + 2f_{xy}''(x,y)\mathcal{K}_{n,n}^{\rho,\rho}((t-x)(s-y);x,y)\right) \\ &+ \mathcal{K}_{n,n}^{\rho,\rho}\left(r(t,s;x,y)\sqrt{(t-x)^4+(s-y)^4};x,y\right).\end{aligned} \tag{25}$$

Applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\left|\mathcal{K}_{n,n}^{\rho,\rho}\left(r(t,s;x,y)\sqrt{(t-x)^4+(s-y)^4};x,y\right)\right| &\le \left(\mathcal{K}_{n,n}^{\rho,\rho}(r^2(t,s;x,y);x,y)\right)^{1/2}\cdot\left(\mathcal{K}_{n,n}^{\rho,\rho}((t-x)^4+(s-y)^4;x,y)\right)^{1/2} \\ &\le \left(\mathcal{K}_{n,n}^{\rho,\rho}(r^2(t,s;x,y);x,y)\right)^{1/2}\cdot\left(\mathcal{K}_n^{\rho}((t-x)^4;x)+\mathcal{K}_n^{\rho}((s-y)^4;y)\right)^{1/2}.\end{aligned}$$

In view of Theorem 4.3, $\lim_{n\to\infty}\mathcal{K}_{n,n}^{\rho,\rho}(r^2(t,s;x,y);x,y) = 0$ uniformly on J^2 and taking the linearity, respectively Corollary 2.4 into account, it follows $\mathcal{K}_n^{\rho}((t-x)^4;x) = O\left(\frac{1}{n^2}\right)$ and $\mathcal{K}_n^{\rho}((s-y)^4;y) = O\left(\frac{1}{n^2}\right)$, such that

$$\lim_{n\to\infty} n\cdot\mathcal{K}_{n,n}^{\rho,\rho}\left(r(t,s;x,y)\sqrt{(t-x)^4+(s-y)^4};x,y\right) = 0.$$

Using Corollary 4.2, we get

$$\lim_{n\to\infty} n\cdot\mathcal{K}_n^{\rho}((t-x);x) = \tfrac{1}{1+\rho} - x, \qquad \lim_{n\to\infty} n\cdot\mathcal{K}_n^{\rho}((s-y);y) = \tfrac{1}{1+\rho} - y,$$

and

$$\lim_{n\to\infty} n\cdot\mathcal{K}_n^{\rho}((t-x)^2;x) = 2x(1-x), \qquad \lim_{n\to\infty} n\cdot\mathcal{K}_n^{\rho}((s-y)^2;y) = 2y(1-y).$$

Also, using Corollary 4.2, we get

$$\lim_{n\to\infty} n\cdot\mathcal{K}_n^{\rho}((t-x);x)\mathcal{K}_n^{\rho}((s-y);y) = 0.$$

Combining the above results and the equation (25), we get the desired result. □

Our last result proposed for study is the order of approximation for the bivariate Lupaş-Kantorovich type operators, involving the Petree's K-functional.

Theorem 4.9. *For the function $f \in C(J^2)$, the following inequality*

$$\begin{aligned}\left|\mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}(f;x,y) - f(x,y)\right| &\le C_1\cdot\mathcal{K}(f;A_{n_1,n_2}(x,y)) + \omega^{(c)}\left(f;\sqrt{\left(\frac{1-(1+\rho_1)x}{(n_1+1)(1+\rho_1)}\right)^2+\left(\frac{1-(1+\rho_2)y}{(n_2+1)(1+\rho_2)}\right)^2}\right) \\ &\le M_1^*\cdot\left(\overline{\omega_2}\left(f;\sqrt{A_{n_1,n_2}(x,y)}\right) + A_{n_1,n_2}(x,y)\|f\|_{C(J^2)}\right) + \omega^{(c)}\left(f;\sqrt{\left(\frac{1-(1+\rho_1)x}{(n_1+1)(1+\rho_1)}\right)^2+\left(\frac{1-(1+\rho_2)y}{(n_2+1)(1+\rho_2)}\right)^2}\right)\end{aligned}$$

holds. The constant C is independent of f and $A_{n_1,n_2}(x,y)$, where $A_{n_1,n_2}(x,y) = \frac{1}{(n_1+1)}\delta_{n_1}^2(x) + \frac{1}{(n_2+1)}\delta_{n_2}^2(y)$, $\delta_{n_k}^2(z) = z(1-z) + \frac{1}{(n_k+1)(1+\rho_k)}$, $k = 1,2$ and $z \in J$.

Proof. We introduce the auxiliary bivariate operators as follows

$$\mathcal{T}_{n_1,n_2}^{\rho_1,\rho_2}(f;x,y) = \mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}(f;x,y) + f(x,y) - f\left(x + \frac{1-(1+\rho_1)x}{(n_1+1)(1+\rho_1)}, y + \frac{1-(1+\rho_2)y}{(n_2+1)(1+\rho_2)}\right). \tag{26}$$

Then, using Corollary 4.2, we get $\mathcal{T}_{n_1,n_2}^{\rho_1,\rho_2}(t-x;x,y)=0$ and $\mathcal{T}_{n_1,n_2}^{\rho_1,\rho_2}(s-y;x,y)=0$.
Let $g \in C^2(J^2)$ and $(t,s)\in J^2$. Using the Taylor's expansion bivariate formula, we can write

$$\begin{aligned} g(t,s)-g(x,y) &= g(t,y)-g(x,y)+g(t,s)-g(t,y) \\ &= \frac{\partial g(x,y)}{\partial x}(t-x)+\int_x^t (t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du+\frac{\partial g(x,y)}{\partial y}(s-y)+\int_y^s (s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv. \end{aligned} \tag{27}$$

Applying the operator $\mathcal{T}_{n_1,n_2}^{\rho_1,\rho_2}$ on the relation (27), we get

$$\begin{aligned} &\mathcal{T}_{n_1,n_2}^{\rho_1,\rho_2}(g;x,y)-g(x,y)=\mathcal{T}_{n_1,n_2}^{\rho_1,\rho_2}\Big(\int_x^t (t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du;x,y\Big)+\mathcal{T}_{n_1,n_2}^{\rho_1,\rho_2}\Big(\int_y^s (s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv;x,y\Big) \\ &= \mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}\Big(\int_x^t (t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du;x,y\Big)-\int_x^{x+\frac{1-(1+\rho_1)x}{(n_1+1)(1+\rho_1)}}\Big(x+\frac{1-(1+\rho_1)x}{(n_1+1)(1+\rho_1)}-u\Big)\frac{\partial^2 g(u,y)}{\partial u^2}du \\ &+\mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}\Big(\int_y^s (s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv;x,y\Big)-\int_y^{y+\frac{1-(1+\rho_2)y}{(n_2+1)(1+\rho_2)}}\Big(y+\frac{1-(1+\rho_2)y}{(n_2+1)(1+\rho_2)}-v\Big)\frac{\partial^2 g(x,v)}{\partial v^2}dv. \end{aligned}$$

Hence, applying the Cauchy-Schwarz inequality, it follows

$$\begin{aligned} &\left|\mathcal{T}_{n_1,n_2}^{\rho_1,\rho_2}(g;x,y)-g(x,y)\right| \\ &\le \mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}\Big(\int_x^t |t-u|\left|\frac{\partial^2 g(u,y)}{\partial u^2}\right|du;x,y\Big)+\int_x^{x+\frac{1-(1+\rho_1)x}{(n_1+1)(1+\rho_1)}}\left|x+\frac{1-(1+\rho_1)x}{(n_1+1)(1+\rho_1)}-u\right|\cdot\left|\frac{\partial^2 g(u,y)}{\partial u^2}\right|du \\ &+\mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}\Big(\left|\int_y^s |s-v|\left|\frac{\partial^2 g(x,v)}{\partial v^2}\right|dv\right|;x,y\Big)+\int_y^{y+\frac{1-(1+\rho_2)y}{(n_2+1)(1+\rho_2)}}\left|y+\frac{1-(1+\rho_2)y}{(n_2+1)(1+\rho_2)}-v\right|\cdot\left|\frac{\partial^2 g(x,v)}{\partial v^2}\right|dv \\ &\le \left(\mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}\left((t-x)^2;x,y\right)+\Big(x+\frac{1-(1+\rho_1)x}{(n_1+1)(1+\rho_1)}-x\Big)^2\right)\cdot\|g\|_{C^2(J^2)} \\ &+\left(\mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}((s-y)^2;x,y)+\Big(y+\frac{1-(1+\rho_2)y}{(n_2+1)(1+\rho_2)}-y\Big)^2\right)\cdot\|g\|_{C^2(J^2)} \\ &\le \left(\frac{C}{(n_1+1)}\delta_{n_1}^2(x)+\Big(\frac{1-(1+\rho_1)x}{(n_1+1)(1+\rho_1)}\Big)^2+\frac{C}{(n_2+1)}\delta_{n_2}^2(y)+\Big(\frac{1-(1+\rho_2)y}{(n_2+1)(1+\rho_2)}\Big)^2\right)\cdot\|g\|_{C^2(J^2)}. \end{aligned}$$

Thus, we get

$$\left|\mathcal{T}_{n_1,n_2}^{\rho_1,\rho_2}(g;x,y)-g(x,y)\right|\le\left(\frac{C}{(n_1+1)}\delta_{n_1}^2(x)+\frac{C}{n_2+1}\delta_{n_2}^2(y)\right)\cdot\|g\|_{C^2(J^2)}.$$

Also

$$\left|\mathcal{T}_{n_1,n_2}^{\rho_1,\rho_2}(f;x,y)\right|\le\left|\mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}(f;x,y)\right|+|f(x,y)|+\left|f\Big(\frac{1-(1+\rho_1)x}{(n_1+1)(1+\rho_1)},\frac{1-(1+\rho_2)y}{(n_2+1)(1+\rho_2)}\Big)\right|\le 3\cdot\|f\|_{C(J^2)}.$$

Hence, in view of the relation (4), we get

$$\begin{aligned} &\left|\mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}(f;x,y)-f(x,y)\right| \\ &=\left|\mathcal{T}_{n_1,n_2}^{\rho_1,\rho_2}(f;x,y)-f(x,y)+f\Big(x+\frac{1-(1+\rho_1)x}{(n_1+1)(1+\rho_1)},y+\frac{1-(1+\rho_2)y}{(n_2+1)(1+\rho_2)}\Big)-f(x,y)\right| \\ &\le\left|\mathcal{T}_{n_1,n_2}^{\rho_1,\rho_2}(f-g;x,y)\right|+\left|\mathcal{T}_{n_1,n_2}^{\rho_1,\rho_2}(g;x,y)-g(x,y)\right|+|g(x,y)-f(x,y)| \\ &+\left|f\Big(x+\frac{1-(1+\rho_1)x}{(n_1+1)(1+\rho_1)},y+\frac{1-(1+\rho_2)y}{(n_2+1)(1+\rho_2)}\Big)-f(x,y)\right|. \end{aligned}$$

Further

$$\begin{aligned}\left|\mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}(f;x,y)-f(x,y)\right| &\leq 4\cdot\|f-g\|_{C(J^2)}+\left|\mathcal{T}_{n_1,n_2}^{\rho_1,\rho_2}(g;x,y)-g(x,y)\right|\\ &+\left|f\Big(x+\frac{1-(1+\rho_1)x}{(n_1+1)(1+\rho_1)},y+\frac{1-(1+\rho_2)y}{(n_2+1)(1+\rho_2)}\Big)-f(x,y)\right|\\ &\leq 4\cdot\|f-g\|_{C(J^2)}+\Big(\frac{C}{n_1+1}\delta_{n_1}^2(x)+\frac{C}{n_2+1}\delta_{n_2}^2(y)\Big)\cdot\|g\|_{C^2(J^2)}\\ &+\left|f\Big(x+\frac{1-(1+\rho_1)x}{(n_1+1)(1+\rho_1)},y+\frac{1-(1+\rho_2)y}{(n_2+1)(1+\rho_2)}\Big)-f(x,y)\right|\\ &\leq\Big(4\cdot\|f-g\|_{C(J^2)}+C\cdot A_{n_1,n_2}(x,y)\cdot\|g\|_{C^2(J^2)}\Big)\\ &+\omega^{(c)}\Big(f;\sqrt{\Big(\frac{1-(1+\rho_1)x}{(n_1+1)(1+\rho_1)}\Big)^2+\Big(\frac{1-(1+\rho_2)y}{(n_2+1)(1+\rho_2)}\Big)^2}\Big).\end{aligned}$$

Taking the infimum on the right hand side over all $g\in C^2(J^2)$ and using (21), we get

$$\begin{aligned}&\left|\mathcal{K}_{n_1,n_2}^{\rho_1,\rho_2}(f;x,y)-f(x,y)\right|\\ &\leq C_1\cdot K(f;A_{n_1,n_2}(x,y))+\omega^{(c)}\left(f;\sqrt{\Big(\frac{1-(1+\rho_1)x}{(n_1+1)(1+\rho_1)}\Big)^2+\Big(\frac{1-(1+\rho_2)y}{(n_2+1)(1+\rho_2)}\Big)^2}\right)\\ &\leq M_1^*\cdot\Big(\overline{\omega_2}\Big(f;\sqrt{A_{n_1,n_2}(x,y)}\Big)+\min\{1,A_{n_1,n_2}(x,y)\}\cdot\|f\|_{C(J^2)}\Big)\\ &+\omega^{(c)}\left(f;\sqrt{\Big(\frac{1-(1+\rho_1)x}{(n_1+1)(1+\rho_1)}\Big)^2+\Big(\frac{1-(1+\rho_2)y}{(n_2+1)(1+\rho_2)}\Big)^2}\right)\\ &\leq M_1^*\Big(\overline{\omega_2}\Big(f;\sqrt{A_{n_1,n_2}(x,y)}\Big)+A_{n_1,n_2}(x,y)\cdot\|f\|_{C(J^2)}\Big)\\ &+\omega^{(c)}\left(f;\sqrt{\Big(\frac{1-(1+\rho_1)x}{(n_1+1)(1+\rho_1)}\Big)^2+\Big(\frac{1-(1+\rho_2)y}{(n_2+1)(1+\rho_2)}\Big)^2}\right).\end{aligned}$$

Hence, the proof is completed. □

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